# Groups with 2-generated Sylow subgroups and their character tables

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### Abstract

Let $G$ be a finite group with a Sylow $p$-subgroup $P$. We show that the character table of $G$ determines whether $P$ has maximal nilpotency class and whether $P$ is a minimal non-abelian group. The latter result is obtained from a precise classification of the corresponding groups $G$ in terms of their composition factors. For $p$-constrained groups $G$ we prove further that the character table determines whether $P$ can be generated by two elements.

**Keywords:** maximal class; minimal non-abelian; Sylow subgroup; fusion system; character table
**AMS classification:** 20C15, 20D20

## 1 Introduction

Recently, Navarro and the second author [24] have investigated finite groups $G$ with a Sylow $p$-subgroup $P$ such that $|P : P'| = p^2$ or $|P : \mathrm{Z}(P)| = p^2$ where $P' = [P, P]$ denotes the commutator subgroup and $\mathrm{Z}(P)$ is the center of $P$. It was proved that both properties can be read off from the character table $X(G)$ of $G$. This was another contribution to Richard Brauer's Problem 12 of [3], which asks what properties of a Sylow $p$-subgroup $P$ are determined by $X(G)$. We refer the reader to the introduction of [24] and [32] for a collection of the known results on this problem. We just mention that one important property is that $X(G)$ knows whether $P$ is abelian. While there is an elementary proof of the case $p = 2$ by Camina–Herzog [4], the full solution has required the classification of finite simple groups (see [17, 26, 19]).

After dealing with $P'$ and $\mathrm{Z}(P)$, it is natural to turn our attention to the Frattini subgroup $\Phi(P)$ of $P$. Recall that $|P : \Phi(P)| \leq p$ holds if and only if $P$ is cyclic. It is easy to show that this property can be read off from $X(G)$ (see [21, Corollary 3.12]). In the first part of the present paper we consider groups $G$ with $|P : \Phi(P)| = p^2$, i.e. $P$ is generated by two elements, but not by one. For $p = 2$ this property is detectable by $X(G)$ as was shown by Navarro et al. [23]. We obtain the corresponding result for odd primes $p$ provided that $G$ is $p$-constrained in Corollary 5. In the general case we offer a partial solution depending on the socle of $G$ (see Proposition 6 and the subsequent remark).

Our next objective are groups with Sylow $p$-subgroups $P$ of maximal nilpotency class. For $p = 2$, this property is equivalent to $|P : P'| = 4$. This case was previously handled in an elementary fashion by Navarro–Sambale–Tiep [25]. The general result is our first main theorem.

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**Theorem A.** The character table of a finite group $G$ determines whether $G$ has Sylow $p$-subgroups of maximal nilpotency class.

It is known that $X(G)$ determines the isomorphism types of abelian Sylow subgroups. Of course we cannot expect this for maximal class Sylow subgroups as $X(D_8) = X(Q_8)$. Perhaps surprisingly, $X(G)$ does not even determine $X(P)$. Counterexamples for $p = 3$ arise as semidirect products of non-equivalent faithful actions of $\mathrm{SL}(2,3)$ on $C_9 \times C_9$ (the groups are `SmallGroup`$(2^3 3^5, a)$ with $a \in \{2289, 2290\}$ in GAP [9]). Here $P$ indeed has maximal class. This is related to [22, Question E].

We obtain Theorem A as a consequence of the following structure description, which might be of independent interest.

**Theorem B.** Let $G$ be a finite group with a Sylow $p$-subgroup $P$ of maximal class. Suppose that $\mathrm{O}_{p'}(G) = 1$ and $\mathrm{O}^{p'}(G) = G$. Then one of the following holds:

  (i) There exists $x \in P$ such that $|\mathrm{C}_G(x)|_p = p^2$.

  (ii) $G$ is quasisimple and $|\mathrm{Z}(G)| \leq p$.

The proof uses recent work by Grazian–Parker [12] on fusion systems and is given in Section 3.

In the final part of the paper we study groups with minimal non-abelian Sylow $p$-subgroups $P$, i.e. $P$ is non-abelian, but every proper subgroup of $P$ is abelian. It is easy to see that this happens if and only if $|P : \mathrm{Z}(P)| = |P : \Phi(P)| = p^2$ (see Lemma 9 below). Refining [24, Theorem 7.5], we obtain in Section 4 the following description:

**Theorem C.** Let $G$ be a finite group with a minimal non-abelian Sylow $p$-subgroup $P$ and $\mathrm{O}_{p'}(G) = 1$. Then one of the following holds:

  (i) $p = 2$, $P \in \{D_8, Q_8\}$ and $\mathrm{O}^{2'}(G) \in \{\mathrm{SL}(2,q), \mathrm{PSL}(2,q'), A_7\}$ where $q \equiv \pm 3 \pmod 8$ and $q' \equiv \pm 7 \pmod{16}$.

  (ii) $|P| = p^3$ and $\exp(P) = p > 2$.

  (iii) $G = P \rtimes Q$ where $Q \leq \mathrm{GL}(2,p)$.

  (iv) $p > 2$, $\mathrm{O}^{p'}(G) = S \rtimes C_{p^a}$ where $S$ is a simple group of Lie type with cyclic Sylow $p$-subgroups. The image of $C_{p^a}$ in $\mathrm{Out}(S)$ has order $p$.

  (v) $p = 2$ and $G = \mathrm{PSL}(2, q^f) \rtimes C_{2^a d}$ where $q$ is a prime, $q^f \equiv \pm 3 \pmod 8$ and $d \mid f$. Moreover, $C_{2^a}$ acts as a diagonal automorphism of order 2 on $\mathrm{PSL}(2, q^f)$ and $C_d$ induces a field automorphism of order $d$.

  (vi) $p = 3$ and $\mathrm{O}^{3'}(G) = \mathrm{PSL}^\epsilon(3, q^f) \rtimes C_{3^a}$ where $\epsilon = \pm 1$, $q$ is a prime, $(q^f - \epsilon)_3 = 3$ and $G/\mathrm{O}^{3'}(G) \leq C_f \times C_2$.

Here, $\mathrm{PSL}^\epsilon$ stands for PSL if $\epsilon = 1$ and PSU otherwise. Again the proof is based on the classification of the corresponding fusion systems. To show that Case (iv) in Theorem C occurs for all odd primes $p$, we will exhibit appropriate examples after the proof.

**Corollary D.** The character table of a finite group $G$ determines whether $G$ has minimal non-abelian Sylow $p$-subgroups.

# 2 2-generated Sylow subgroups

In the following $G$ will always denote a finite group. The exponent of $G$ is denoted by $\exp(G)$. The core of a subgroup $H \le G$ is defined by $\operatorname{core}_G(H) := \bigcap_{g \in G} gHg^{-1} \trianglelefteq G$. For $x, y \in G$ let $[x, y] := xyx^{-1}y^{-1}$. The Fitting subgroup and the generalized Fitting subgroup of $G$ are denoted by $\mathrm{F}(G)$ and $\mathrm{F}^*(G) = \mathrm{F}(G)\mathrm{E}(G)$ respectively. We write $\operatorname{Irr}(G)$ to denote the set of ordinary complex irreducible characters of $G$. For $g \in G$ and $\chi \in \operatorname{Irr}(G)$ let

$$\mathbb{Q}(g) := \mathbb{Q}(\chi(g) : \chi \in \operatorname{Irr}(G)),$$
$$\mathbb{Q}(\chi) := \mathbb{Q}(\chi(g) : g \in G).$$

It is well-known that $\mathbb{Q}(\chi)$ lies in the cyclotomic field $\mathbb{Q}_n$ where $n = |G|$. Let $f_\chi$ be the smallest positive integer such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_{f_\chi}$ ($f_\chi$ is called the *Feit number* in [21]). Let $\operatorname{Irr}_{p'}(G) := \{\chi \in \operatorname{Irr}(G) : p \nmid \chi(1)\}$ as usual. The $p$-part and the $p'$-part of an integer $n$ are denoted by $n_p$ and $n_{p'}$ respectively.

Our first lemma is applied multiple times throughout the paper.

**Lemma 1.** *Let $A$ be an abelian normal subgroup of $G$ such that $G = \langle x \rangle A$ for some $x \in G$. Then the map $A \to G'$, $a \mapsto [x, a]$ is an epimorphism with kernel $\mathrm{C}_A(x)$. In particular, $|G'| = |A/\mathrm{C}_A(x)|$.*

*Proof.* See [16, Lemma 4.6]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

To get from $P'$ to $\Phi(P)$ we need the following variant.

**Lemma 2.** *Let $P$ be a $p$-group with a proper normal subgroup $Q$ and $x \in P$ such that $P = \langle x \rangle Q$ and $\langle x \rangle \cap Q \le P'$. Then $|P : \Phi(P)| = p^2$ if and only if $|\mathrm{C}_{Q/\Phi(Q)}(x)| = p$.*

*Proof.* Since $\langle x \rangle \cap Q \le P' \le \Phi(P)$ and $Q < P$, we have

$$P/\Phi(P) = Q\Phi(P)/\Phi(P) \times \langle x \rangle \Phi(P)/\Phi(P) \cong Q/(Q \cap \Phi(P)) \times C_p.$$

Moreover,

$$\Phi(P) \cap Q = P'\Phi(Q)\langle x^p \rangle \cap Q = P'\Phi(Q)(\langle x^p \rangle \cap Q) = P'\Phi(Q).$$

Now $|P : \Phi(P)| = p^2$ if and only if

$$|Q/\Phi(Q) : (P/\Phi(Q))'| = |Q : P'\Phi(Q)| = p.$$

By Lemma 1 applied to $Q/\Phi(Q) \trianglelefteq P/\Phi(Q)$, this is equivalent to $|\mathrm{C}_{Q/\Phi(Q)}(x)| = p$. $\qquad\square$

The next result is a variation of [24, Theorem 6.1].

