GENUS AND CROSSCAP OF SOLVABLE CONJUGACY CLASS GRAPHS OF FINITE GROUPS

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Abstract. The solvable conjugacy class graph of a finite group G, denoted by $\Gamma_{sc}(G)$, is a simple undirected graph whose vertices are the non-trivial conjugacy classes of G and two distinct conjugacy classes C, D are adjacent if there exist $x \in C$ and $y \in D$ such that $\langle x, y\rangle$ is solvable. In this paper, we discuss certain properties of genus and crosscap of $\Gamma_{sc}(G)$ for the groups D_{2n}, Q_{4n}, S_n, A_n, and $\mathrm{PSL}(2, 2^d)$. In particular, we determine all positive integers n such that their solvable conjugacy class graphs are planar, toroidal, double-toroidal or triple-toroidal. We shall also obtain a lower bound for the genus of $\Gamma_{sc}(G)$ in terms of order of the center and number of conjugacy classes for certain groups. As a consequence, we shall derive a relation between the genus of $\Gamma_{sc}(G)$ and the commuting probability of certain finite non-solvable group.

1. Introduction

The genus of a graph Γ, denoted by $\gamma(\Gamma)$, is the smallest non-negative integer n such that the graph can be embedded on the surface obtained by attaching n handles to a sphere. Graphs having genus zero are called planar, while those having genus one are called toroidal. Graphs having genus two and three are called double-toroidal triple-toroidal respectively. Let N_k be the connected sum of k projective planes. A simple graph which can be embedded in N_k but not in N_{k-1}, is called a graph of crosscap k. The crosscap of a graph Γ is denoted by $\bar{\gamma}(\Gamma)$. A graph Γ is called projective if $\bar{\gamma}(\Gamma) = 1$. It is well-known that $\gamma(\Gamma) \geq \gamma(\tilde{\Gamma})$ and $\bar{\gamma}(\Gamma) \geq \bar{\gamma}(\tilde{\Gamma})$, if $\tilde{\Gamma}$ is a subgraph of Γ. Also, for $n \geq 3$ and $r, s \geq 2$ we have

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil, \tag{1}$$

$$\gamma(K_{r,s}) = \left\lceil \frac{(r-2)(s-2)}{4} \right\rceil \tag{2}$$

and

$$\bar{\gamma}(K_n) = \begin{cases} \lceil \frac{1}{6}(n-3)(n-4) \rceil, & \text{if } n \neq 7 \\ 3, & \text{if } n = 7, \end{cases} \tag{3}$$

where K_n is the complete graph on n vertices and $K_{r,s}$ is the complete bipartite graph with two parts of sizes r and s.

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The solvable conjugacy class graph (SCC-graph) of a finite group G, denoted by $\Gamma_{sc}(G)$, is a simple undirected graph whose vertices are the non-trivial conjugacy classes of G and two distinct conjugacy classes C, D are adjacent if there exist $x \in C$ and $y \in D$ such that $\langle x, y\rangle$ is solvable. The SCC-graph of G is introduced and studied in [3] extending the notions of commuting conjugacy class graph [13] and nilpotent conjugacy class graph [14].

It was shown in [3, Theorem 3.6] that there are only finitely many finite groups G whose SCC-graph has given genus. Therefore, if some bounds for $|G|$ are known in terms of genus of $\Gamma_{sc}(G)$ then one may have characterizations of finite groups such that their SCC-graphs are planar, toroidal, double-toroidal or triple-toroidal. It is worth mentioning that the genus of the commuting graph and commuting conjugacy class graph of various classes of finite groups have been computed in [1, 9, 4] and characterized finite non-abelian groups such that their commuting/ commuting conjugacy class graphs are planar, toroidal, double-toroidal or triple-toroidal. In [8], all finite non-nilpotent groups are characterized such that their nilpotent graphs are planar or toroidal. Further, in [5, 6], it was shown that solvable and non-solvable graphs of finite non-solvable groups are neither planar, toroidal, double-toroidal or triple-toroidal.

In this paper, we discuss certain properties of genus and crosscap of $\Gamma_{sc}(G)$ for the groups D_{2n}, Q_{4n}, S_n, A_n and $\mathrm{PSL}(2, 2^d)$. In particular, we determine all cases for which their SCC-graphs are planar, toroidal, double-toroidal or triple-toroidal. We shall also obtain a lower bound for $\gamma(\Gamma_{sc}(G))$ in terms of order of the center and number of conjugacy classes for certain groups. As a consequence, we shall derive a relation between $\gamma(\Gamma_{sc}(G))$ and commuting probability of certain finite non-solvable groups.

2. Generalities on SCC-graphs

As noted earlier, it was shown in the earlier paper [3] that the order of a group G is bounded in terms of the clique number of its SCC-graph. We mentioned in that paper the problem of finding an explicit bound.

We noted in the preceding section that the genus and crosscap of the complete graph K_n are known explicitly, and are functions which tend to infinity with n. It follows immediately that

Proposition 2.1. *Given k, there are only finitely many finite groups G such that the genus or crosscap of $\Gamma_{sc}(G)$ is equal to k.*

We pose the analogous question:

Problem 2.2. *Find an explicit bound for the order of a finite group G for which $\gamma(\Gamma_{sc}(G)) = k$. Same problem for G for which $\bar{\gamma}(\Gamma_{sc}(G)) = k$.*

In particular, if G is a solvable group, then $\Gamma_{sc}(G)$ is a complete graph K_k, where k is one less than the number of conjugacy classes of G; so the genus and crosscap of $\Gamma_{sc}(G)$ are determined by the results in the preceding section.

