ON RESTRICTION OF CHARACTERS TO DEFECT GROUPS

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ABSTRACT. We put forward a blockwise version of a recent conjecture of [Giannelli–Navarro, 2018] on finite groups. Let B be a p-block of a finite group G with defect group D. Let $\chi \in \mathrm{Irr}(B)$ be a character with positive height. In this note we conjecture that the number of distinct linear constituents of the restriction χ_D is 0 or at least p. We prove that this is indeed the case for various classes of finite groups and p-blocks

1. INTRODUCTION

Let G be a finite group and let P be a Sylow p-subgroup of G. In [6], the first author and G. Navarro study the number of linear constituents of the restriction to P of any irreducible character χ of G. When $\chi(1)$ is coprime to p then it is immediate to see that χ_P necessarily has a linear constituent. In the opposite case, when p divides $\chi(1)$ it is proposed (and proved for various classes of groups) that χ_P has either 0 or at least p distinct linear constituents.

In this note we introduce blocks in the picture. Let B be a p-block of the finite group G and let D be a defect group of B. The purpose of this article is to study the restriction to D of any irreducible character χ of G lying in the p-block B. Our first result generalizes to this setting the trivial observation mentioned above about characters of degree coprime to p.

Theorem A. *Let $\chi \in \mathrm{Irr}(G)$ be a character of height 0 in a p-block B with defect group D. Then χ_D has a linear constituent.*

Inspired by the results in [6], most of this note is devoted to study the opposite situation, namely the restriction to D of irreducible characters of positive height in B. When G is a symmetric or alternating group, we prove the following.

Theorem B. *Let n be a natural number, let p be a prime and let B be a p-block of $\mathfrak{S}_n$ or $\mathfrak{A}_n$ with defect group D. If $\chi \in \mathrm{Irr}(B)$ has positive height, then the restriction χ_D has at least p different linear constituents.*

In general, χ_D does not necessarily have a linear constituent (take $G = D$ non-abelian for instance). Nevertheless, we believe that the following blockwise analogue of [6, Conjecture D] holds.

Conjecture C. *Let B be a p-block of a finite group G with defect group D. Suppose that $\chi \in \mathrm{Irr}(B)$ has positive height. Them χ_D has 0 or at least p different linear constituents.*

If D is abelian, then the statement vacuously holds, since by Kessar–Malle [8] there are no irreducible characters of positive height. Furthermore, in Section 2 we show that Conjecture C holds when G is a sporadic simple group, except possibly for the following four: Fi_{23}, Fi'_{24}, B, M. More evidence in support of Conjecture C is given by the following result.

Theorem D. *Let B be a p-block of a finite group G with defect group D. Then Conjecture C holds in the following cases.*

- *B is nilpotent,*
- *D is normal in G,*
- *D is a metacyclic 2-group.*

We also checked Conjecture C for all blocks of groups from the small group library and the library of perfect groups in GAP [3]. We remark that in many situations the number of linear constituents of χ_D is in fact a multiple of p, but this is not true in general.

For the principal block, Conjecture C is a consequence of [6, Conjecture D]. However, in general we do not know if Conjecture C follows from (or implies) [6, Conjecture D].

We fix below some of the notation that will be used often throughout this note.

Notation. Let G be a finite group and let P be a Sylow p-subgroup of G. In this article we will sometimes denote by $\mathrm{Lin}(G)$ the set of linear characters of G. Moreover, if B is a p-block of G with defect group D then we let $\mathrm{Irr}(B)$ be the set of irreducible characters of G lying in B. For $\chi \in \mathrm{Irr}(B)$ we let $h(\chi)$ be the p-height of χ. In particular we have that $\chi(1)_p = p^{a-d+h(\chi)}$, where $p^a = |P|$, $p^d = |D|$ and where n_p denotes the highest power of p dividing the natural number $n \in \mathbb{N}$.

2. Theorems A and D

We start by giving a concise proof of Theorem A of the introduction.