**Lemma 3.** *Let $G$ be a finite group with a Sylow $p$-subgroup $P$ and $\mathrm{O}_{p'}(G) = 1$. Then*

$$K := \bigcap_{\substack{\chi \in \operatorname{Irr}_{p'}(G) \\ p^2 \nmid f_\chi}} \operatorname{Ker}(\chi) = \operatorname{core}_G(\Phi(P)).$$

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*Proof.* Let $n := |G|$. If $n_p = 1$, then the claim holds since $\bigcap_{\chi \in \mathrm{Irr}(G)} \mathrm{Ker}(\chi) = 1 = P$. Thus, let $n_p \neq 1$. Then $\mathcal{G} := \mathrm{Gal}(\mathbb{Q}_n | \mathbb{Q}_{pn_{p'}})$ is a $p$-group. Let $N := \mathrm{core}_G(\Phi(P))$ and $\chi \in \mathrm{Irr}_{p'}(G)$ with $p^2 \nmid f_\chi$. Since $\mathbb{Q}(\chi_P) \subseteq \mathbb{Q}(\chi) \subseteq \mathbb{Q}_{pn_{p'}}$, $\mathcal{G}$ permutes the irreducible constituents of $\chi_P$. Since the sizes of the $\mathcal{G}$-orbits are $p$-powers and $p \nmid \chi(1)$, there must be a linear constituent $\lambda \in \mathrm{Irr}(P|\chi)$ fixed by $\mathcal{G}$, i.e. $\mathbb{Q}(\lambda) \subseteq \mathbb{Q}_p$. It follows that $N \subseteq \Phi(P) \subseteq \mathrm{Ker}(\lambda)$. By Clifford theory, $\chi_N$ is a sum of conjugates of $\lambda_N$. Hence, $N \subseteq \mathrm{Ker}(\chi)$. This shows that $N \leq K$.

Now let $\lambda \in \mathrm{Irr}(P/\Phi(P))$. This time, $\mathcal{G}$ acts on the irreducible constituents of $\lambda^G$. Since $p \nmid |G : P| = \lambda^G(1)$, there must be a constituent $\chi \in \mathrm{Irr}_{p'}(G|\lambda)$ fixed by $\mathcal{G}$, i.e. $p^2 \nmid f_\chi$. This implies $\chi_{P \cap K} = \chi(1) 1_{P \cap K}$. On the other hand, $\lambda_{P \cap K}$ is a constituent of $\chi_{P \cap K}$. Therefore, $P \cap K \subseteq \mathrm{Ker}(\lambda)$. Since $\lambda \in \mathrm{Irr}(P/\Phi(P))$ was arbitrary, we obtain $P \cap K \leq \Phi(P)$. Now Tate's theorem (see [15, Satz IV.4.7]) yields that $K$ is $p$-nilpotent. By hypothesis, $\mathrm{O}_{p'}(K) \leq \mathrm{O}_{p'}(G) = 1$ and $K$ is a $p$-group. Finally, $K \leq \mathrm{O}_p(G) \cap K \leq P \cap K \leq \Phi(P)$ and $K \leq N$. $\qquad\square$

We mention that the characters $\chi$ with $p^2 \nmid f_\chi$ are precisely the *almost p-rational* characters introduced in [14]. Lemma 3 allows to read off $K := \mathrm{core}_G(\Phi(P))$ from the character table. Since $|P/K : \Phi(P/K)| = |P : \Phi(P)|$, it is therefore no loss to assume that $K = 1$. The next theorem comes close to [24, Theorem 3.1].

**Theorem 4.** *Let $G$ be a finite group with a non-abelian Sylow $p$-subgroup $P$ such that $|P : \Phi(P)| = p^2$ and $\mathrm{O}_{p'}(G) = 1 = \mathrm{core}_G(\Phi(P))$. Then $\mathrm{F}^*(G)$ is the unique minimal normal subgroup of $G$ and $P\mathrm{F}^*(G)/\mathrm{F}^*(G)$ is cyclic. If $\mathrm{F}^*(G)$ is non-abelian, then $P$ permutes the simple components of $\mathrm{F}^*(G)$ transitively. In particular, their number is a $p$-power in this case.*

*Proof.* Let $N$ be a minimal normal subgroup of $G$. Then

$$|PN/N : \Phi(PN/N)| = |P/P \cap N : \Phi(P/P \cap N)| = |P/P \cap N : \Phi(P)(P \cap N)/P \cap N|$$
$$= |P : \Phi(P)(P \cap N)| \leq |P : \Phi(P)| = p^2,$$

where the second equality follows from [16, Lemma 4.5], for instance. Suppose first that $P \cap N \leq \Phi(P)$. Then by Tate's theorem (see [15, Satz IV.4.7]), $N$ is a $p$-group and $N \leq \Phi(P)$. This contradicts $\mathrm{core}_G(\Phi(P)) = 1$. Consequently, $|PN/N : \Phi(PN/N)| \leq p$ and $PN/N$ is cyclic. Let $M \neq N$ be another minimal normal subgroup of $G$. Then $G/N$ and similarly $G/M$ have cyclic Sylow $p$-subgroups. Since $G$ is isomorphic to a subgroup of $G/M \times G/N$, $G$ has abelian Sylow $p$-subgroups, which we have excluded explicitly. This shows that $N$ is the unique minimal normal subgroup.

Assume now that $N$ is non-abelian. Then $\mathrm{F}(G) \cap N = 1$ implies $\mathrm{F}(G) = 1 = \mathrm{Z}(G)$ and $\mathrm{F}^*(G) = \mathrm{E}(G) = N$. Write $N = T_1 \times \ldots \times T_n$ with non-abelian simple groups $T_1 \cong \ldots \cong T_n$. If $P \leq N$, using that $P$ is 2-generated and non-abelian, we conclude that $n = 1$ and $P$ certainly acts transitively on $\{T_1, \ldots, T_n\}$. Hence, we may assume that $P \not\subseteq N$ and $n \geq 2$. Let $Q_i := P \cap T_i$ for $i = 1, \ldots, n$. Let $x \in P$ such that $PN/N = \langle xN \rangle$. Since $P \cap N \not\subseteq \Phi(P)$, there exists some $1 \leq i \leq n$ with $Q_i \not\subseteq \Phi(P)$. Without loss of generality, let $i = 1$. Choose $y \in Q_1 \setminus \Phi(P)$. For all $j \in \mathbb{Z}$ we note that $xy^j \notin N \supseteq \Phi(P)$. Since $|P : \Phi(P)| = p^2$, it follows that $P = \langle x, y \rangle$. Without loss of generality, let $T_1, \ldots, T_k$ be the orbit of $T_1$ under $P$. Suppose by way of contradiction that $k < n$. Then $Q_1 \ldots Q_k \trianglelefteq P$ and $Q_{k+1} \times \ldots \times Q_n \leq P/Q_1 \ldots Q_k = \langle xQ_1 \ldots Q_k \rangle$ is cyclic. This is only possible if $n = k + 1$ and $Q_n$ is cyclic. Moreover, $Q_n = \langle x^{p^a} z \rangle$ for some $a \geq 1$ and $z \in Q_1 \cdots Q_k$. Since a non-abelian simple group cannot have a cyclic Sylow 2-subgroup, $p > 2$. It follows from [13, Theorem A] that $x$ induces an inner automorphism on $T_n$. This is impossible since $x^{p^a}$ induces an inner automorphism of order $|T_n|_p$. This contradiction shows that $P$ permutes the $T_i$ transitively.

Finally, assume that $N$ is elementary abelian. Since $O_{p'}(G) = 1$, we have $F := F(G) = O_p(G)$. Suppose that $N < F$. Then $\Phi(F) \le \Phi(P)$ yields $\Phi(F) \le \mathrm{core}_G(\Phi(P)) = 1$, i.e. $F$ is elementary abelian. Now the existence of an element of order $p$ in $P \setminus N$ implies the existence of a (cyclic) complement of $N$ in $P$. By a theorem of Gaschütz (see [15, Hauptsatz I.17.4]), $N$ has a complement $K$ in $G$. Since $F$ centralizes $N$, we obtain $1 \ne K \cap F \trianglelefteq NK = G$. This contradicts the fact that $N$ is the unique minimal normal subgroup of $G$. Hence, $F = N$. Suppose that $E(G) \ne 1$ and choose a central product $M \trianglelefteq G$ of quasisimple components. Then $N \le Z(M)$, because $1 \ne N \cap M \trianglelefteq G$. Since $M/N$ has cyclic Sylow $p$-subgroups, the order of the Schur multiplier of $M/N$ is not divisible by $p$. This contradicts $N \le Z(M)$. We have therefore shown that $N = F^*(G)$. $\qquad\square$

In order to decide whether $|P : \Phi(P)| = p^2$, we may assume that the hypotheses of Theorem 4 are fulfilled. The situation now splits into two cases. When $F^*(G)$ is abelian, the group $G$ is $p$-constrained (recall that in general a group $G$ is called $p$-constrained if $C_{\overline{G}}(O_p(\overline{G})) \le O_p(\overline{G})$ where $\overline{G} := G/O_{p'}(G)$). In this case we solve the problem completely. To do so, we will use a result of Higman (see [21, Corollary 7.18]) that allows to locate the $p$-elements in $X(G)$.