The analysis is complicated by the fact that, if G and H are groups with $G \leq H$, it is not necessarily true that $\Gamma_{sc}(G)$ is a subgraph of $\Gamma_{sc}(H)$. For example, the cyclic group of order 5 has five conjugacy classes but the dihedral group of order 10 only has four.

This example also shows that a natural monotonic graph parameter (the clique number) does not induce a monotonic parameter on groups.

However, for one important class of groups, things are better.

Proposition 2.3. *For any n, there is a natural embedding f_n from the vertex set of $\Gamma_{sc}(S_n)$ to that of $\Gamma_{sc}(S_{n+1})$, which embeds the first graph as a subgraph of the second.*

Proof. Under the natural embedding of S_n into S_{n+1} as the stabilizer of $n+1$, two elements of S_n are conjugate in S_{n+1} if and only if they are conjugate in S_n. (This is because elements of the symmetric group are conjugate if they have the same cycle structure; and the natural embedding simply adds one cycle of length 1 to each element of S_n.) So the set of conjugacy classes of S_n embeds naturally into that for S_{n+1}.

If two conjugacy classes in S_n contain elements which generate a solvable group, then this is still true in S_{n+1}. So the embedding above maps edges to edges. (Note that non-edges are not necessarily mapped to non-edges; the classes of a 3-cycle and a 5-cycle are non-adjacent in S_n for $n = 5, 6, 7$ but are adjacent for $n \geq 8$. So the embedding is not as induced subgraph in general.) □

It follows that both $\gamma(\Gamma_{sc}(S_n))$ and $\bar{\gamma}(\Gamma_{sc}(S_n))$ are non-decreasing functions of n. We can observe further that, if we embed S_n in S_{n+k} by composing the maps f_n, f_{n+1}, …, f_{n+k-1}, then for sufficiently large k, the image of the composite map is a complete graph. (It suffices to take $k = n$; for, given any two elements of S_n, we can find conjugates of them in S_{2n} with disjoint support, and hence commuting.) In particular, we get the following fractional exponential lower bounds, using the known results on the genus and crosscap of the complete graph. Here $p(n)$ denotes the number of partitions of the integer n (which is the number of conjugacy classes of the symmetric group S_n). Thus, if $k = p(\lfloor n/2 \rfloor) - 1$ then $\Gamma_{sc}(S_n)$ has a subgraph isomorphic to a complete graph having k vertices. Hence, using (1) and (3) we get the following result.

Theorem 2.4. *Given $n \geq 10$, let $k = p(\lfloor n/2 \rfloor) - 1$. Then*

$$\gamma(\Gamma_{sc}(S_n)) \geq \left\lceil \frac{(k-3)(k-4)}{12} \right\rceil \quad \text{and} \quad \bar{\gamma}(\Gamma_{sc}(S_n)) \geq \left\lceil \frac{(k-3)(k-4)}{6} \right\rceil.$$

Inspection of the proof shows that the same bound holds for the commuting and nilpotent conjugacy class graphs of the symmetric group [13, 14].

Recall that $k(G)$ denotes the number of conjugacy classes of G. The following lemma is useful in obtaining a lower bound for $\gamma(\Gamma_{sc}(G))$ as mentioned above.

Lemma 2.5. *Let G be a finite non-solvable group with non-trivial center $Z(G)$. Then $\Gamma_{sc}(G)$ has a subgraph isomorphic to $K_{|Z(G)|-1,\ k(G)-|Z(G)|}$.*

Proof. Let $S = \{x^G : x \in Z(G) \setminus \{1\}\}$ and $T = \{y^G : y \in G \setminus Z(G)\}$. We consider the subgraph S_Γ of $\Gamma_{sc}(G)$ by removing edges between the vertices of S as well as removing edges between the vertices of T. Then the subgraph thus obtained is isomorphic to $K_{|Z(G)|-1,\ k(G)-|Z(G)|}$. □

Theorem 2.6. *Let G be a finite non-solvable group with non-trivial center $Z(G)$. Then*

$$4\gamma(\Gamma_{sc}(G)) \geq (|Z(G)| - 3)(k(G) - |Z(G)| - 2).$$

Proof. By Lemma 2.5, it follows that $\Gamma_{sc}(G)$ has a subgraph isomorphic to $K_{|Z(G)|-1,k(G)-|Z(G)|}$. We have

$$\gamma(\Gamma_{sc}(G)) \geq \gamma(K_{|Z(G)|-1,k(G)-|Z(G)|}).$$

Therefore, by (2), we have

$$\begin{aligned}\gamma(\Gamma_{sc}(G)) &\geq \left\lceil \frac{(|Z(G)|-3)(k(G)-|Z(G)|-2)}{4} \right\rceil \\ &\geq \frac{(|Z(G)|-3)(k(G)-|Z(G)|-2)}{4}.\end{aligned}$$

Hence, the result follows on simplification. □

We conclude this section with the following relation between commuting probability (which is the probability that a randomly chosen pair of elements of G commute) and genus of SCC-graph of finite non-solvable group with non-trivial center.