Theorem A. *Let $\chi \in \mathrm{Irr}(G)$ be a character of height 0 in a block B with defect group D. Then χ_D has a linear constituent.*

Proof. Let b be an extended Brauer main correspondent of B in $D\mathbf{C}_G(D)$ (see [10, Theorem 9.7]). By a result of Blau (cf. [10, Theorem 6.8]), $\chi_{D\mathbf{C}_G(D)}$ has an irreducible constituent of height 0 in b. Hence, we may assume that $G = D\mathbf{C}_G(D)$. Then by [10, Theorem 9.12], we have $\chi_D = n\xi$ where $\xi \in \mathrm{Irr}(D)$ and $n_p = |G : D|_p$. Since χ has height 0, we obtain $\chi(1)_p = |G : D|_p = n_p$ and $\xi(1) = 1$. $\qquad\square$

Now we prove a strong form of Conjecture C for nilpotent blocks.

Theorem 2.1. *Let B be a nilpotent p-block of G with defect group D, and let $\chi \in \mathrm{Irr}(B)$ with positive height. Then the number of linear constituents of χ_D is divisible by p.*

Proof. Let $\psi \in \mathrm{Irr}(B)$ of height 0. By Broué–Puig [1] there exists a non-linear $\theta \in \mathrm{Irr}(D)$ such that $\chi = \theta * \psi$. It follows that $\chi_D = \theta\psi_D$. We may assume that ψ_D has a constituent $\rho \in \mathrm{Irr}(D)$ such that $\theta\rho$ has a linear constituent $\lambda \in \mathrm{Irr}(D)$. Then $[\lambda\overline{\theta}, \rho] = [\lambda, \theta\rho] \neq 0$

and we conclude that $\lambda\overline{\theta} = \rho$ since $\lambda\overline{\theta}$ is irreducible. Hence, $[\lambda, \theta\rho] = 1$. We showed that every linear constituent of $\theta\rho$ occurs with multiplicity 1. Since $(\theta\rho)(1)$ is divisible by p, there must be at least p linear constituents in $\theta\rho$. Moreover, if $\rho' \in \mathrm{Irr}(D) \setminus \{\rho\}$ is another constituent of ψ_D, then $[\lambda, \theta\rho'] = [\lambda\overline{\theta}, \rho'] = 0$. The claim follows easily. $\square$

The following verifies our conjecture for blocks with normal defect group.

Proposition 2.2. *Let B be a p-block of G with normal defect group D and $\chi \in \mathrm{Irr}(B)$. If χ_D has a linear constituent, then χ has height 0.*

Proof. If χ_D has a linear constituent, then χ_D is a sum of linear characters by Clifford theory. Then $D' \subseteq \ker(\chi)$ and $\chi \in \mathrm{Irr}(G/D')$. Since D/D' is an abelian normal subgroup of G/D', Ito's theorem shows that $\chi(1)$ divides $|G : D|$. Hence, χ has height 0. $\square$

Now we deal with the 2-blocks with metacyclic defect groups.

Theorem 2.3. *Let B be a 2-block of G with metacyclic defect group D, and let $\chi \in \mathrm{Irr}(B)$ with positive height such that χ_D has a linear constituent. Then χ_D has at least two distinct linear constituents. In particular, Conjecture C holds for tame blocks.*

Proof. By Theorem 2.1, we may assume that B is non-nilpotent. By Kessar–Malle [8], we may assume that D is non-abelian. Then it follows from [14, Theorem 8.1] that D is dihedral, semidihedral or quaternion. Let $x \in D$ such that $|D : \langle x \rangle| = 2$ and $|\langle x \rangle| = 2^n$. Recall that the generalized decomposition numbers $d_{\chi\phi}^x$ (for $\phi \in \mathrm{IBr}(\mathbf{C}_G(x))$) are algebraic integers in the cyclotomic field $\mathbb{Q}_{2^n}$. In particular, there exist uniquely determined integers $a_0, \ldots, a_{2^{n-1}-1} \in \mathbb{Z}$ such that

$$d_{\chi\phi}^x = \sum_{i=0}^{2^{n-1}-1} a_i \zeta^i$$

where $\zeta \in \mathbb{C}$ is a primitive 2^n-th root of unity. It follows from the orthogonality relations for generalized decomposition numbers that $a_0 = 0$ (see [2, Theorem 0]). In particular,

$$\chi(x) = \sum_{\phi \in \mathrm{IBr}(\mathbf{C}_G(x))} d_{\chi\phi}^x \phi(1)$$

has no rational part. The same argument applies to the group D where the generalized decomposition numbers are just the character values. This means that for every non-linear $\theta \in \mathrm{Irr}(D)$, $\theta(x)$ has no rational part. On the other hand, every linear character $\theta \in \mathrm{Irr}(D)$ satisfies $\theta(x) = \pm 1$ since D/D' is a Klein four-group. Hence, if χ_D has just one linear constituent, then $\chi(x)$ would have a non-zero rational part, but this is not the case. $\square$

The following lemma is used to handle some of the sporadic groups.