**Corollary 5.** *The character table of a $p$-constrained group $G$ determines whether a Sylow $p$-subgroup $P$ is generated by two elements.*

*Proof.* Let $P$ be a Sylow $p$-subgroup of $G$. Since the character table $X(G)$ determines $X(G/O_{p'}(G))$, we may assume that $O_{p'}(G) = 1$. Since $G$ is $p$-constrained, $O_p(G) > 1$. By Lemma 3, we may assume that $\mathrm{core}_G(\Phi(P)) = 1$. Moreover, the orders and embeddings of the normal subgroups of $G$ can be read off from $X(G)$. Hence by Theorem 4, we may assume that $N = O_p(G) = F(G)$ is the only minimal normal subgroup of $G$. If $P = N$, then $|P : \Phi(P)| = |P|$ and we are done. Hence, let $N < P$. By [21, Corollary 3.12], $X(G/N)$ detects whether $P/N$ is cyclic. By Theorem 4, we can assume that this is the case. Choose $x \in P$ with $P/N = \langle xN \rangle$ (note that $x$ can be spotted in $X(G)$ using [21, Corollary 3.12]). Since $P = N\langle x \rangle = O_p(G)\langle x \rangle$ is the only Sylow $p$-subgroup of $G$ containing $x$, $C_P(x) = C_N(x)\langle x \rangle$ is a Sylow $p$-subgroup of $C_G(x)$. In particular, $|C_N(x)| = \frac{|C_G(x)|_p}{|P/N|}$ is determined by $X(G)$. By Lemma 1, we have
$$P' = [x, N] = \{[x, y] : y \in N\} \tag{2.1}$$
and $|P'| = |N/C_N(x)|$ can be computed from $X(G)$. Let $|P/N| = p^a$ and $|N/P'| = p^n$. If $x^{p^a} \in P'$, then $P/P' \cong C_{p^a} \times C_p^n$ and otherwise $P/P' \cong C_{p^{a+1}} \times C_p^{n-1}$. Since $\mathbb{Q}(x)$ can be read off from $X(G)$, it suffices to show that
$$p|\mathbb{Q}(x) : \mathbb{Q}|_p = \exp(P/P').$$
Taking only $X(G/N)$ into account, we obtain $\mathbb{Q}(xN) = \mathbb{Q}_{p^a}$ or equivalently $|\mathbb{Q}(xN) : \mathbb{Q}|_p = p^{a-1}$ by [21, Theorem 3.11]. It follows that $|\mathbb{Q}(x) : \mathbb{Q}|_p \ge p^{a-1}$. If $x^{p^a} = 1$, then $p|\mathbb{Q}(x) : \mathbb{Q}|_p = p^a = \exp(P/P')$ as desired. Now let $|\langle x \rangle| = p^{a+1}$. If $x^{p^a} \in P'$, then there exists $y \in N$ with $x^{p^a} = [x, y] = xyx^{-1}y^{-1}$ by (2.1). It follows that $yxy^{-1} = x^{1-p^a}$ and $|N_G(\langle x \rangle) : C_G(x)|_p = p$. Again by [21, Theorem 3.11], we have $p|\mathbb{Q}(x) : \mathbb{Q}|_p = p^a = \exp(P/P')$. Assume conversely that $|\mathbb{Q}(x) : \mathbb{Q}|_p = p^{a-1}$. Then there exists $y \in G$ with $yxy^{-1} = x^{1+kp^a}$ for some $0 < k < p$. We observe that $y \in N_G(\langle x \rangle N) = N_G(P)$. Replacing $y$ by its $p$-part, we get $y \in P$. Now $x^{-kp^a} = [x, y] \in P'$ and $\exp(P/P') = p^a$ as desired. $\qquad\square$

If $G$ is $p$-solvable in the situation of Corollary 5 (recall that every $p$-solvable group is $p$-constrained), then $O_p(G)$ has a complement $K$ in $O_{pp'}(G)$ by the Schur–Zassenhaus theorem. Using the Frattini argument, it is easy to show that $N_G(K)$ is a complement of $N$ in $G$. In this situation, $G$ is a primitive permutation group on $N$ of affine type.

On the other hand, every non-abelian simple group $S$ gives rise to a non-split extension $G = N.S$ where $N = \Phi(G)$ is elementary abelian without complement (see [7, Theorem B.11.8]). Garrison [10] has exhibited examples to show that $X(G)$ does not determine whether $G$ splits over $N$. For instance, `PerfectGroup(7500, 1)` $\cong C_5^3 \rtimes A_5$ and `PerfectGroup(7500, 2)` $\cong C_5^3.A_5$ in GAP [9] have the same character table and the Sylow 5-subgroup is 2-generated in both cases.

Now assume that $N = \mathrm{F}^*(G)$ in the situation of Theorem 4 is non-abelian. If $N \cap P$ is abelian, then $N$ has a complement in $PN$ by [15, Satz IV.3.8]. In this case $PN$ is a twisted wreath product. The non-split extension $M_{10} = A_6.C_2$ with $P = SD_{16}$, a semidihedral group, shows that this is not always the case. Even when $N$ is not simple, $P \cap N$ is not always abelian (as in [24, Theorem 3.1]). One example is

$$G = \mathrm{PSL}(2,7)^2 \rtimes \langle x \rangle \cong \mathrm{PSL}(2,7)^2 \rtimes C_4 \leq \mathrm{PGL}(2,7) \wr C_2$$

where $x^2$ acts as a diagonal automorphism on both factors $\mathrm{PSL}(2,7)$ simultaneously. Here $P = D_8^2 \rtimes C_4$ is 2-generated. Nevertheless, we provide the following reduction theorem.

**Proposition 6.** *Let $G$ be a finite group with Sylow $p$-subgroup $P$ such that $\mathrm{O}_{p'}(G) = 1$ and $N = \mathrm{F}^*(G)$ is the unique minimal normal subgroup of $G$. Suppose that $N$ is non-abelian and $PN/N$ is cyclic. Let $S$ be a simple component of $N$. Assume that $|G : \mathrm{N}_G(S)|$ is a $p$-power. Then the following hold:*

*(i) $G = \mathrm{N}_G(S)P$.*

*(ii) $\tilde{P} := \mathrm{N}_P(S)\mathrm{C}_G(S)/\mathrm{C}_G(S)$ is a Sylow $p$-subgroup of the almost simple group $\mathrm{N}_G(S)/\mathrm{C}_G(S)$ with socle $\tilde{S} := S\mathrm{C}_G(S)/\mathrm{C}_G(S) \cong S$. Moreover, $\tilde{P}\tilde{S}/\tilde{S}$ is cyclic.*

*(iii) $|P : \Phi(P)| \leq p^2$ if and only if $|\tilde{P} : \Phi(\tilde{P})| \leq p^2$.*

*(iv) $S$ and $|\tilde{P}|$ are determined by $X(G)$.*

*Proof.*

(i) Since $|G : \mathrm{N}_G(S)|$ is a $p$-power, $|\mathrm{N}_G(S)P| = |\mathrm{N}_G(S) : \mathrm{N}_P(S)||P| = |G|$ and $G = \mathrm{N}_G(S)P$.

(ii) By (i), $\mathrm{N}_P(S)$ is a Sylow $p$-subgroup of $\mathrm{N}_G(S)$. Hence, $\tilde{P}$ is a Sylow $p$-subgroup of $\mathrm{N}_G(S)/\mathrm{C}_G(S)$. Let $Q := N \cap P \trianglelefteq P$. Then $P/Q \cong PN/N$ is cyclic by hypothesis. Let $x \in P$ such that $P = \langle x \rangle Q$. Then $\tilde{P}\tilde{S}/\tilde{S} \cong \mathrm{N}_P(S)S\mathrm{C}_G(S)/S\mathrm{C}_G(S) \leq \langle x \rangle S\mathrm{C}_G(S)/S\mathrm{C}_G(S)$ is cyclic.

(iii) If $P \leq N \leq \mathrm{N}_G(S)$, then $S \trianglelefteq G$ and $N = S$. Here, $P \cong \tilde{P}$, so we are done. Now assume $PN/N \neq 1$. As in (ii), let $Q := N \cap P \trianglelefteq P$. Since $\mathrm{O}^p(PN) = N$, there exists $x \in P$ such that $P = \langle x \rangle Q$ and $\langle x \rangle \cap Q \leq P'$ (see [2, Satz 3.3]). Lemma 2 yields $|P : \Phi(P)| = p^2$ if and only if $|\mathrm{C}_{Q/\Phi(Q)}(x)| = p$.

By (i), we may write $N = T_1 \times \ldots \times T_{p^a}$ such that $T_i = x^{i-1}Sx^{1-i}$ for $i = 1, \ldots, p^a$. Let $Q_i := T_i \cap P \leq Q$. Then $\tilde{Q} := Q_1\mathrm{C}_G(S)/\mathrm{C}_G(S) \cong Q_1$ is a normal subgroup of $\tilde{P}$. Since $\mathrm{N}_P(S) = \langle x^{p^a} \rangle Q$, we have $\tilde{P} = \langle \tilde{x} \rangle \tilde{Q}$ where $\tilde{x} := x^{p^a}\mathrm{C}_G(S)$. It is easy to see that the map

$$\mathrm{C}_{Q_1/\Phi(Q_1)}(x^{p^a}) \to \mathrm{C}_{Q/\Phi(Q)}(x), \quad y\Phi(Q_1) \mapsto \prod_{i=0}^{p^a-1} x^i y x^{-i} \Phi(Q)$$

is an isomorphism. In particular, $|\mathrm{C}_{Q/\Phi(Q)}(x)| = |\mathrm{C}_{Q_1/\Phi(Q_1)}(x^{p^a})|$. Assume for the moment that $x^{p^a} \in Q$. Then $\tilde{P} = \tilde{Q} \leq \tilde{S}$ and $|\mathrm{C}_{Q_1/\Phi(Q_1)}(x^{p^a})| = |Q_1/\Phi(Q_1)| = |\tilde{P}/\Phi(\tilde{P})|$. In this case, $|P : \Phi(P)| = p^2$ if and only if $\tilde{P}$ is cyclic, i.e. $|\tilde{P} : \Phi(\tilde{P})| = p$. Now let $x^{p^a} \notin Q$. By way of

contradiction, suppose that $x^{p^a} \in Q_1 \mathrm{C}_G(S)$. Then there exists $y \in Q_1$ such that $x^{p^a} y \in \mathrm{C}_G(S)$. Now also

$$z := x^{p^a} \prod_{i=0}^{p^a-1} x^i y x^{-i} \in \mathrm{C}_G(S).$$

Since $z$ is centralized by $x$, it follows that $z \in x^i \mathrm{C}_G(S) x^{-i} = \mathrm{C}_G(T_i)$ for $i = 1, \ldots, p^a$. Hence, $z \in \mathrm{C}_G(N) = 1$ and $x^{p^a} \in Q$, a contradiction. Thus, $\tilde{Q} < \tilde{P}$ and

$$\tilde{Q} \cap \langle \tilde{x} \rangle = (Q \cap \langle x^{p^a} \rangle) \mathrm{C}_G(S) / \mathrm{C}_G(S) \leq P' \mathrm{C}_G(S) / \mathrm{C}_G(S) = \tilde{P}'.$$

Lemma 2 shows that $|\tilde{P} : \Phi(\tilde{P})| = p^2$ if and only if

$$|\mathrm{C}_{Q_1/\Phi(Q_1)}(x^{p^a})| = |\mathrm{C}_{\tilde{Q}/\Phi(\tilde{Q})}(\tilde{x})| = p.$$

Now the claim follows.