Corollary 2.7. *Let G be a finite non-solvable group and $|Z(G)| > 3$. If $\Pr(G)$ is the commuting probability of G then*

$$\Pr(G) \leq \frac{4\gamma(\Gamma_{sc}(G)) + (|Z(G)|-3)(|Z(G)|+2)}{|G|(|Z(G)|-3)}.$$

Proof. The result follows from Theorem 2.6 and the fact that $\Pr(G) = \frac{k(G)}{|G|}$ as noted in [12]. □

It is worth mentioning that many bounds for $\Pr(G)$ have been obtained using various group theoretic notions over the years (see [11, 15]). However, the bound for $\Pr(G)$ obtained in Corollary 2.7 is the first of its kind involving genus of certain graph defined on groups though it is difficult to compute genus of $\Gamma_{sc}(G)$ in general.

3. Genus and crosscap of SCC-graphs

We begin with the characterizations of dihedral groups ($D_{2n} = \langle a, b : a^n = b^2 = 1, bab^{-1} = a^{-1}\rangle$) and quaternion groups ($Q_{4n} = \langle a, b : a^{2n} = 1, b^2 = x^n, bab^{-1} = a^{-1}\rangle$) such that their SCC-graphs are planar, toroidal, double-toroidal or triple-toroidal.

Since the dihedral and quaternion groups are solvable, their SCC graphs are complete graphs K_k where k is one less than the number of conjugacy classes.

Theorem 3.1. (a) *$\Gamma_{sc}(D_{2n})$ is planar if and only if $n = 2, 3, 4, 5$ and 7.*
(b) *$\Gamma_{sc}(D_{2n})$ is toroidal if and only if $n = 6, 8, 9, 10, 11$ and 13.*
(c) *$\Gamma_{sc}(D_{2n})$ is double-toroidal if and only if $n = 12$ and 15.*
(d) *$\Gamma_{sc}(D_{2n})$ is triple-toroidal if and only if $n = 14$ and 17.*
(e) $$\gamma(\Gamma_{sc}(D_{2n})) = \begin{cases} \left\lceil \frac{(n-5)(n-7)}{48} \right\rceil, & \text{when } n \geq 19 \text{ and } n \text{ is odd} \\ \left\lceil \frac{(n-2)(n-4)}{48} \right\rceil, & \text{when } n \geq 16 \text{ and } n \text{ is even.} \end{cases}$$
(f) *$\bar{\gamma}(\Gamma_{sc}(D_{2n})) = 0$ if and only if $n = 2, 3, 4, 5$ and 7.*
(g) *$\Gamma_{sc}(D_{2n})$ is projective if and only if $n = 6, 8, 9$ and 11.*
(h) $$\bar{\gamma}(\Gamma_{sc}(D_{2n})) = \begin{cases} \left\lceil \frac{(n-5)(n-7)}{24} \right\rceil, & \text{when } n \geq 13 \text{ and } n \text{ is odd} \\ \left\lceil \frac{(n-2)(n-4)}{24} \right\rceil, & \text{when } n \geq 10 \text{ and } n \text{ is even.} \end{cases}$$

Proof. We consider the following cases.
Case 1. If n is odd.
The non-trivial conjugacy classes in D_{2n} are $a^{D_{2n}}$, $(a^2)^{D_{2n}}$, …, $(a^{\frac{n-1}{2}})^{D_{2n}}$, $b^{D_{2n}}$. There are $\frac{n+1}{2}$ such conjugacy classes which give $\Gamma_{sc}(D_{2n}) = K_{\frac{n+1}{2}}$.

For $n = 3, 5$ and 7 we have $\Gamma_{sc}(D_{2n}) = K_2, K_3$ and K_4 respectively; and so $\gamma(\Gamma_{sc}(D_{2n})) = 0$ as well as $\bar{\gamma}(\Gamma_{sc}(D_{2n})) = 0$. For $n = 9, 11$ and 13 we have $\Gamma_{sc}(D_{2n}) = K_5, K_6$ and K_7 respectively; so $\gamma(\Gamma_{sc}(D_{2n})) = 1$ and $\bar{\gamma}(\Gamma_{sc}(D_{2\times 9})) = \bar{\gamma}(\Gamma_{sc}(D_{2\times 11})) = 1$. For $n = 15$ we have $\Gamma_{sc}(D_{2n}) = K_8$ and so $\gamma(\Gamma_{sc}(D_{2n})) = 2$. For $n = 17$ we have $\Gamma_{sc}(D_{2n}) = K_9$ and so $\gamma(\Gamma_{sc}(D_{2n})) = 3$. For $n \geq 19$, by (1), we have

$$\gamma(\Gamma_{sc}(D_{2n})) = \left\lceil \frac{(n-5)(n-7)}{48} \right\rceil \geq 4.$$

For $n \geq 13$, by (3), we have

$$\bar{\gamma}(\Gamma_{sc}(D_{2n})) = \left\lceil \frac{(n-5)(n-7)}{24} \right\rceil \geq 2.$$

Case 2. If n is even.
The non-trivial conjugacy classes in D_{2n} are $a^{D_{2n}}$, $(a^2)^{D_{2n}}$, …, $(a^{\frac{n}{2}})^{D_{2n}}$, $b^{D_{2n}}$, $(ab)^{D_{2n}}$. There are $\frac{n+4}{2}$ such conjugacy classes which give $\Gamma_{sc}(D_{2n}) = K_{\frac{n+4}{2}}$.