Lemma 2.4. *Let B be a block of G with defect group D, and let $\chi \in \mathrm{Irr}(B)$. Suppose that there exists $x \in D$ such that $|\mathbf{C}_D(x)| = |D : D'|$ and $\chi(x) = 0$. Then χ_D has 0 or at least p linear constituents.*

Proof. By the second orthogonality relation, we have

$$|\mathbf{C}_D(x)| = \sum_{\theta \in \mathrm{Irr}(D)} |\theta(x)|^2 \geq \sum_{\theta \in \mathrm{Irr}(D/D')} 1 = |D : D'| = |\mathbf{C}_D(x)|.$$

Hence, $\theta(x) = 0$ for every non-linear $\theta \in \mathrm{Irr}(D)$. Suppose that χ_D has linear constituents $\lambda_1, \ldots, \lambda_s$ with multiplicities $a_1, \ldots, a_s > 0$ respectively. Then $a_1\lambda_1(x) + \ldots + a_s\lambda_s(x) = 0$. The $\lambda_i(x)$ are $|\langle x \rangle|$-th roots of unity and they form orbits under the Galois group of the cyclotomic field $\mathbb{Q}_{|\langle x \rangle|}$. Now it is easy to see that $s \geq p$. $\qquad\square$

Proposition 2.5. *Conjecture C holds for the blocks of sporadic groups, except possibly the principal 2-blocks and the principal 3-blocks of Fi_{23}, Fi'_{24}, B or M.*

Proof. We first consider the blocks with non-maximal defect. The non-principal 2-blocks were determined by Landrock [9]. Most of them have abelian or metacyclic defect groups. Hence, by Kessar–Malle [8] and Theorem 2.3, only the following blocks need to be considered where the labeling is taken from GAP [3]:

  (i) second 3-block of Co_1 (defect 3),
 (ii) second 2-block of Ly (defect 7),
(iii) fourth 3-block of M (defect 3).

For Co_1 and M all characters of positive height vanish on an element $x \in D \setminus \mathbf{Z}(D)$. Hence, the claim follows from Lemma 2.4. For Ly the claim can be proved with GAP by restricting to the maximal subgroup $2.\mathfrak{A}_{11}$ which has odd index in Ly and therefore contains every defect group.

Now we deal with the blocks of maximal defect. With GAP we can restrict to various subgroups where the class fusion is known. This approach works for all blocks except the ones listed in the statement. $\qquad\square$

3. Symmetric and Alternating groups

The goal of this section is to prove Theorem B from the introduction.

We recall that if D is a Sylow p-subgroup of $\mathfrak{S}_n$ (respectively $\mathfrak{A}_n$), then Conjecture C holds by [6, Theorem A] (respectively [5]). We usually denote by P_n a chosen Sylow p-subgroup of $\mathfrak{S}_n$ and by Q_n a fixed Sylow p-subgroup of $\mathfrak{A}_n$ such that $Q_n \leq P_n$.

For any natural number n let $\mathcal{P}(n)$ be the set of partitions of n. Let $\mathcal{H}(n)$ be the subset of $\mathcal{P}(n)$ consisting of all the hook partitions of n. Given $\lambda = (\lambda_1, \ldots, \lambda_t) \in \mathcal{P}(n)$ we denote by $\mathcal{Y}(\lambda)$ the *Young diagram* of λ. As usual, we think of $\mathcal{Y}(\lambda)$ as the subset of the Cartesian plane defined by $\mathcal{Y}(\lambda) = \{(a, b) \mid 1 \leq a \leq t, \ 1 \leq b \leq \lambda_a\}$. Given $(x, y) \in \mathcal{Y}(\lambda)$ we let $H_{x,y}(\lambda) := \{(a, y), (x, b) \in \mathcal{Y}(\lambda) \mid a \geq x, \ \mathrm{and} \ b \geq y\}$ be the (x, y)-hook of λ and we let $h_{x,y}(\lambda) = |H_{x,y}(\lambda)|$. Given any $e \in \mathbb{N}$ we let $\mathcal{H}_e(\lambda)$ be the subset of $\mathcal{Y}(\lambda)$ consisting of all the nodes (x, y) such that $h_{x,y}(\lambda)$ is a multiple of e. Finally, given any hook H of λ we denote by $\lambda - H$ the partition of $n - |H|$ obtained by removing the *rim of H* from $\mathcal{Y}(\lambda)$ (we refer the reader to [11, Section 1] for a precise description of this process). The *e-core* of λ (denoted by $C_e(\lambda)$) is the (unique) partition obtained from λ by successively removing hooks of length e.