(iv) The isomorphism types of $N$ and $S$ are determined by $X(G)$ according to [24, Theorem 4.1]. We obtain $|\mathrm{N}_P(S)|$ from $|N| = |S|^{|P:\mathrm{N}_P(S)|}$. Arguing as in (iii), shows that $\mathrm{C}_P(S) = \mathrm{C}_Q(S) = Q_2 \ldots Q_{p^a}$. Hence, $|\mathrm{C}_P(S)| = |S|_p^{p^a-1}$ is computable from $X(G)$. The claim follows from $\tilde{P} \cong \mathrm{N}_P(S)/\mathrm{C}_P(S)$. $\square$

To decide whether $|P : \Phi(P)| = p^2$ holds, it suffices to obtain the structure of $\tilde{P}$ with the notation from Proposition 6. If $p \geq 5$ and $S$ is neither a linear nor a unitary group, then $\mathrm{Out}(S)$ has a cyclic Sylow $p$-subgroup by [5, Table 5]. In this case the isomorphism type of $\tilde{P}$ is uniquely determined by $X(G)$ and the problem is solved. On the other hand, the proof of [24, Lemma 5.1] shows that for linear and unitary groups $S$ the condition $|P : \Phi(P)| = p^2$ is not determined by $|\tilde{P}|$ alone. It remains a challenge to settle these cases (and $p = 3$ with $S = D_4(q)$, $E_6(q)$ and $^2E_6(q)$).

# 3 $p$-groups of maximal class

We start by introducing some terminology of (saturated, non-exotic) fusion systems. Let $P$ be a Sylow $p$-subgroup of $G$ as before. The *fusion system* $\mathcal{F} = \mathcal{F}_P(G)$ of $G$ on $P$ is a category whose objects are the subgroups of $P$ and the morphisms of $\mathcal{F}$ have the form $f : S \to T$, $x \mapsto gxg^{-1}$ where $S, T \leq P$ and $g \in G$. Then $\mathrm{Aut}_\mathcal{F}(S) \cong \mathrm{N}_G(S)/\mathrm{C}_G(S)$ and $\mathrm{Out}_\mathcal{F}(S) \cong \mathrm{N}_G(S)/S\mathrm{C}_G(S)$. Elements $x, y \in P$ (or subsets $S, T \subseteq P$) are called $\mathcal{F}$-*conjugate* if there exists a morphism $f$ such that $f(x) = y$ (or $f(S) = T$). A subgroup $S \leq P$ is called

- *fully normalized*, if $|\mathrm{N}_P(T)| \leq |\mathrm{N}_P(S)|$ for all $\mathcal{F}$-conjugates $T$ of $S$.

- *centric*, if $\mathrm{C}_P(T) = \mathrm{Z}(T)$ for all $\mathcal{F}$-conjugates $T$ of $S$.

- *radical*, if $\mathrm{O}_p(\mathrm{Aut}_\mathcal{F}(S)) = \mathrm{Inn}(S)$ (equivalently, $\mathrm{O}_p(\mathrm{Out}_\mathcal{F}(S)) = 1$).

- *essential*, if $S$ is fully normalized, centric and $\mathrm{Out}_\mathcal{F}(S)$ contains a strongly $p$-embedded subgroup (see [1, Definition A.6]). For our purpose, it is enough to know that $S$ is radical in this case.

By Alperin's fusion theorem, every morphism in $\mathcal{F}$ is a composition of restrictions of morphisms $f \in \mathrm{Aut}_\mathcal{F}(S)$ where $S = P$ or $S$ is essential (see [1, Theorem I.3.5]). Note that $\mathrm{Aut}_\mathcal{F}(P)$ permutes the essential subgroups by conjugation. Hence, if $Q \leq P$ does not lie in any essential subgroup, then $Q$ is fully normalized. In this case, $\mathrm{N}_P(Q)$ is a Sylow $p$-subgroup of $\mathrm{N}_G(Q)$ (see [1, Lemma I.1.2]). Consequently, $\mathrm{C}_P(Q) = \mathrm{N}_P(Q) \cap \mathrm{C}_G(P)$ is a Sylow $p$-subgroup of $\mathrm{C}_G(P)$.

We call $\mathcal{F}$ *controlled* if $\mathrm{N}_G(P)$ controls the fusion in $P$ with respect to $G$, i.e. every morphism $S \to T$ has the form $x \mapsto gxg^{-1}$ for some $g \in \mathrm{N}_G(P)$. Abstractly, this means that there are no essential subgroups and $\mathcal{F} = \mathcal{F}_P(P \rtimes A)$ for some Schur–Zassenhaus complement $A$ of $\mathrm{Inn}(P)$ in $\mathrm{Aut}_{\mathcal{F}}(P)$. More generally, $\mathcal{F}$ is called *constrained* if there exists $Q \trianglelefteq P$ such that $\mathrm{C}_P(Q) = \mathrm{Z}(Q)$ and $\mathrm{N}_G(Q)$ controls the fusion in $P$. By the model theorem (see [1, Theorem I.4.9]), a constrained fusion system is realized by a unique group $G$ such that $\mathrm{C}_G(\mathrm{O}_p(G)) \le \mathrm{O}_p(G)$ (note that $G$ is $p$-constrained). The largest subgroup $Q \trianglelefteq P$ such that $\mathrm{N}_G(Q)$ controls the fusion in $P$ is denoted by $\mathrm{O}_p(\mathcal{F})$. Note that $\mathrm{O}_p(G) \le \mathrm{O}_p(\mathcal{F})$.

It is well-known that a $p'$-automorphism of $Q \le P$ acts non-trivially on $Q/\Phi(Q)$. If $Q$ is radical, it follows that $\mathrm{Out}_{\mathcal{F}}(Q)$ acts faithfully on $Q/\Phi(Q)$. Now assume that there exists a series of characteristic subgroups $\Phi(Q) = Q_0 < \ldots < Q_n = Q$ of $Q$. Then $\mathrm{Out}_{\mathcal{F}}(Q)$ acts faithfully on $Q_n/Q_{n-1} \times \ldots \times Q_1/Q_0$ by [11, 5.3.2]. This argument will often be applied in the following to exclude same candidates of essential subgroups.

We say that a $p$-group $P$ of order $p^n$ has *maximal class* if the nilpotency class is $n-1$. This may include the case $|P| = p^2$. The 2-groups of maximal class are the dihedral groups (including $C_2^2$), the semidihedral groups, the (generalized) quaternion groups and $C_4$ (see [15, Satz III.11.9]). Now assume that $n \ge 4$ and $p > 2$ to avoid some degenerate cases. Let $\mathrm{K}_2(P) = P'$ and $\mathrm{K}_{i+1}(P) = [P, \mathrm{K}_i(P)]$ for $i \ge 2$. Let $\mathrm{Z}_0(P) := 1$ and $\mathrm{Z}_{i+1}(P/\mathrm{Z}_i(P)) := \mathrm{Z}(P/\mathrm{Z}_i(P))$ for $i \ge 0$. Then $\mathrm{K}_i(P) = \mathrm{Z}_{n-i}(P)$ is the only normal subgroup of $P$ of index $p^i$ by [15, Hilfssatz III.14.2]. It is easy to see that the characteristic subgroups $P_1 := \mathrm{C}_P(\mathrm{K}_2(P)/\mathrm{K}_4(P))$ and $P_2 := \mathrm{C}_P(\mathrm{Z}_2(P))$ are maximal in $P$.

**Lemma 7.** *Let $P$ be a $p$-group with a non-abelian subgroup $Q \le P$ of order $p^3$ and exponent $p$. If $\mathrm{C}_P(Q) = \mathrm{Z}(Q)$, then $\mathrm{Z}_2(P) \le Q$.*

*Proof.* Since $\mathrm{Z}(P) \le \mathrm{C}_P(Q)$, we have $Z := \mathrm{Z}(P) = \mathrm{Z}(Q) \cong C_p$. Let $xZ \in \mathrm{C}_{P/Z}(Q/Z)$. Then $x \in \mathrm{N}_P(Q)$. By [35], $\mathrm{N}_P(Q)/Q \le \mathrm{Out}(Q) \cong \mathrm{GL}(2, p)$. As mentioned above, the kernel of the action of $\mathrm{Aut}(Q)$ on $Q/Z$ is a $p$-group. Since $\mathrm{O}_p(\mathrm{GL}(2, p)) = 1$, we obtain $x \in Q$. Hence, $\mathrm{Z}_2(P)/Z = \mathrm{Z}(P/Z) \le \mathrm{C}_{P/Z}(Q/Z) = Q/Z$ and $\mathrm{Z}_2(P) \le Q$. $\qquad\square$

**Lemma 8.** *Let $G$ be a finite group with Sylow $p$-subgroup $P$ of maximal class. Let $N \trianglelefteq G$ such that $p^2 \le |N|_p < |P|$. Then there exists $x \in P$ such that $|\mathrm{C}_G(x)|_p = p^2$.*

*Proof.* By hypothesis, $|P| \ge p|N|_p \ge p^3$. In particular, $\mathrm{Z}(P)$ is the unique normal subgroup of order $p$ of $P$. Since $M := P \cap N \trianglelefteq P$, we have $\mathrm{Z}(P) \le N$. If $|P| = p^3$, every element $x \in P \setminus N$ cannot be conjugate to an element of $\mathrm{Z}(P) \le N$. Hence, $|\mathrm{C}_G(x)|_p = p^2$. Now assume that $|P| \ge p^4$. If $p = 2$, $P$ is a dihedral, semidihedral or quaternion group and we choose $x \in P$ outside the cyclic maximal subgroup of $P$. For $p > 2$, let $x \in P \setminus (P_1 \cup P_2)$. By [15, Hilfssatz III.14.13], we have $|\mathrm{C}_P(x)| = p^2$. Since $|P| \ge p^4$, $\mathrm{Z}_2(P)$ is the unique normal subgroup of order $p^2$ in $P$. In particular, $\mathrm{Z}_2(P) \le M$ since $|M| \ge p^2$. If $p = 2$, we may assume that $x \notin M$. For $p > 2$, we have $P_1 \cup P_2 \cup M \subsetneq P$. Again we may choose $x \notin M$.