For $n = 2$ and 4 we have $\Gamma_{sc}(D_{2n}) = K_3$ and K_4 respectively; and so $\gamma(\Gamma_{sc}(D_{2n})) = 0$ as well as $\bar{\gamma}(\Gamma_{sc}(D_{2n})) = 0$. For $n = 6, 8$ and 10 we have $\Gamma_{sc}(D_{2n}) = K_5, K_6$ and K_7 respectively; so $\gamma(\Gamma_{sc}(D_{2n})) = 1$ and $\bar{\gamma}(\Gamma_{sc}(D_{2\times 6})) = \bar{\gamma}(\Gamma_{sc}(D_{2\times 8})) = 1$. For $n = 12$ we have $\Gamma_{sc}(D_{2n}) = K_8$ and so $\gamma(\Gamma_{sc}(D_{2n})) = 2$. For $n = 14$ we have $\Gamma_{sc}(D_{2n}) = K_9$ and so $\gamma(\Gamma_{sc}(D_{2n})) = 3$. For $n \geq 16$, by (1), we have

$$\gamma(\Gamma_{sc}(D_{2n})) = \left\lceil \frac{(n-2)(n-4)}{48} \right\rceil \geq 4.$$

For $n \geq 10$, by (3), we have

$$\bar{\gamma}(\Gamma_{sc}(D_{2n})) = \left\lceil \frac{(n-2)(n-4)}{24} \right\rceil \geq 2.$$

Hence, the result follows. □

Theorem 3.2. (a) $\Gamma_{sc}(Q_{4n})$ *is planar if and only if* $n = 1$ *and* 2.
(b) $\Gamma_{sc}(Q_{4n})$ *is toroidal if and only if* $n = 3, 4$ *and* 5.
(c) $\Gamma_{sc}(Q_{4n})$ *is double-toroidal if and only if* $n = 6$.
(d) $\Gamma_{sc}(Q_{4n})$ *is triple-toroidal if and only if* $n = 7$.
(e) $\gamma(\Gamma_{sc}(Q_{4n})) = \left\lceil \frac{(n-1)(n-2)}{12} \right\rceil$ *for* $n \geq 8$.
(f) $\bar{\gamma}(\Gamma_{sc}(Q_{4n})) = 0$ *if and only if* $n = 1$ *and* 2.
(g) $\Gamma_{sc}(Q_{4n})$ *is projective if and only if* $n = 3$ *and* 4.
(h) $\bar{\gamma}(\Gamma_{sc}(Q_{4n})) = \begin{cases} 3, & \text{when } n = 5 \\ \left\lceil \frac{(n-1)(n-2)}{6} \right\rceil, & \text{when } n \geq 6. \end{cases}$

Proof. The non-trivial conjugacy classes in Q_{4n} are $a^{Q_{4n}}$, $(a^2)^{Q_{4n}}$, …, $(a^n)^{Q_{4n}}$, $b^{Q_{4n}}$ and $(ab)^{Q_{4n}}$. There are $n+2$ such conjugacy classes which give $\Gamma_{sc}(Q_{4n}) = K_{n+2}$.

For $n = 1$ and 2 we have $\Gamma_{sc}(Q_{4n}) = K_3$ and K_4 respectively; and so $\gamma(\Gamma_{sc}(Q_{4n})) = 0$ as well as $\bar{\gamma}(\Gamma_{sc}(Q_{4n})) = 0$. For $n = 3, 4$ and 5 we have $\Gamma_{sc}(D_{2n}) = K_5, K_6$ and K_7 respectively; so $\gamma(\Gamma_{sc}(Q_{4n})) = 1$ and $\bar{\gamma}(\Gamma_{sc}(Q_{4\times 3})) = \bar{\gamma}(\Gamma_{sc}(Q_{4\times 4})) = 1$.

For $n = 6$ we have $\Gamma_{sc}(D_{2n}) = K_8$ and so $\gamma(\Gamma_{sc}(Q_{4n})) = 2$. For $n = 7$ we have $\Gamma_{sc}(Q_{4n}) = K_9$ and so $\gamma(\Gamma_{sc}(Q_{4n})) = 3$. For $n \geq 8$, by (1), we have

$$\gamma(\Gamma_{sc}(Q_{4n})) = \left\lceil \frac{(n-1)(n-2)}{12} \right\rceil \geq 4.$$

By (3), we have $\bar{\gamma}(\Gamma_{sc}(Q_{4\times 5})) = 3$ and for $n \geq 6$,

$$\bar{\gamma}(\Gamma_{sc}(Q_{4n})) = \left\lceil \frac{(n-1)(n-2)}{6} \right\rceil \geq 4.$$

Hence, the result follows. □

In general, we have the following result for the SCC-graph of a finite solvable group.

Theorem 3.3. *Let G be a finite solvable group with $k(G)$ conjugacy classes. Then*

(a) *$\Gamma_{sc}(G)$ is planar if and only if $k(G) = 1, 2, 3, 4$ and 5.*
(b) *$\Gamma_{sc}(G)$ is toroidal if and only if $k(G) = 6, 7$ and 8.*
(c) *$\Gamma_{sc}(G)$ is double-toroidal if and only if $k(G) = 9$.*
(d) *$\Gamma_{sc}(G)$ is triple-toroidal if and only if $k(G) = 10$.*
(e) *$\gamma(\Gamma_{sc}(G)) = \left\lceil \frac{(k(G)-4)(k(G)-5)}{12} \right\rceil$ for $k(G) \geq 11$.*
(f) *$\bar{\gamma}(\Gamma_{sc}(G)) = 0$ if and only if $k(G) = 1, 2, 3, 4$ and 5.*
(g) *$\Gamma_{sc}(G)$ is projective if and only if $k(G) = 6$ and 7.*
(h) $\bar{\gamma}(\Gamma_{sc}(G)) = \begin{cases} 3, & \text{when } k(G) = 8 \\ \left\lceil \frac{(k(G)-4)(k(G)-5)}{6} \right\rceil, & \text{when } k(G) \geq 9. \end{cases}$