Keeping λ a partition of n, we denote by $|\lambda| = n$ the size of λ and we let χ^λ be the corresponding irreducible character of $\mathfrak{S}_n$. Let p be a prime number. The p-blocks of $\mathfrak{S}_n$ are naturally labeled by p-core partitions. In particular given a p-core partition γ and a natural number w such that $n = |\gamma| + pw$, we denote by $B(\gamma, w)$ the p-block of $\mathfrak{S}_n$ consisting of all the irreducible characters χ^λ such that $C_p(\lambda) = \gamma$. We remind the reader that the defect group D of $B(\gamma, w)$ is a Sylow p-subgroup of $\mathfrak{S}_{wp} \leq \mathfrak{S}_n$.

Keeping the notation described at the end of the introduction we let $\mathrm{Irr}_0(B)$ be the set of irreducible characters of height zero in the block B. The following description of $\mathrm{Irr}_0(B(\gamma, w))$ follows from [13, Lemma 3.1].

Theorem 3.1. *Let $B = B(\gamma, w)$ be a p-block of $\mathfrak{S}_n$. Let $wp = a_t p^t + \cdots + a_1 p$, be the p-adic expansion of wp where $a_t \neq 0$. Let $\chi^\lambda \in \mathrm{Irr}(B)$. The following are equivalent.*

(i) $\chi^\lambda \in \mathrm{Irr}_0(B)$.
(ii) $|\mathcal{H}_{p^t}(\lambda)| = a_t$ *and* $\chi^{C_{p^t}(\lambda)} \in \mathrm{Irr}_0(B(\gamma, w - a_t p^{t-1}))$.

Definition 3.2. *Let $m = a_t p^t + \cdots + a_1 p$ be the p-adic expansion of m. Let n be a natural number such that $m \leq n$. An element $g \in \mathfrak{S}_n$ has cycle type corresponding to m, if g is the product of a_j cycles of length p^j, for all $j \in \{1, \ldots, t\}$. In particular g has $n - m$ fixed points.*

The Murnaghan–Nakayama rule (see [7, 2.4.7]) used together with Theorem 3.1 imply the following statement.

Corollary 3.3. *Let $B = B(\gamma, w)$ be a p-block of $\mathfrak{S}_n$, let D be a defect group of B and let $\chi^\lambda \in \mathrm{Irr}(B)$ be such that $h(\chi^\lambda) > 0$. Let $g \in D$ be an element of cycle type corresponding to wp. Then $\chi^\lambda(g) = 0$.*

The last ingredient needed to prove Theorem B for $\mathfrak{S}_n$ is the following Lemma.

Lemma 3.4. *Let $B = B(\gamma, w)$ be a p-block of $\mathfrak{S}_n$. Let $wp = a_t p^t + \cdots + a_1 p$ be the p-adic expansion of wp where $a_t \neq 0$. Let D be a defect group of B. Then there exists $g \in D$ of cycle type corresponding to wp such that the following hold.*

(i) $\theta(g)$ *is a p-th root of unity for every linear character θ of D.*
(ii) $\delta(g) = 0$ *for all $\delta \in \mathrm{Irr}(D)$ such that $p \mid \delta(1)$.*

Proof. Since $D \in \mathrm{Syl}_p(\mathfrak{S}_{wp})$ we have that $D \cong (P_{p^t})^{\times a_t} \times \cdots \times (P_p)^{\times a_1}$. Hence there exists an element $g \in D$ of cycle type corresponding to wp. Statements (i) and (ii) are now a direct consequence of [6, Lemma 3.11]. $\qquad\square$