Let $\mathcal{F}$ be the fusion system of $G$ on $P$. If $x$ is not contained in any essential subgroup, then $\langle x \rangle$ is fully normalized as explained above. It follows that $|\mathrm{C}_G(x)|_p = |\mathrm{C}_P(x)| = p^2$ and we are done. Now let $Q < P$ be essential containing $x$. By [12, Theorem D], $Q$ is a so-called pearl, i.e. $Q$ is elementary abelian of order $p^2$ or non-abelian of order $p^3$ and exponent $p$ (or $Q = Q_8$ if $p = 2$, see [12, Lemma 6.1]). As an essential subgroup, $Q$ is centric and $\mathrm{C}_P(Q) = \mathrm{Z}(Q)$. Assume first that $|Q| = p^2$. Then

$$Z := \mathrm{Z}(P) = M \cap Q = N \cap Q \trianglelefteq \mathrm{N}_G(Q).$$

Since $Q$ is radical, $\mathrm{Out}_{\mathcal{F}}(Q) \cong \mathrm{N}_G(Q)/Q$ acts faithfully on $Z \times Q/Z \cong C_p^2$. But then $\mathrm{Out}_{\mathcal{F}}(Q)$ would be a $p'$-group in contradiction to $Q < \mathrm{N}_P(Q)$. Next let $|Q| = p^3$. Here, Lemma 7 shows that $\mathrm{Z}_2(P) = M \cap Q = N \cap Q \trianglelefteq \mathrm{N}_G(Q)$. Then $\mathrm{Out}_{\mathcal{F}}(Q)$ acts faithfully on $\mathrm{Z}_2(P)/Z \times Q/\mathrm{Z}_2(P) \cong C_p^2$ and we derive another contradiction. $\qquad \square$

**Theorem B.** Let $G$ be a finite group with a Sylow $p$-subgroup $P$ of maximal class. Suppose that $\mathrm{O}_{p'}(G) = 1$ and $\mathrm{O}^{p'}(G) = G$. Then one of the following holds:

(i) There exists $x \in P$ such that $|\mathrm{C}_G(x)|_p = p^2$.

(ii) $G$ is quasisimple and $|\mathrm{Z}(G)| \leq p$.

*Proof.* We may assume that $G$ is not simple and $|P| \geq p^3$. Let $N < G$ be a maximal normal subgroup. Then $1 < |N|_p < |P|$ as $\mathrm{O}_{p'}(G) = 1$ and $\mathrm{O}^{p'}(G) = G$. If $|N|_p \geq p^2$, then the claim follows from Lemma 8. Hence, let $|N|_p = p$. Then $P \cap N \trianglelefteq P$ has index $p^s \geq p^2$ and therefore $P \cap N = \mathrm{K}_s(P) \leq P'$. By Tate's theorem (see [15, Satz IV.4.7]), $N$ has a normal $p$-complement. Since $\mathrm{O}_{p'}(G) = 1$, this forces $|N| = p$. Since $|G : \mathrm{C}_G(N)|$ divides $p - 1$, we further have $N \leq \mathrm{Z}(G)$. Since $G/N$ is simple, $G$ is quasisimple with $|\mathrm{Z}(G)| \leq p$. $\qquad \square$

If Case (ii) in Theorem B applies with $|\mathrm{Z}(G)| = p$ and (i) fails, then Robinson's ordinary weight conjecture predicts the existence of an irreducible character $\chi$ in the principal $p$-block such that $p^2\chi(1)_p = |G|_p$ (see [28, Lemma 4.7]). Conversely, such a character can only appear when $P$ has maximal class. Examples are $\mathrm{SL}(2,9)$ for $p = 2$, $\mathrm{SL}(3,19)$ for $p = 3$ and $\mathrm{SL}(p,q)$ for $p \geq 5$ where $q - 1$ is divisible by $p$ just once. Our proof of Theorem A does however not rely on any conjecture.

**Theorem A.** The character table of a finite group $G$ determines whether $G$ has Sylow $p$-subgroups of maximal class.

*Proof.* Let $P$ be a Sylow $p$-subgroup of $G$. We may assume that $\mathrm{O}_{p'}(G) = 1$ and $|P| \geq p^3$. Let $K := \mathrm{O}^{p'}(G)$. The character table detects elements $x \in P$ such that $|\mathrm{C}_G(x)|_p = |\mathrm{C}_K(x)|_p = p^2$. In this case $|\mathrm{C}_P(x)| = p^2$ and $P$ has maximal class by [15, Satz III.14.23]. Hence, by Theorem B we may assume that $K$ is quasisimple with $|\mathrm{Z}(K)| \leq p$. Note that the character table of $G$ determines the isomorphism type of the simple chief factor $K/\mathrm{Z}(K)$ (see [24, Theorem 4.1]). In this way we confirm that the Sylow $p$-subgroup $P/\mathrm{Z}(K)$ of $K/\mathrm{Z}(K)$ has maximal class. If $\mathrm{Z}(K) = 1$, then we are done. Otherwise, $P$ has maximal class if and only if $\mathrm{Z}(K) = \mathrm{Z}(P)$. This happens if and only if $|\mathrm{C}_G(x)|_p < |P|$ for all $x \in P \setminus \mathrm{Z}(K)$. $\qquad \square$

# 4 Minimal non-abelian Sylow subgroups

The following elementary lemma underlines the importance of minimal non-abelian groups. For elements $x, y, z$ of a group we use the commutator convention $[x, y, z] := [x, [y, z]]$.

**Lemma 9.** *For a $p$-group $P$ the following assertions are equivalent:*

*(1) $P$ is minimal non-abelian.*

*(2) $|P : \Phi(P)| = |P : \mathrm{Z}(P)| = p^2$.*

*(3) $|P : \Phi(P)| = p^2$ and $|P'| = p$.*

*Proof.* (1) $\Rightarrow$ (2): Since $P$ is non-abelian, there exist non-commuting elements $x, y \in P$. Since $\langle x, y \rangle$ is non-abelian, we have $P = \langle x, y \rangle$. By Burnside's basis theorem, $|P : \Phi(P)| = p^2$. Choose distinct maximal subgroups $S, T < P$. Since $S$ and $T$ are abelian and $P = ST$, it follows that $\Phi(P) = S \cap T \subseteq Z(P)$. It is well-known that $P/Z(P)$ cannot be a non-trivial cyclic group. In particular, $|P : Z(P)| \geq p^2$ and $\Phi(P) = Z(P)$.

(2) $\Rightarrow$ (3): Let $Z(P) < S < P$. Since $S/Z(P)$ is cyclic and $Z(P) \leq Z(S)$, we obtain that $S$ is abelian. Pick $x \in P \setminus S$. Then Lemma 1 yields that $|P'| = |S : Z(P)| = p$.

(3) $\Rightarrow$ (1): Obviously, $P$ is non-abelian since $P' \neq 1$. For $g, x \in P$ we have $gxg^{-1} = [g, x]x \in P'x$. Thus, every conjugacy class lies in a coset of $P'$. The hypothesis $|P'| = p$ implies $|P : C_P(x)| \leq p$ for every $x \in P$. Since $\Phi(P)$ is the intersection of the maximal subgroups of $P$, we deduce $\Phi(P) \leq \bigcap_{x \in P} C_P(x) = Z(P)$. Now for every maximal subgroup $S < P$, we see that $S/Z(S)$ is cyclic and $S$ must be abelian. We conclude that $P$ is minimal non-abelian. $\qquad \square$

The non-nilpotent, minimal non-abelian groups were classified by Miller–Moreno [20]. The nilpotent ones are $p$-groups and have been determined by Rédei [30]. For the convenience of the reader we give a proof.

**Lemma 10** (Rédei). *Every minimal non-abelian $p$-group belongs to one of the following classes:*

(i) $\Gamma(a, b) := \langle x, y \mid x^{p^a} = y^{p^b} = 1, \; yxy^{-1} = x^{1+p^{a-1}} \rangle$ *a metacyclic group where $a \geq 2$ and $b \geq 1$.*

(ii) $\Delta(a, b) := \langle x, y \mid x^{p^a} = y^{p^b} = [x, y]^p = [x, x, y] = [y, x, y] = 1 \rangle$ *where $a \geq b \geq 1$.*

(iii) $Q_8$.

*Proof.* Let $P$ be minimal non-abelian. By Lemma 9, there exist $x, y \in P$ such that $P/P' = \langle xP' \rangle \times \langle yP' \rangle \cong C_{p^a} \times C_{p^b}$. Since $|P'| = p$, we have $P' = \langle z \rangle$ where $z := [x, y]$. Note that $P' \leq \Phi(P) = Z(P)$ and $[x, z] = [y, z] = 1$. We distinguish three cases:

**Case (1):** $x^{p^a} = y^{p^b} = 1$.
Here $P$ fulfills the same relations as $\Delta(a, b)$, so it must be a quotient of the latter group. Moreover, every element of $P$ can be written uniquely in the form $x^i y^j z^k$ with $1 \leq i \leq p^a$, $1 \leq j \leq p^b$ and $1 \leq k \leq p$. Consequently, $|P| = p^{a+b+1}$. For the same reason we have $|\Delta(a, b)| \leq p^{a+b+1}$. Therefore, $P \cong \Delta(a, b)$.

**Case (2):** Either $x^{p^a} = 1$ or $y^{p^b} = 1$.
Without loss of generality, let $x^{p^a} \neq 1$ and $y^{p^b} = 1$. Then $P' \leq \langle x \rangle \trianglelefteq P$ and $yxy^{-1} = x^k$ for some $k \in \mathbb{Z}$. Since $\langle x^p, y \rangle < P$ is abelian, $x^p = yx^p y^{-1} = x^{kp}$ and $p \equiv kp \pmod{p^{a+1}}$ as $|\langle x \rangle| = p^{a+1}$. Hence, we may assume that $k = 1 + p^a l$ for some $0 < l < p$. Let $0 < l' < p$ such that $ll' \equiv 1 \pmod{p}$. Then $y^{l'} x y^{-l'} = x^{(1+p^a l)^{l'}} = x^{1+p^a}$. Thus, after replacing $y$ by $y^{l'}$, we obtain $yxy^{-1} = x^{1+p^a}$. Now $P$ satisfies the relations of $\Gamma(a+1, b)$. It is clear that these groups have the same order, so $P \cong \Gamma(a+1, b)$.