Proof. We have $\Gamma_{sc}(G) = K_{k(G)-1}$. If $k(G) = 1$ then $\Gamma_{sc}(G)$ is the null graph. For $k(G) = 2, 3, 4$ and 5 we have $\Gamma_{sc}(G) = K_1, K_2, K_3$ and K_4 respectively; and so $\gamma(\Gamma_{sc}(G)) = 0$ as well as $\bar{\gamma}(\Gamma_{sc}(G)) = 0$. For $k(G) = 6, 7$ and 8 we have $\Gamma_{sc}(G) = K_5, K_6$ and K_7 respectively; and so $\gamma(\Gamma_{sc}(G)) = 1$. If $k(G) = 9$ then we have $\Gamma_{sc}(G) = K_8$ and so $\gamma(\Gamma_{sc}(G)) = 2$. If $k(G) = 10$ then $\Gamma_{sc}(G) = K_9$ and so $\gamma(\Gamma_{sc}(G)) = 3$. For $k(G) \geq 11$, by (1), we have

$$\gamma(\Gamma_{sc}(G)) = \left\lceil \frac{(k(G)-4)(k(G)-5)}{12} \right\rceil \geq 4.$$

For $k(G) = 6, 7$, by (3), we have $\bar{\gamma}(\Gamma_{sc}(G)) = 1$. If $k(G) = 8$ then $\bar{\gamma}(\Gamma_{sc}(G)) = 3$. For $k(G) \geq 9$, by (3), we have

$$\bar{\gamma}(\Gamma_{sc}(G)) = \lceil \frac{1}{6}(k(G)-4)(k(G)-5) \rceil \geq 4.$$

Hence, the result follows. □

The groups S_3, S_4, A_3 and A_4 are solvable, with respectively 3, 5, 3 and 4 conjugacy classes; so their SCC-graphs are complete graphs on 2, 4, 2 and 3 vertices respectively. All these graphs are planar. Also, $\bar{\gamma}(\Gamma_{sc}(G)) = 0$, if G is one of the groups S_3, S_4, A_3 and A_4 . The SCC-graphs of other small symmetric and alternating groups are shown in the following figures, where a vertex is labelled with a representative of its conjugacy class.