Proof of Theorem B for $\mathfrak{S}_n$. Let $B = B(\gamma, w)$ be a p-block of $\mathfrak{S}_n$ and let $D \in \mathrm{Syl}_p(\mathfrak{S}_{wp})$ be a defect group of B, chosen such that $D \leq P_n$. Let $\chi \in \mathrm{Irr}(B)$ be such that $h(\chi) > 0$. From [6, Theorem 3.1] we know that χ_{P_n} has a linear constituent. Hence there exists a positive integer ℓ such that χ_D has ℓ distinct linear constituents. Call these $\theta_1, \ldots, \theta_\ell$. Suppose for a contradiction that $\ell < p$. Let $g \in D$ be an element of cycle type corresponding to wp as prescribed in Lemma 3.4. Then Corollary 3.3 and Lemma 3.4 imply that

$$0 = \chi_D(g) = c_1 \theta_1(g) + \cdots + c_\ell \theta_\ell(g).$$

This is a contradiction, since $\theta_j(g)$ is a p-th root of unity for all $j \in \{1, \ldots, \ell\}$ by Lemma 3.4, and no $\mathbb{N}$-linear combination of ℓ p-th roots of unity can be equal to 0. $\qquad\square$

3.1. **Alternating groups.** We start by recalling that if $\phi \in \operatorname{Irr}(\mathfrak{A}_n)$ and $\lambda \vdash n$ is such that ϕ is an irreducible constituent of $(\chi^\lambda)_{\mathfrak{A}_n}$, then $\phi = (\chi^\lambda)_{\mathfrak{A}_n}$ if and only if $\lambda \neq \lambda'$. If $\lambda = \lambda'$ then $(\chi^\lambda)_{\mathfrak{A}_n} = \phi + \phi^g$, where $g \in \mathfrak{S}_n \smallsetminus \mathfrak{A}_n$.

Let $\tilde{B}$ be a p-block of $\mathfrak{A}_n$ with defect group $\tilde{D}$ covered by the p-block B of $\mathfrak{S}_n$ with defect group D. We choose the defect groups in order to have $\tilde{D} \leq D \leq P_n$ and $\tilde{D} = D \cap Q_n$. It is shown in [5, Theorem A] that for any $\chi \in \operatorname{Irr}(\mathfrak{A}_n)$ such that p divides $\chi(1)$ we have that χ_{Q_n} has at least p distinct linear constituents. Hence we can assume for the rest of this section that D is strictly contained in P_n. In particular we have that there exists $w \in \mathbb{N}$ such that $D \in \operatorname{Syl}_p(\mathfrak{S}_{wp})$ and $\tilde{D} \in \operatorname{Syl}_p(\mathfrak{A}_{wp})$, where $\mathfrak{A}_{wp} \leq \mathfrak{S}_{wp} \leq \mathfrak{S}_n$. Fix $\phi \in \operatorname{Irr}(\tilde{B})$ such that $h(\phi) > 0$ and let $\lambda \vdash n$ be such that ϕ is an irreducible constituent of $(\chi^\lambda)_{\mathfrak{A}_n}$. As usual let $\gamma = C_p(\lambda)$ and hence we have that $B = B(\gamma, w)$.

3.1.1. **Odd primes.** When p is an odd prime, we have that $P_n = Q_n$ and hence that $D = \tilde{D}$. Moreover, by [12, Proposition 4.3] we have that $h(\chi^\lambda) = h(\phi)$. For this reason we can use an argument that is totally similar to the one given for symmetric groups.

Proof of Theorem B for $\mathfrak{A}_n$ at odd primes. We start by observing that $\phi_{\tilde{D}}$ has a linear constituent θ. This follows directly from [5, Theorem A], where it is shown that ϕ_{Q_n} has a linear constituent. Let $\theta_1, \ldots, \theta_\ell$ be all the distinct linear constituents of $\phi_{\tilde{D}}$. Suppose for a contradiction that $\ell < p$. Let $g \in D = \tilde{D}$ be an element of cycle type wp as in Lemma 3.4. By Corollary 3.3 we have that $\chi^\lambda(g) = 0$, since $h(\chi^\lambda) = h(\phi) > 0$. We deduce that $\phi(g) = 0$. This is clear if $\lambda \neq \lambda'$ because in this case $\phi = (\chi^\lambda)_{\mathfrak{A}_n}$; otherwise, if $\lambda = \lambda'$ then we have that $\phi(g) = (\chi^\lambda(g))/2 = 0$, by [7, 2.5.13].