**Case (3):** $x^{p^a} \neq 1 \neq y^{p^b}$.
Without loss of generality, let $a \geq b$. Let $x^{p^a} = z^i$ and $y^{p^b} = z^j$ where $0 < i, j < p$. Then $(x^j)^{p^a} = z^{ij}$, $(y^i)^{p^b} = z^{ij}$ and $[x^j, y^i] = z^{ij}$ by [15, Hilfssatz III.1.3] (using $z \in Z(P)$). Hence, replacing $x$ by $x^j$ and $y$ by $y^i$, we may assume that $x^{p^a} = z = y^{p^b}$. Again by [15, Hilfssatz III.1.3],

$$(x^{p^{a-b}} y^{-1})^{p^b} = x^{p^a} y^{-p^b} [y^{-1}, x^{p^{a-b}}]^{\binom{p^b}{2}} = z^{p^{a-b}\binom{p^b}{2}} = 1$$

unless $p^b = p^a = 2$. In this exceptional case, $P \cong Q_8$. Otherwise, we replace $y$ by $x^{p^{a-b}} y^{-1}$. Afterwards we still have $P/P' = \langle xP' \rangle \times \langle yP' \rangle$, but now $y^{p^b} = 1$. Thus, we are in Case (2). $\qquad \square$

The metacyclic groups $\Gamma(a, b)$ can of course be constructed as semidirect products, while the groups $\Delta(a, b)$ can be constructed as subgroups of $\Gamma(a, b) \times C_{p^a}$. For $p = 2$, note that $\Gamma(2, 1) \cong D_8 \cong \Delta(1, 1)$. Apart from that, the groups in Lemma 10 are pairwise non-isomorphic (for different parameters $a, b$).

We digress slightly to present a counterexample to a related question. Since for $p$-groups $P$ in general we have $\Phi(P) = P'\mho(P)$ where $\mho(P) = \langle x^p : x \in P \rangle$, one might wonder if $X(G)$ determines the property $|P : \mho(P)| = p^2$. For $p = 2$, it is well-known that $\mho(P) = \Phi(P)$, so the answer is yes in this case. For $p > 2$, $|P : \mho(P)| = p^2$ holds if and only if $P$ is metacyclic (see [15, Satz III.11.4]). The following example shows that this is not even determined by $X(P)$.

**Proposition 11.** *For $a \geq 2$ and all primes $p$ the groups $\Gamma(2, a)$ and $\Delta(a, 1)$ have the same character table.*

*Proof.* We denote the generators of $P := \Gamma(2, a)$ by $x, y$ and those of $\tilde{P} := \Delta(a, 1)$ by $\tilde{x}, \tilde{y}$ as in Lemma 10. Additionally, let $\tilde{z} := [\tilde{x}, \tilde{y}]$. We consider the maximal subgroups $Q := \langle x^p, y \rangle \leq P$ and $\tilde{Q} := \langle \tilde{x}, \tilde{z} \rangle \leq \tilde{P}$. Since $xyx^{-1} = x^{-p}y$ and $\tilde{y}\tilde{x}\tilde{y}^{-1} = \tilde{z}^{-1}\tilde{y}$, the map

$$Q \to \tilde{Q}, \qquad x^p \mapsto z, \ y \mapsto \tilde{x}$$

is an isomorphism compatible with the action of $P$ and $\tilde{P}$. The irreducible characters of $P$ of degree $p$ are induced from linear characters of $Q$, which are not $P$-invariant. Since these characters vanish outside $Q$, they correspond naturally to irreducible characters of $\tilde{P}$. On the other hand, the linear characters of $P$ are extensions of characters of $Q$ with $x^p$ in their kernel. For $\lambda \in \mathrm{Irr}(Q/P')$ the extensions $\hat{\lambda}$ are determined by $\hat{\lambda}(x) = \zeta$ where $\zeta$ is a $p$-th root of unity. Similarly, for $\lambda \in \mathrm{Irr}(\tilde{Q}/\tilde{P}')$ the extensions are determined by $\hat{\lambda}(\tilde{y}) = \zeta$. Therefore, the bijection $P \to \tilde{P}$, $x^{i+jp}y^k \mapsto \tilde{x}^k\tilde{y}^i\tilde{z}^j$ where $0 \leq i, j < p$ and $0 \leq k < p^a$ induces the equality of the matrices $X(P)$ and $X(\tilde{P})$. $\square$

The second author has investigated fusion systems on minimal non-abelian 2-groups in order to classify blocks with such defect groups (see e. g. [31]). We now determine the fusion systems for odd primes too (partial results were obtained in [36]). It turns out that they all come from finite groups unless $|P| = 7^3$. We make use of the Frobenius group $M_9 \cong \mathrm{PSU}(3, 2) \cong C_3^2 \rtimes Q_8$ with $\mathrm{Out}(M_9) \cong S_3$.

**Theorem 12.** *Let $\mathcal{F}$ be a saturated fusion system on a minimal non-abelian $p$-group $P$. Then one of the following holds*

(i) $P \in \{D_8, Q_8\}$ and $\mathcal{F} = \mathcal{F}_P(G)$ where $G \in \{P, S_4, \mathrm{GL}(3, 2), \mathrm{SL}(2, 3)\}$.

(ii) $|P| = p^3$, $\exp(P) = p > 2$ and the possibilities for $\mathcal{F}$ are given in [29].

(iii) $P \cong \Gamma(a, b)$, $a \geq 2$, $b \geq 1$ and $\mathcal{F} = \mathcal{F}_P(C_{p^a} \rtimes C_{p^b d})$ for some $d \mid p - 1$.

(iv) $P \cong \Delta(a, b)$, $a > b$ and $\mathcal{F} = \mathcal{F}_P(P \rtimes Q)$ where $Q \leq C_{p-1}^2$.

(v) $P \cong \Delta(a, a)$, $a \geq 2$ and $\mathcal{F} = \mathcal{F}_P(P \rtimes Q)$ for some $p'$-group $Q \leq \mathrm{GL}(2, p)$.

(vi) $p = 2$, $P \cong \Delta(a, 1)$, $a \geq 2$ and $\mathcal{F} = \mathcal{F}_P(A_4 \rtimes C_{2^a})$ where $C_{2^a}$ acts as a transposition in $\mathrm{Aut}(A_4) = S_4$.

(vii) $p = 3$, $P \cong \Delta(a, 1)$, $a \geq 2$ and $\mathcal{F} = \mathcal{F}_P(G)$ where $G \in \{M_9 \rtimes C_{3^a}, M_9 \rtimes D_{2 \cdot 3^a}\}$. Here the image of $C_{3^a}$ and $D_{2 \cdot 3^a}$ in $\mathrm{Out}(M_9)$ is $C_3$ and $S_3$ respectively.

*Proof.* The case $P \in \{D_8, Q_8\}$ is well-known and can be found in [6, Theorem 5.3], for instance. If $p = 2$ and $P = \Gamma(a, b)$ with $|P| \geq 16$, then $\mathcal{F}$ is trivial, i.e. $\mathcal{F} = \mathcal{F}_P(P)$ by [6, Theorem 3.7]. Then (iii) holds. Now suppose that $p > 2$ and $P = \Gamma(a, b)$. Then $\mathcal{F}$ is controlled, i.e. $\mathcal{F} = \mathcal{F}_P(P \rtimes Q)$ for some $p'$-group $Q \leq \mathrm{Aut}(P)$ by [34] (see also [6, Theorem 3.10]). By [33, Lemma 2.4], $\mathrm{Aut}(P) = A \rtimes \langle \sigma \rangle$ where $A$ is a $p$-group, $|\langle \sigma \rangle| = p - 1$, $\sigma(x) \in \langle x \rangle$ and $\sigma(y) = y$. Hence, $Q$ is conjugate to a subgroup of $\langle \sigma \rangle$. After renaming the generators of $P$, we may assume that $Q \leq \langle \sigma \rangle$. Now (iii) holds.

Next assume that $P \cong \Delta(a, b)$ for some $a \geq b \geq 1$. If $a = 1$ and $p > 2$, then $|P| = p^3$ and $\exp(P) = p$, so (ii) holds. Hence, let $a \geq 2$. Set $z := [x, y] \in P$. Since the $p'$-group $\mathrm{Out}_{\mathcal{F}}(P)$ acts faithfully on $P/\Phi(P) \cong C_p^2$, we have $\mathrm{Out}_{\mathcal{F}}(P) \leq \mathrm{GL}(2, p)$. If $a > b$, then $\mathrm{Out}_{\mathcal{F}}(P)$ acts on $P/\Omega_{a-1}(P) \times \Omega_{a-1}(P)/\Phi(P)$ where $\Omega_{a-1}(P) = \langle g \in P : g^{p^{a-1}} = 1 \rangle = \langle x^p, y, z \rangle$. In this case $\mathrm{Out}_{\mathcal{F}}(P) \leq C_{p-1}^2$. If $\mathcal{F}$ is controlled, then we are in Case (iv) or (v). Hence, we may assume that $\mathcal{F}$ is not controlled. Then there exists an essential subgroup $Q \leq P$. Since $Q$ is centric and $\Phi(P) = \mathrm{Z}(P) \leq \mathrm{C}_P(Q) \leq Q$, $Q$ is a maximal subgroup. Those are given by

$$\langle xy^i, y^p, z \rangle \cong C_{p^a} \times C_{p^{b-1}} \times C_p \qquad (i = 0, \ldots, p-1),$$
$$\langle x^p, y, z \rangle \cong C_{p^{a-1}} \times C_{p^b} \times C_p.$$

By [11, Theorem 5.2.4], $A := \mathrm{Aut}_{\mathcal{F}}(Q)$ acts faithfully on $\Omega(Q) = \{g \in Q : g^p = 1\}$. Since $P/Q \leq A$, this implies $\Omega(Q) \not\subseteq \mathrm{Z}(P)$ and $Q = \langle x^p, y, z \rangle$ with $b = 1$. Now $Q$ is the only maximal subgroup of $P$ isomorphic to $C_{p^{a-1}} \times C_p^2$. In particular, $Q$ is characteristic in $P$. By Alperin's fusion theorem, $\mathcal{F}$ is constrained with $\mathrm{O}_p(\mathcal{F}) = Q$. By the model theorem, there exists a unique $p$-constrained group $H$ with $P \in \mathrm{Syl}_p(H)$, $\mathrm{O}_{p'}(H) = 1$ and $\mathcal{F} = \mathcal{F}_P(H)$. We will construct $H$ in the following.