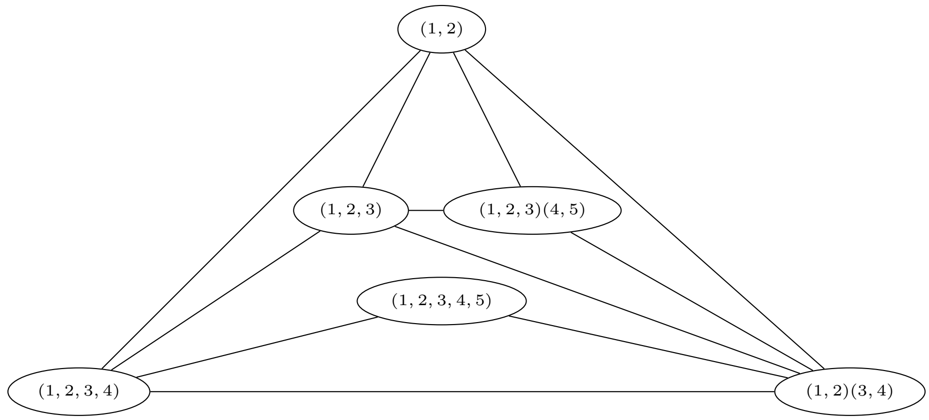

Figure 1. $\Gamma_{sc}(S_5)$

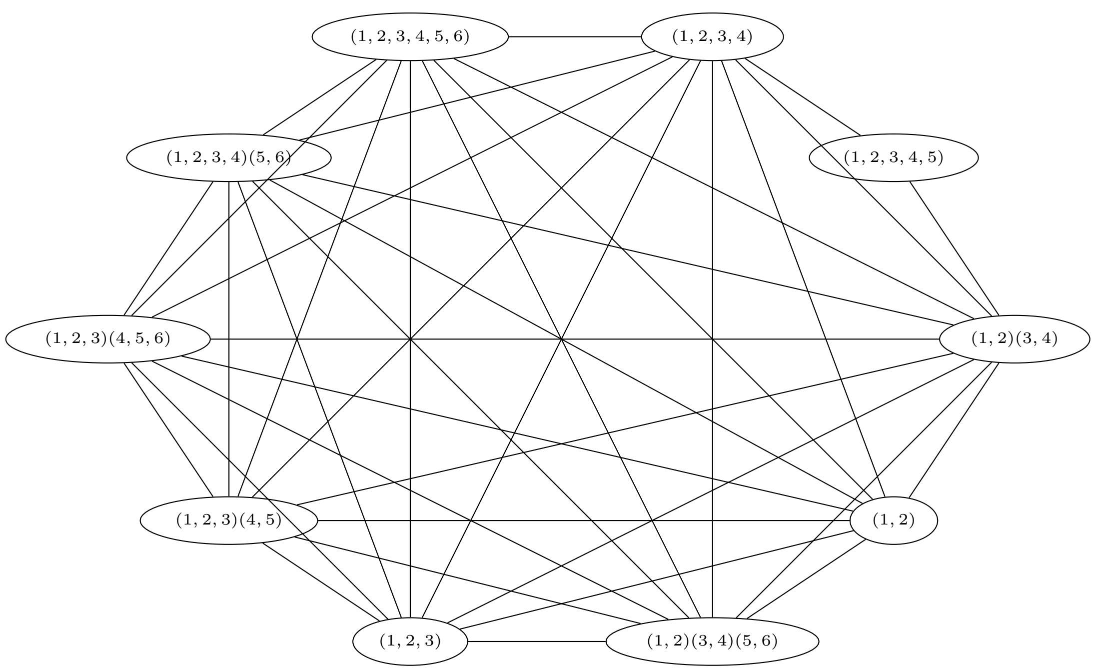

Figure 2. $\Gamma_{sc}(S_6)$

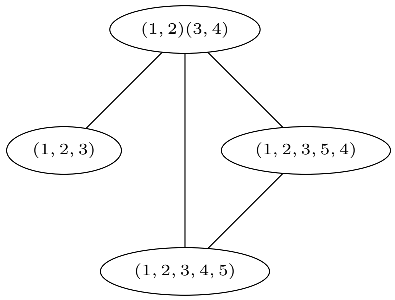

Figure 3. $\Gamma_{sc}(A_5)$

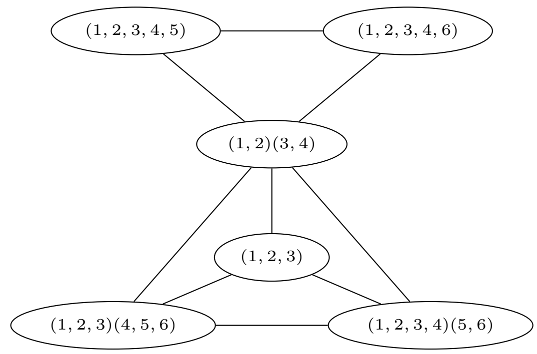

Figure 4. $\Gamma_{sc}(A_6)$

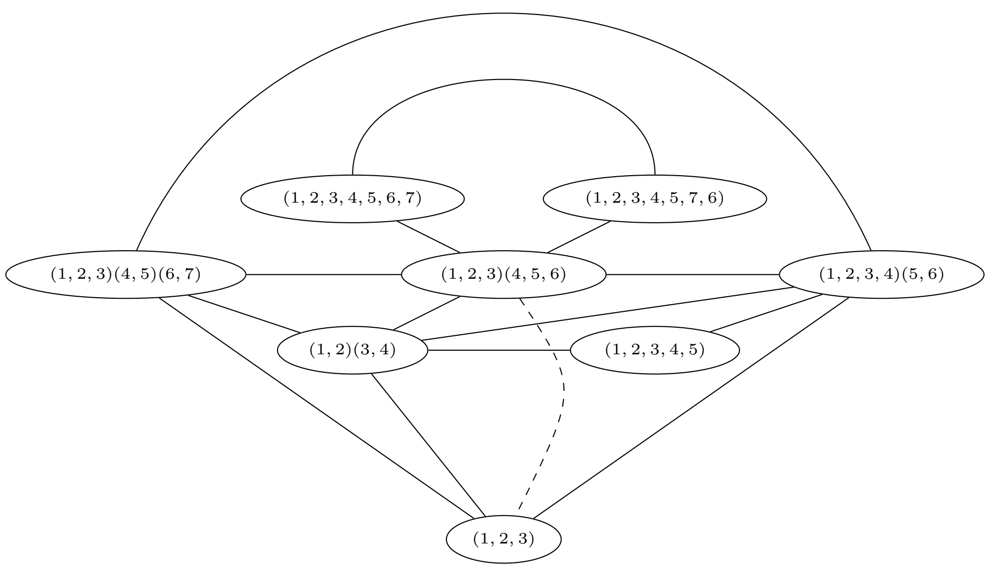

Figure 5. $\Gamma_{sc}(A_7)$

The symmetric and alternating groups whose SCC-graphs have small genus or are projective are given in the following results. First, the symmetric groups.

Theorem 3.4. (a) $\Gamma_{sc}(S_n)$ *is planar if and only if* $n \leq 5$.

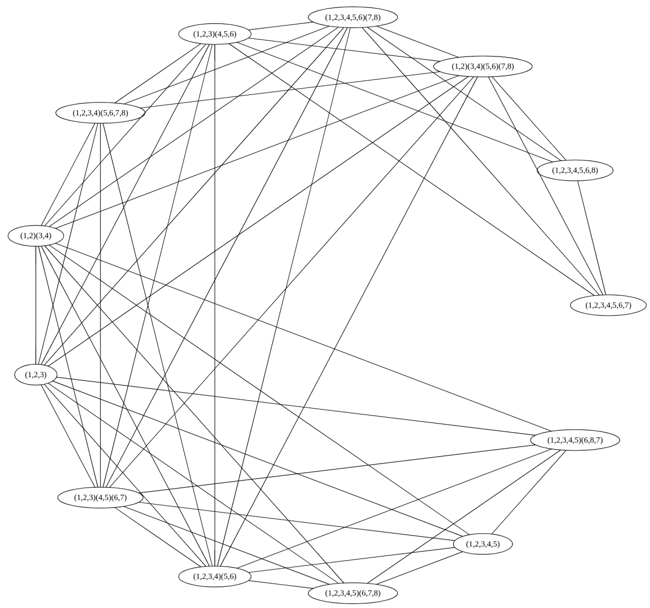

Figure 6. $\Gamma_{sc}(A_8)$

(b) *If* $n \geq 7$ *then* $\Gamma_{sc}(S_n)$ *is neither planar, toroidal, double-toroidal nor triple-toroidal.*
(c) $\Gamma_{sc}(S_6)$ *is not toroidal.*
(d) *If* $n \geq 6$ *then* $\Gamma_{sc}(S_n)$ *is not projective.*