Using Lemma 3.4 it follows that

$$0 = \phi_{\tilde{D}}(g) = c_1 \theta_1(g) + \cdots + c_\ell \theta_\ell(g).$$

This is a contradiction, by Lemma 3.4. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.1.2. **The prime** 2. Let $p = 2$ and let $n \in \mathbb{N}$, for $k \in \{0, 1, \ldots, n-1\}$ we let g_k be the element of $\mathfrak{S}_{2^n}$ defined by

$$g_k = \prod_{i=1}^{2^k} (i, i + 2^k).$$

It is easy to check that $P_{2^n} := \langle g_0, \ldots, g_{n-1} \rangle$ is a Sylow 2-subgroup of $\mathfrak{S}_{2^n}$. Moreover the element $\gamma_n \in P_{2^n}$ defined by $\gamma_n = g_1 g_2 \cdots g_{n-1}$ has cycle type $(2^{n-1}, 2^{n-1})$. In particular $\gamma_n \in \mathfrak{A}_{2^n} \cap P_{2^n} = Q_{2^n}$. Similarly, the element $\omega_n \in P_{2^n}$ defined by $\omega_n = g_0 \gamma_n$ is a 2^n-cycle.

The following Lemma was first proved in [5, Lemma 3.4].

Lemma 3.5. *Let γ_n and ω_n be the elements defined above and let $g \in \{\gamma_n, \omega_n\}$. Then:*
   (i) *$\theta(g) \in \{-1, +1\}$ for every linear character θ of P_{2^n}.*
   (ii) *$\delta(g) = 0$ for all $\delta \in \operatorname{Irr}(P_{2^n})$ such that $\delta(1)$ is even.*

Remark 3.6. We recall that when $p = 2$ then $h(\phi) = h(\chi^\lambda)$ if $\lambda \neq \lambda'$. On the other hand we have that $h(\phi) = h(\chi^\lambda) - 1$ if $\lambda = \lambda'$. This follows from [12, Proposition 4.5]. Moreover, in this case we have that $\tilde{D}$ and Q_n are proper subgroups of index 2 of D and P_n respectively.

Theorem 3.1 has an easier form when $p = 2$. We restate this result here.

Proposition 3.7. *Let $B = B(\gamma, w)$ be a 2-block of $\mathfrak{S}_n$ and let $2w = 2^{w_1} + \cdots + 2^{w_t}$ be the binary expansion of $2w$, with $w_1 > \cdots > w_t > 0$. Let $\chi^\lambda \in \mathrm{Irr}(B)$. Then $\chi^\lambda \in \mathrm{Irr}_0(B)$ if and only if there exists a unique 2^{w_1}-hook h in λ and $\chi^{\lambda - h} \in \mathrm{Irr}_0(B(\gamma, w - 2^{w_1 - 1}))$.*

Proposition 3.8. *Let $\chi^\lambda \in \mathrm{Irr}(B(\gamma, w))$ be such that $h(\chi^\lambda) > 0$. Suppose that $2w$ is not a power of 2. Then there exists an element $\Omega_\lambda \in \tilde{D}$ such that the following hold.*

(i) *$\theta(\Omega_\lambda) \in \{-1, +1\}$ for all $\theta \in \mathrm{Lin}(D)$.*
(ii) *$\delta(\Omega_\lambda) = 0$ for all $\delta \in \mathrm{Irr}(D) \smallsetminus \mathrm{Lin}(D)$.*
(iii) *$\chi^\lambda(\Omega_\lambda) = 0$.*