By [27, Lemma 1.11], there exists an $A$-invariant decomposition $Q = Q_1 \times Q_2$ with $Q_1 \cong C_p^2$ and $Q_2 \cong C_{p^{a-1}}$. Moreover, $\mathrm{O}^{p'}(A) \cong \mathrm{SL}(2, p)$ acts faithfully on $Q_1$ and trivially on $Q_2$. Since $P/Q \leq \mathrm{O}^{p'}(A)$, it follows that $Q_2 \leq \mathrm{Z}(P) = \langle x^p, z \rangle$. Moreover, $xyx^{-1} = yz$ implies $z \in Q_1$. Let $\alpha \in A$ be a $p'$-automorphism acting trivially on $Q_1$. Then $\alpha$ commutes with the action of $P/Q$. Since $Q$ is receptive (see [1, Definition I.2.2]), $\alpha$ extends to an automorphism of $P$. Suppose that $\alpha \neq 1$. Since $Q_2 \leq \mathrm{Z}(P) = \Phi(P)$, $\alpha$ must act non-trivially on $P/Q_2$. Note that $P/Q_2$ is non-abelian of order $p^3$ as $z \in Q_1$. An analysis of $\mathrm{Aut}(P/Q_2)$ reveals that $\alpha$ cannot act trivially on $Q/Q_2 \cong Q_1$. Hence, $\alpha = 1$ and $A$ acts faithfully on $Q_1$. In particular, $A \leq \mathrm{GL}(2, p)$. If $p = 2$, then $A \cong \mathrm{SL}(2, 2) = \mathrm{GL}(2, 2) \cong S_3$. It is easy to see that (vi) holds here. If $p = 3$, then $\mathrm{SL}(2, 3) \cong Q_8 \rtimes C_3$, $\mathrm{GL}(2, 3) \cong Q_8 \rtimes S_3$ and (vii) is satisfied. Thus, let $p \geq 5$. Then the Sylow normalizer in $\mathrm{SL}(2, p)$ acts non-trivially on a Sylow $p$-subgroup of $\mathrm{SL}(2, p)$. Hence, there exists $\alpha \in \mathrm{O}^{p'}(A)$ acting non-trivially $P/Q$. But then $\alpha$ acts non-trivially on $\langle x^p \rangle Q_1/Q_1 = Q/Q_1 \cong Q_2$. This contradicts [27, Lemma 1.11]. $\qquad\square$

The groups $A_4 \rtimes C_4$, $M_9 \rtimes C_9$ and $M_9 \rtimes D_{18}$ can be constructed in GAP [9] as `SmallGroup`$(n, k)$ where $(n, k) \in \{(48, 39), (648, 534), (6^4, 2892)\}$ respectively.

**Corollary 13.** *Let $\mathcal{F}$ be a fusion system on a minimal non-abelian $p$-group $P$ with $|P| \geq p^4$. Then $\mathcal{F}$ is constrained. If $p \geq 5$, then $\mathcal{F}$ is controlled.*

We now gather some information on simple groups in order to prove Theorem C. As customary, if $q$ is a prime power, let

$$\mathrm{PSL}^{\epsilon}(n, q) := \begin{cases} \mathrm{PSL}(n, q) & \text{if } \epsilon = 1, \\ \mathrm{PSU}(n, q) & \text{if } \epsilon = -1. \end{cases}$$

The following is certainly known, but included for convenience.

**Lemma 14.** *Let $q$ be a prime power. Let $S = \mathrm{PSL}^\epsilon(n, q)$ with a cyclic Sylow $p$-subgroup and $n \geq 3$. Then there exists a unique integer $2 \leq d \leq n$ such that $p$ divides $q^d - \epsilon^d$.*

*Proof.* Since a non-abelian simple group cannot have cyclic Sylow 2-subgroups, we have $p > 2$. If $p \mid q$, then a Sylow $p$-subgroup of $S$ is given by the set of unitriangular matrices. This subgroup is non-abelian since $n \geq 3$. Now let $p \nmid q$. If $q \equiv \epsilon \pmod{p}$, then $S$ contains a subgroup of diagonal matrices isomorphic to $C_p^2$. Hence, let $q \not\equiv \epsilon \pmod{p}$. In the following we write $q^* := q$ if $\epsilon = 1$ and $q^* := q^2$ if $\epsilon = -1$. Let $x \in S$ be a generator of a Sylow $p$-subgroup of $S$. We identify $x$ with a preimage in $\mathrm{GL}(n, q^*)$. We may assume that $x$ has order $p^k$. Let $e$ be the order of $q^*$ modulo $p^k$. Then $x$ has an eigenvalue $\zeta \in \mathbb{F}_{(q^*)^e}^\times$ of order $p^k$. Since $\mathrm{tr}(x) \in \mathbb{F}_{q^*}$, the elements $\zeta^{(q^*)^i}$ for $i = 0, \ldots, e-1$ are distinct eigenvalues of $x$. In particular, $e \leq n$. If $\epsilon = 1$, then $e \geq 2$ we can choose $d := e$ in the statement. If $2e \leq n$, we obtain $q^d \equiv 1 \equiv \epsilon^d \pmod{p}$ for $d := 2e$.

Now suppose that $\epsilon = -1$ and $2e > n$. Since $x$ is a unitary matrix, we have $\bar{x}x^{\mathrm{t}} = 1$ where $\bar{x} = (x_{ij}^q)_{i,j}$ and $x^{\mathrm{t}}$ is the transpose of $x$. It follows that $\zeta^{-q}$ is an eigenvalue of $x$. Since $n < 2e$, there must be some $i$ with $\zeta^{q^{2i}} = \zeta^{-q}$. This shows that $q^{2i-1} \equiv -1 \equiv \epsilon^{2i-1} \pmod{p^k}$. Since $q^{2(2i-1)} \equiv 1 \pmod{p^k}$, we have $e \mid 2i - 1 \leq 2(e-1) - 1 < 2e$ and $e = 2i - 1$. Hence, we can set $d := e$.

For the uniqueness of $d$, we note that

$$|S| = \frac{q^{n(n-1)/2}}{\gcd(n, q - \epsilon)} \prod_{i=2}^n (q^i - \epsilon^i),$$

is not divisible by $p^{k+1}$, since $p^k = |\langle x \rangle| = |S|_p$. $\qquad\square$

**Lemma 15.** *Let $S$ be a finite simple group with Sylow 3-subgroup $C_3^2$ and outer automorphism of order 3. Then $S \cong \mathrm{PSL}^\epsilon(3, p^f)$ where $\epsilon = \pm 1$, $p$ is a prime and $(p^f - \epsilon)_3 = 3$. Moreover, $\mathrm{Out}(S) \cong C_3 \rtimes (C_f \times C_2)$.*

*Proof.* The simple groups with Sylow 3-subgroup $C_3^2$ were classified in [18, Proposition 1.2]. The alternating groups and sporadic groups do not have outer automorphisms of order 3. Now let $S$ be a classical group of dimension $d$ over $\mathbb{F}_{p^f}$. Then $p^f \not\equiv \pm 1 \pmod 9$. This implies $3 \nmid f$ and $S$ does not have field automorphisms of order 3. According to [5, Table 5], there must be a diagonal automorphism of order 3. This forces $d = 3$ and $S = \mathrm{PSL}^\epsilon(3, p^f)$ such that $(p^f - \epsilon)_3 = 3$. If $\epsilon = 1$, then $\mathrm{Out}(S) = C_3 \rtimes (C_f \times C_2)$ as desired. If $\epsilon = -1$, then there is no graph automorphism and instead we have a field automorphism of order $2f$. However, since $p^f \equiv 2, 5 \pmod 9$, $f$ must be odd and $C_{2f} \cong C_f \times C_2$. $\qquad\square$

**Theorem C.** *Let $G$ be a finite group with a minimal non-abelian Sylow $p$-subgroup $P$ and $\mathrm{O}_{p'}(G) = 1$. Then one of the following holds:*

(i) *$p = 2$, $P \in \{D_8, Q_8\}$ and $\mathrm{O}^{2'}(G) \in \{\mathrm{SL}(2, q), \mathrm{PSL}(2, q'), A_7\}$ where $q \equiv \pm 3 \pmod 8$ and $q' \equiv \pm 7 \pmod{16}$.*

(ii) *$|P| = p^3$ and $\exp(P) = p > 2$.*

(iii) *$G = P \rtimes Q$ where $Q \leq \mathrm{GL}(2, p)$.*

(iv) *$p > 2$, $\mathrm{O}^{p'}(G) = S \rtimes C_{p^a}$ where $S$ is a simple group of Lie type with cyclic Sylow $p$-subgroups. The image of $C_{p^a}$ in $\mathrm{Out}(S)$ has order $p$.*

(v) *$p = 2$ and $G = \mathrm{PSL}(2, q^f) \rtimes C_{2^a d}$ where $q$ is a prime, $q^f \equiv \pm 3 \pmod 8$ and $d \mid f$. Moreover, $C_{2^a}$ acts as a diagonal automorphism of order 2 on $\mathrm{PSL}(2, q^f)$ and $C_d$ induces a field automorphism of order $d$.*

(vi) $p = 3$ and $\mathrm{O}^{3'}(G) = \mathrm{PSL}^\epsilon(3, q^f) \rtimes C_{3^a}$ where $\epsilon = \pm 1$, $q$ is a prime, $(q^f - \epsilon)_3 = 3$ and $G/\mathrm{O}^{3'}(G) \le C_f \times C_2$.