Proof. (a) If $n \leq 5$ then, from our earlier remarks and Figure 1, it follows that $\Gamma_{sc}(S_n)$ is planar. If $n \geq 6$ then it is easy to show that the elements $(1,2)$, $(1,2,3)$, $(1,2)(3,4)$, $(1,2,3,4)$, $(1,2,3)(4,5)$ induce a clique in $\Gamma_{sc}(S_n)$. Hence,

$$\gamma(\Gamma_{sc}(S_n)) \geq \gamma(K_5) = 1$$

and so $\Gamma_{sc}(S_n)$ is not planar.

(b) One can show that the ten elements

$$(1,2), (1,2,3), (1,2)(3,4), (1,2,3,4), (1,2,3)(4,5), (1,2)(3,4)(5,6),$$
$$(1,2,3,4)(5,6), (1,2,3)(4,5,6), (1,2,3)(4,5)(6,7), (1,2,3,4)(5,6,7)$$

induce a clique in $\Gamma_{sc}(S_n)$. Hence,

$$\gamma(\Gamma_{sc}(S_n)) \geq \gamma(K_{10}) = 4$$

and so $\Gamma_{sc}(S_n)$ is neither planar, toroidal, double-toroidal nor triple-toroidal.

(c) From Figure 2, it follows that $\Gamma_{sc}(S_6)$ contains a subgraph isomorphic to K_9 (which is induced by $V(\Gamma_{sc}(S_6)) \setminus \{(1,2,3,4,5)^{S_6}\}$). Therefore,

$$\gamma(\Gamma_{sc}(S_6)) \geq \gamma(K_9) = 3.$$

Hence, the result follows from (a) and (b).

(d) In addition to the five permutations listed in the proof of (a), also the elements

$$(1,2),(1,2)(3,4)(5,6),(1,2,3,4)(5,6),(1,2,3)(4,5,6),(1,2,3,4,5,6)$$

induce a clique. Consequently, $\Gamma_{sc}(S_n)$ contains two copies of K_5 which share a single vertex. This subgraph is isomorphic to the graph denoted by A_1 in [10]. Therefore, $\Gamma_{sc}(S_n)$ is not projective. □

Here is the analogous results for alternating groups.

Theorem 3.5. (a) *$\Gamma_{sc}(A_n)$ is planar if and only if $n \leq 6$.*
(b) *If $n \geq 9$ then $\Gamma_{sc}(A_n)$ is neither planar, toroidal, double-toroidal nor triple-toroidal.*
(c) *$\Gamma_{sc}(A_n)$ is toroidal if and only if $n = 7$.*
(d) *If $n \geq 8$ then $\Gamma_{sc}(A_n)$ is not projective.*

Proof. (a) If $n \leq 6$ then, as shown in Figures 3 and 4, it follows that $\Gamma_{sc}(A_n)$ is planar.

If $n \geq 7$ then the permutations

$$(1,2,3),(1,2)(3,4),(1,2,3,4)(5,6),(1,2,3)(4,5,6),(1,2,3)(4,5)(6,7)$$

induce a clique in $\Gamma_{sc}(A_n)$ (note that the elements have pairwise distinct cycle types). Therefore,

$$\gamma(\Gamma_{sc}(A_n)) \geq \gamma(K_5) = 1.$$

and so $\Gamma_{sc}(A_n)$ is not planar.

(b) The ten elements

$$(1,2,3),(1,2)(3,4),(1,2,3,4)(5,6),(1,2,3)(4,5,6),(1,2,3)(4,5)(6,7),(1,2)(3,4)(5,6)(7,8),$$
$$(1,2,3,4,5,6)(7,8),(1,2,3,4)(5,6,7)(8,9),(1,2,3)(4,5,6)(7,8,9),(1,2,3,4,5,6,7,8,9)$$

induce a clique in $\Gamma_{sc}(A_n)$. Thus, the result follows as in Theorem 3.4(b).

(c) The fact that $\Gamma_{sc}(A_7)$ is toroidal follows from Figure 5 and part (a).

It is easy to see in Figure 6, that the subgraph induced by the permutations

$$(1,2,3)(4,5,6),(1,2,3,4,5,6,7),(1,2,3,4,5,6,8),(1,2)(3,4)(5,6)(7,8),(1,2,3,4,5,6)(7,8)$$

and

$$(1,2,3,4,5),(1,2,3,4)(5,6),(1,2,3)(4,5)(6,7),(1,2,3,4,5)(6,7,8),(1,2,3,4,5)(6,8,7)$$

contains a subgraph isomorphic to $K_5 \sqcup K_5$, where $\sqcup$ stands for disjoint union. Therefore,

$$\gamma(\Gamma_{sc}(A_8)) \geq \gamma(K_5 \sqcup K_5) = 2.$$

Hence, the result follows from parts (a) and (b).