Proof. Let $2w = 2^{w_1} + \cdots + 2^{w_t}$ be the binary expansion of $2w$, where $w_1 > \cdots > w_t \geq 1$ and where $t \geq 2$. Assume without loss of generality that $D \in \mathrm{Syl}_2(\mathfrak{S}_{2w})$ fixes pointwise the set $\{2w + 1, 2w + 2, \ldots, n\}$. Then $D = P_{2^{w_1}} \times \cdots \times P_{2^{w_t}}$. For all $j \in \{1, \ldots, t\}$ let ω_{w_j} and γ_{w_j} be the elements of the component $P_{2^{w_j}}$ described in Lemma 3.5 (and in the paragraph right above it). In particular ω_{w_j} has cycle type (2^{w_j}) while γ_{w_j} has cycle type $(2^{w_j - 1}, 2^{w_j - 1})$. Let Ω_0, Ω_1 and Ω_2 be the elements of D defined by

$$\Omega_0 = \omega_{w_1} \cdots \omega_{w_t}, \quad \Omega_1 := \gamma_{w_1} \omega_{w_1}^{-1} \Omega_0 \quad \text{and} \quad \Omega_2 = \gamma_{w_2} \omega_{w_2}^{-1} \Omega_0.$$

Suppose that the binary length t of $2w$ is even. Set $\Omega_\lambda = \Omega_0$. Then $\Omega_\lambda \in \tilde{D}$ and statements (i) and (ii) hold by Lemma 3.5. Moreover, Ω_λ is an element of cycle type corresponding to $2w$ (in the sense of Definition 3.2). Hence Corollary 3.3 implies that $\chi^\lambda(\Omega_\lambda) = 0$.

Suppose that t is odd. In this case $\Omega_0 \notin \tilde{D}$, hence we must change our choice of Ω_λ. Since $|\lambda| < 2 \cdot 2^{w_1}$ we clearly have that $|\mathcal{H}_{2^{w_1}}(\lambda)| \leq 1$. If $|\mathcal{H}_{2^{w_1}}(\lambda)| = 0$ then we set $\Omega_\lambda = \Omega_2$. Otherwise, if $|\mathcal{H}_{2^{w_1}}(\lambda)| = 1$ then we set $\Omega_\lambda = \Omega_1$. It is now routine to check that in both cases $\Omega_\lambda \in \tilde{D}$ and that conditions (i), (ii) and (iii) are fulfilled. This follows again from Lemma 3.5 and from Proposition 3.7 used together with the Murnaghan–Nakayama rule. $\square$

If the weight w is a power of 2, then the knowledge of the 2-height of the irreducible character χ^λ allows us to give a very precise description of the shape of the labeling partition λ. The following statement is an immediate consequence of [13, Lemma 3.1].

Proposition 3.9. *Let $k \geq 2$ and let $w = 2^{k-1}$. Let $B = B(\gamma, w)$ be a 2-block of $\mathfrak{S}_n$ and let $\chi^\lambda \in \mathrm{Irr}(B)$. Then the following hold.*

(i) *$h(\chi^\lambda) = 0$ if and only if $|\mathcal{H}_{2^k}(\lambda)| = 1$.*
(ii) *$h(\chi^\lambda) = 1$ if and only if $|\mathcal{H}_{2^k}(\lambda)| = 0$ and $|\mathcal{H}_{2^{k-1}}(\lambda)| = 2$.*
(iii) *$h(\chi^\lambda) \geq 2$ if and only if $|\mathcal{H}_{2^{k-1}}(\lambda)| \leq 1$.*

Proof of Theorem B for $\mathfrak{A}_n$ at $p = 2$. If $\tilde{D} = Q_n$ then the statement follows from [5, Theorem A]. Hence we assume that $\tilde{D} < Q_n$. Equivalently, $\gamma := C_2(\lambda)$ is such that $|\gamma| \geq 3$.

Since $h(\phi) > 0$ we deduce from Remark 3.6 that $h(\chi^\lambda) > 0$. Hence from Theorem B for $\mathfrak{S}_n$ we know that there exists $\psi \in \mathrm{Lin}(D)$ such that ψ is an irreducible constituent of $(\chi^\lambda)_D$. Let $\theta = \psi_{\tilde{D}}$. We claim that θ is a linear constituent of $\phi_{\tilde{D}}$. If $\lambda \neq \lambda'$, this is obvious

because $\phi_{\tilde{D}} = (\chi^\lambda)_{\tilde{D}}$. Otherwise, if $\lambda = \lambda'$ then let σ be an element of $D \smallsetminus \tilde{D}$. Then $(\chi^\lambda)_{\mathfrak{A}_n} = \phi + \phi^\sigma$ and there exists $\Phi \in \{\phi, \phi^\sigma\}$ such that θ is an irreducible constituent of $\Phi_{\tilde{D}}$. It follows that θ^σ is a constituent of $(\Phi^\sigma)_{\tilde{D}}$. The claim now follows by observing that $\theta^\sigma = (\psi^\sigma)_{\tilde{D}} = \theta$.