*Proof.* By Lemma 9, $|P : \mathrm{Z}(P)| = p^2$ and $G$ is described in [24, Theorem 7.5]. We go through the various cases in the notation used there:

In Case (A), using that $P$ is 2-generated and $\mathrm{O}_p(G)$ is not cyclic, we deduce that $S = 1$. Here $P = \mathrm{F}^*(G) \trianglelefteq G$ and $\mathrm{C}_G(P) \le P$. Since $G/P$ acts faithfully on $P/\Phi(P) \cong C_p^2$, we have $G/P \le \mathrm{GL}(2, p)$ and (iii) holds. Assume now that $P < G$. In Case (B), the quasisimple group $C$ has a non-abelian Sylow $p$-subgroup of order $p^3$ which must coincide with $P$. If $P = D_8$, then (i) or (v) holds by the Gorenstein–Walter theorem (there are no field automorphisms of order 2) [11, p. 462]. If $P = Q_8$, the claim follows from the Brauer–Suzuki theorem [11, Theorem 12.1.1] and Walter's theorem [11, p. 485]. If $p > 2$, then we must have $\exp(P) = p$, since otherwise the focal subgroup theorem [16, Theorem 5.21] and Theorem 12 lead to the contradiction $|P| = |P : P \cap G'| \ge p$. Thus, (ii) holds. Case (D) is impossible, since then $P$ has a non-abelian maximal subgroup.

Now consider Case (C), i.e. $\mathrm{F}^*(G) = \mathrm{O}_p(G) \times S$ has abelian Sylow $p$-subgroups, $S$ is a direct product of simple groups and $|G : \mathrm{F}^*(G)|_p = p$. Let $x \in P \setminus \mathrm{F}^*(G)$.

**Case 1:** $S = 1$.
Since $S = 1$, $\mathrm{F}^*(G) = \mathrm{O}_p(G)$ and so $\mathrm{C}_G(\mathrm{F}^*(G)) = \mathrm{C}_G(\mathrm{O}_p(G)) \le \mathrm{O}_p(G)$. Therefore, $\mathrm{C}_P(\mathrm{O}_p(G)) = \mathrm{O}_p(G)$ and we have $\mathrm{C}_G(\mathrm{O}_p(G)) = \mathrm{O}_p(G) \times K$ where $K \le \mathrm{O}_{p'}(G) = 1$. Hence, $G$ is $p$-constrained and $\mathcal{F}_P(G)$ is given by (vi) or (vii) of Theorem 12. By the model theorem, the isomorphism type of $G$ is uniquely determined by $\mathcal{F}_P(G)$. Since $\mathrm{PSL}(2, 3) \cong A_4$ and $\mathrm{PSU}(3, 2) \cong M_9$, we obtain (v) or (vi).

**Case 2:** $S \ne 1$ is not simple.
By Lemma 10, the maximal subgroups of $P$ are generated by at most three elements. Hence, $S$ is a direct product of two or three simple groups, say $S = T_1 \times T_2$ or $T_1 \times T_2 \times T_3$. Since a Sylow 2-subgroup of a simple group cannot be generated by less than 2 elements, we deduce that $p > 2$ and the $T_i$ have cyclic Sylow $p$-subgroups. If $x$ does not normalize some $T_i$, then $p = 3$ and $x$ permutes $T_1 \cong T_2 \cong T_3$. However, $C_{3^n} \wr C_3$ is not minimal non-abelian. Hence, $x$ acts on each $T_i$. If $x$ acts non-trivially on $\mathrm{O}_p(G)$, then $\mathrm{O}_p(G)\langle x \rangle$ is non-abelian and $P = \mathrm{O}_p(G)\langle x \rangle$. But then $S$ would be simple. Similarly, if $x$ acts non-trivially on $Q_1 := P \cap T_1$, then $P = Q_1\langle x \rangle$. Write $Q_2 := P \cap T_2 = \langle y \rangle$ such that $x^p \in yQ_1$. Then $x$ centralizes $y$. By [13, Theorem B], this implies that $x$ induces an inner automorphism on $T_2$. However, $x^p$ induces the inner automorphism by $y$. Hence, $x$ cannot have order greater than $|T_2|_p$. Another contradiction.

**Case 3:** $S$ is simple.
Let $Q := P \cap S \trianglelefteq P$ be a Sylow $p$-subgroup of $S$. Arguing as in Case 2, we see that $x$ acts non-trivially on $Q$ and therefore $P = Q\langle x \rangle$. First let $Q$ be cyclic. Then $p > 2$ and $P$ is metacyclic. Since $\mathrm{Out}(S)$ needs to have an element of order $p$, $S$ must be of Lie type. To obtain (iv), it remains to show that $PS$ is normal in $G$. Assume the contrary. By the structure of $\mathrm{Out}(S)$ (see [5, Table 5]), $P$ induces a field or graph automorphism of order $p$ on $S$ which acts non-trivially on the subgroup of outer diagonal automorphisms of $S$. In particular, the diagonal automorphism group must have order at least $p + 1$, in fact $2p + 1 \ge 7$ since $p > 2$. This excludes all families of simple groups except $S = \mathrm{PSL}^\epsilon(d, q^f)$ where $p \mid f$ and $d \ge 2p + 1$. Since $Q$ is cyclic and $f > 1$, we have $q \ne p$. By Fermat's little theorem, $q^{(p-1)f} \equiv q^{2(p-1)f} \equiv 1 \pmod p$. This contradicts Lemma 14 (note that $p - 1$ is even). Hence, $PS \trianglelefteq G$ and (iv) holds.

Let $Q$ be non-cyclic. Recall that in general $Q$ is homocyclic and $\mathrm{N}_S(Q)$ acts irreducibly on $\Omega(Q)$ (see [8, Proposition 2.5]). This implies that $P$ cannot be metacyclic, as otherwise the fusion in $P$ is controlled by $\mathrm{N}_G(P)$ and $\mathrm{N}_G(Q) = \mathrm{N}_G(P)\mathrm{C}_G(Q)$ acts reducibly on $Q$ according to Theorem 12.

Hence, let $P \cong \Delta(a, b)$. Then $P'$ is a direct factor of $Q$ and we obtain $Q = \Omega(Q)$. If $Q$ has rank 3, then $P \cong \Delta(2, 1)$. However, by Theorem 12, $\mathrm{N}_G(Q)/\mathrm{C}_G(Q) \leq \mathrm{GL}(2, p)$ does not act irreducibly on $Q$. Hence, we may assume that $Q$ has rank 2. Now $P \cong \Delta(a, 1)$ with $a \geq 2$. If $\mathrm{N}_G(P)$ controls the fusion in $P$, then $\mathrm{N}_G(Q)$ would fix $P'$. Hence, we are in Case (vi) or (vii) of Theorem 12. Consider $p = 2$ first. By Walter's theorem (see [11, p. 485]), $S \cong \mathrm{PSL}(2, q^f)$ with $q^f \equiv \pm 3 \pmod 8$. It follows that $f$ is odd and $G/PS \leq \mathrm{Out}(S) \leq C_{2f}$ by [5, Table 5]. Here $C_2$ induces a diagonal automorphism and $C_f$ is caused by a field automorphism. So (v) holds. Finally, let $p = 3$. Here the claim follows easily from Lemma 15. $\qquad\square$

Examples for Theorem C(iv) can be constructed as follows: Let $p > 2$ and $a \geq 2$. By Dirichlet's theorem, there exists a prime $q \equiv 1 + p^{a-1} \pmod{p^{a+1}}$. Then $q^p \equiv 1 + p^a \pmod{p^{a+1}}$ and $S := \mathrm{PSL}(2, q^p)$ has a cyclic Sylow $p$-subgroup $Q$ of order $p^a$. Let $R \cong C_{p^b}$ and construct $G := S \rtimes R$ where $R$ acts as the field automorphism $\mathbb{F}_{q^p} \to \mathbb{F}_{q^p}$, $\lambda \mapsto \lambda^q$ on $S$. By [13], $R$ acts non-trivially on $Q$ and $P := Q \rtimes R \cong \Gamma(a, b)$. A different example is $G = \mathrm{Sz}(2^5) \rtimes C_5$ for $p = 5$.

**Corollary 16.** *Let $G$ be a finite group with a minimal non-abelian Sylow $p$-subgroup and $\mathrm{O}_{p'}(G) = 1$. Then $G$ has at most one non-abelian composition factor.*

*Proof.* We may assume that $G$ is non-solvable. If $|G|_p = p^3$, then $\mathrm{F}^*(G)$ is quasisimple and $G/\mathrm{F}^*(G) \leq \mathrm{Aut}(\mathrm{F}^*(G)) \leq \mathrm{Aut}(\mathrm{F}^*(G)/\mathrm{Z}(\mathrm{F}^*(G)))$ is solvable by Schreier's conjecture. Otherwise we have $\mathrm{F}^*(G) = S \times C_{p^b}$ for a simple group $S$ and $b \geq 0$ by the proof of Theorem C. Since $\mathrm{Aut}(C_{p^b})$ is abelian, the claim follows again from Schreier's conjecture. $\qquad\square$

**Corollary D.** *The character table of a finite group $G$ determines whether $G$ has minimal non-abelian Sylow $p$-subgroups.*

*Proof.* Let $P$ be a Sylow $p$-subgroup of $G$. We may assume that $\mathrm{O}_{p'}(G) = 1$. By [24, Theorem B], the character table determines whether $|P : \mathrm{Z}(P)| = p^2$. Suppose that this is the case. By Lemma 9, it remains to detect whether $|P : \Phi(P)| = p^2$. This is true for $|P| = p^3$, so let $|P| \geq p^4$. By Theorem 4 and Corollary 5, we may assume that $\mathrm{O}_p(G) = 1$. Now by Theorem C we expect that $\mathrm{O}^{p'}(G) = S \rtimes C_p$ for a simple group $S$ with a cyclic Sylow $p$-subgroup $Q$. As usual, $X(G)$ determines the isomorphism type of $S$. If $Q$ is indeed cyclic, then clearly $P$ is 2-generated and we are done. $\qquad\square$

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