(d) There are two 5-cliques induced by

$$(1,2,3),(1,2)(3,4),(1,2,3,4,5),(1,2,3,4)(5,6),(1,2,3)(4,5)(6,7),$$
$$(1,2,3),(1,2,3)(4,5,6),(1,2)(3,4)(5,6)(7,8),(1,2,3,4,5,6)(7,8),(1,2,3,4)(5,6,7,8),$$

which share a single vertex. Thus, the claim follows as in Theorem 3.4(d). □

We conclude with one more family of groups to show that the results can be extended, namely the groups PSL$(2,q)$, where q is a power of 2. This group is isomorphic to S_3 if $q = 2$, and to A_5 if $q = 4$; these cases have already been considered. Relevant group-theoretic information can be found in the $\mathbb{ATLAS}$ of finite groups [7].

The group $G = \mathrm{PSL}(2,q)$, for $q = 2^d$, has a unique conjugacy class of involutions. The remaining elements lie in cyclic groups of orders $q \pm 1$, whose normalisers are dihedral of orders $2(q \pm 1)$; any element of odd order is conjugate to one in a fixed dihedral group of order $2(q \pm 1)$, and two such elements are conjugate if and only if they are conjugate in this dihedral group. From our analysis of dihedral groups, we see that $\Gamma_{sc}(\mathrm{PSL}(2,q))$ is the union of complete graphs of orders $(q+2)/2$ and $q/2$ with one vertex from each (corresponding to the class of involutions) identified. (The remaining solvable subgroups are the normalizers of Sylow 2-subgroups, of order $q(q-1)$, and give rise to the same edges as the dihedral group of order $2(q-1)$.) For $q = 8$, the graph is shown in Figure 7.

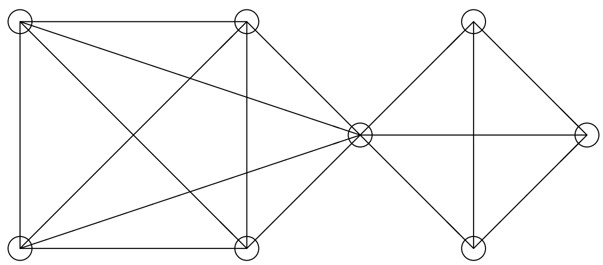

Figure 7. $\Gamma_{sc}(\mathrm{PSL}(2,8))$

Theorem 3.6. *For $q = 2^d$, with $d \geq 3$, the genus of $\Gamma_{sc}(\mathrm{PSL}(2,q))$ is either $\gamma(K_{q/2}) + \gamma(K_{q/2}+1)$ or $\gamma(K_{q/2}) + \gamma(K_{q/2}+1) - 1$.*

Proof. The upper bound comes from the observation that we cam embed each of the two complete graphs in a surface of minimal genus, and then identify one vertex (by deleting a small disc containing the vertex on each surface and putting a cylinder connecting the boundaries of the two discs with one new vertex on it).

The value of this sum is $\lceil (m-3)(m-4)/12\rceil + \lceil (m-2)(m-3)/12\rceil$, where $m = q/2$. Now $m \equiv 4$ or $8 \pmod{12}$. So $(m-3)(m-4) \equiv 0$ or 8, and $(m-2)(m-3) \equiv 2$ or $6 \pmod{12}$. Thus, taking the ceiling means adding 10 to the numerator, giving

$$g \leq \frac{m^2 - 7m + 12 + m^2 - 5m + 6 + 10}{12} = \frac{m^2 - 6m + 14}{6}.$$

For the lower bound, we note that the numbers of vertices and edges are given by $V = 2m$, $E = m^2$, where $m = q/2$. Since a face has at least three edges, the number F of faces satisfies $3F \leq 2E$, so $F \leq \frac{2}{3}m^2$. Now Euler's formula gives

$$V - E + F = 2 - 2g,$$

so $g \geq (m^2 - 6m + 6)/6$. Since g is integral, we can round up. Now again $m \equiv 2$ or $4 \pmod 6$, so we add 2 to the numerator to reach a multiple of 6. Thus

$$g \geq \frac{m^2 - 6m + 8}{6}.$$

The lower and upper bounds differ by 1, giving the result. $\square$

Problem 3.7. *Which of the two values is correct?*

In particular, we see that for $q \leq 4$ the graph is planar; for $q = 8$ it is toroidal, and for $q \geq 16$ it has genus at least 4.

We note in passing that these groups have the property that the unique class of involutions is a dominant vertex in the SCC graph.

Problem 3.8. *Which non-solvable groups G have the property that the SCC graph contains a dominant vertex?*

We note that the Janko group J_1 also has this property.

A similar analysis (with some differences in detail) could be carried out for the groups $\mathrm{PSL}(2, q)$ and $\mathrm{PGL}(2, q)$, where q is an odd prime power; we leave this as an exercise for readers, after working one case, the group $\mathrm{PSL}(2, 7)$. In this case, the two dihedral subgroups of orders 6 and 8 are contained in a subgroup isomorphic to S_4. Since all involutions are conjugate, this gives a triangle on the classes of elements of orders 2, 3 and 4. The elements of order 7 are contained in subgroups of order 21, each meeting both conjugacy classes. So the two classes of order 7 and the class of order 3 form another triangle, with one vertex in common with the first; so $\Gamma_{sc}(\mathrm{PSL}(2, 7))$ is the "bowtie graph", and is planar.

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