Assume for a contradiction that θ is the only linear constituent of $\phi_{\tilde{D}}$, appearing with multiplicity $m \in \mathbb{N}$. In particular this means that all the distinct linear constituents $\psi = \psi_1, \psi_2, \ldots, \psi_\ell$ of ϕ_D are such that $(\psi_j)_{\tilde{D}} = \theta$.

Case 1. Suppose first that $2w$ is not a power of 2. Let $\Omega_\lambda \in \tilde{D}$ be the element prescribed by Proposition 3.8. Then (using [7, 2.5.13] if $\lambda = \lambda'$) we have that

$$0 = \phi(\Omega_\lambda) = c_1\psi_1(\Omega_\lambda) + \cdots + c_\ell\psi_\ell(\Omega_\lambda) = m\theta(\Omega_\lambda) = \pm m,$$

by Proposition 3.8. This is clearly a contradiction, since $m = c_1 + \ldots + c_\ell \neq 0$.

Case 2. Let us now assume that $2w = 2^k$ and that $n = |\gamma| + 2^k$. Notice that since $h(\phi) > 0$ we have that $\tilde{D} \neq 1$. Hence $w > 1$ and therefore $k > 1$.

If $h(\chi^\lambda) \geq 2$ then $|\mathcal{H}_{2^k}(\lambda)| = 0$ and $|\mathcal{H}_{2^{k-1}}(\lambda)| \leq 1$, by Proposition 3.9. Hence if we take $x \in \tilde{D}$ to be an element of cycle type $(2^{k-1}, 2^{k-1})$ then we have that $\chi^\lambda(x) = 0$, by the Murnaghan-Nakayama rule. It follows (using [7, 2.5.13] in case $\lambda = \lambda'$) that $0 = \phi(x) = m\theta(x) = \pm m$, where the second and third equalities follow from Lemma 3.5. This is clearly a contradiction.

If $h(\chi^\lambda) = 1$ then Remark 3.6 implies that $\lambda \neq \lambda'$, because $h(\phi) \neq 0$. Hence $\phi = (\chi^\lambda)_{\mathfrak{A}_n}$. If λ is a hook partition of n then $\gamma = (2, 1)$ and λ has exactly two distinct 2^{k-1}-hooks, by Proposition 3.9. It follows that $\lambda = (2 + 2^{k-1}, 1^{1+2^{k-1}})$. Hence $\lambda = \lambda'$ and this is a contradiction. We conclude that λ is not a hook partition. Equivalently, λ is a partition of n such that $(2, 2) \in \mathcal{Y}(\lambda)$. This in particular shows that $k > 2$ (because the only $\lambda \vdash |\gamma| + 4$ such that $h(\chi^\lambda) = 1$ is such that $\lambda = \lambda'$). Hence there exists $\mu \vdash 2^k$ such that $\mu \subseteq \lambda$ and $(2, 2) \in \mathcal{Y}(\mu)$. Moreover, since $\lambda \neq \lambda'$ we can always choose μ such that $\mu \neq \mu'$. It follows that $(\chi^\mu)_{\mathfrak{A}_{2^k}} \in \mathrm{Irr}(\mathfrak{A}_{2^k})$. Moreover, $\chi^\mu(1)$ is even, since μ is not a hook partition (see for instance [4, Lemma 3.1]). Since $\tilde{D} \in \mathrm{Syl}_2(\mathfrak{A}_{2^k})$, it follows from [5, Theorem A] that $(\chi^\mu)_{\tilde{D}}$ has at least two distinct linear constituents. Observe now that χ^μ is an irreducible constituent of $(\chi^\lambda)_{\mathfrak{S}_{2^k}}$ and that $\tilde{D} \in \mathrm{Syl}_2(\mathfrak{A}_{2^k})$, where $\mathfrak{A}_{2^k} \leq \mathfrak{S}_{2^k} \leq \mathfrak{S}_n$. We conclude that θ can not be the unique linear constituent of $\phi_{\tilde{D}}$. $\qquad\square$

Acknowledgments

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