

# Characterizing inner automorphisms and realizing outer automorphisms

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## Abstract

We give elementary proofs of the following two theorems on automorphisms of a finite group  $G$ : (1) An automorphism of  $G$  is inner if and only if it extends to an automorphism of every finite group containing  $G$ . (2) There exists a finite group, whose outer automorphism group is isomorphic to  $G$ . The first theorem was proved by Pettet using a graph-theoretical construction of Heineken–Liebeck. A Lie-theoretical proof of the second theorem was sketched by Cornulier in a MathOverflow post. Our proofs are purely group-theoretical.

**Keywords:** inner automorphism, outer automorphism

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## 1 Introduction

An automorphism  $\alpha$  of a group  $G$  is called *inner* if there exists some  $g \in G$  such that  $\alpha(x) = gxg^{-1}$  for all  $x \in G$ . If  $G$  is a subgroup of a group  $H$ , it is clear that  $g$  still induces an (inner) automorphism of  $H$ . In 1987, Schupp [27] has shown conversely that inner automorphisms are characterized by this property, i. e. if  $\alpha \in \text{Aut}(G)$  extends to every group containing  $G$ , then  $\alpha$  is inner. According to [8], this has answered a question of Macintyre. The question was asked again much later by Bergman [1, p. 93], who obtained a partial answer in the language of category theory. Using free products and small cancellation theory, Schupp constructs for a non-inner  $\alpha \in \text{Aut}(G)$  an infinite group  $H$  such that  $\alpha$  does not extend to  $H$  (if  $G$  is countable, the construction is already contained in Miller–Schupp [23]). One may ask whether inner automorphisms of *finite* groups  $G$  are characterized by the property that they extend to all *finite* groups containing  $G$ .

The first step in this direction was a paper from 1974 of Heineken–Liebeck [13], who constructed a finite  $p$ -group  $P$  such that the image of the canonical map  $\text{Aut}(P) \rightarrow \text{Aut}(P/\text{Z}(P))$  is isomorphic to  $G$ . Their construction relies on a variation of Frucht’s theorem on the automorphism group of graphs, and requires a treatment of special cases. Using more advanced graph theoretical theorems, Lawton [20] came up with a shorter proof. Subsequently, Webb [28] has refined the construction (again at the cost of more graph theory) to obtain a special  $p$ -group  $P$  (that means  $P' = \text{Z}(P) = \Phi(P)$  is elementary abelian) with the desired property (see also Hughes [14]). Only after Schupp’s paper, Pettet [24] noticed in 1989 that this result implies Schupp’s theorem for finite groups (and more restricted families of groups).

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In the same paper, he also obtained the dual statement for factor groups (see the theorem below). Eventually, Pettet [25] gave a new proof of Schupp’s original theorem with the same graph theoretical approach. An alternative construction of  $P$  was established by Bryant–Kovács [4] in 1978 making use of Lie theory (see also Huppert–Blackburn [16, Theorem VIII.13.5] and Hartley–Robinson [12]).

The first objective of this paper is to provide a new elementary proof of the following theorem, which avoids small cancellation theory, graph theory and Lie theory.

**Theorem (PETTET).** *For every automorphism  $\alpha$  of a finite group  $G$  the following statements are equivalent:*

- (1)  $\alpha$  is an inner automorphism.
- (2)  $\alpha$  extends to every finite group containing  $G$ .
- (3)  $\alpha$  lifts to every finite group  $\hat{G}$  such that  $\hat{G}/N \cong G$  for some characteristic subgroup  $N \leq \hat{G}$ .

The strategy is to replace the  $p$ -group  $P$  in [13] by a semidirect product  $N = Q \rtimes P$ , where  $Q$  is an elementary abelian  $q$ -group and  $P$  is a  $p$ -group of nilpotency class 2. In contrast to the papers cited above, we do not make an effort to minimize  $N$ .

Our second objective concerns the inverse problem on automorphism groups. By a theorem of Ledermann–Neumann [21], there exist only finitely many finite groups with a given automorphism group (for an elementary proof see [26]). However, not every group actually occurs as an automorphism group. For instance, it is a popular (and easy) exercise that a non-trivial cyclic group of odd order cannot be an automorphism group. The situation changes if we instead consider the *outer* automorphism group  $\text{Out}(H) := \text{Aut}(H)/\text{Inn}(H)$  of groups  $H$ . Indeed, Matumoto [22] proved that for every group  $G$  there exists a group  $H$  such that  $\text{Out}(H) \cong G$  (for finite  $G$  this was established earlier by Kojima [19]). Similar to Schupp’s paper, the construction of  $H$  is based on an HNN-extension and yields infinite groups. In later work, it was shown that  $H$  can be chosen to be locally finite, finitely generated (if  $G$  is countable), metabelian or simple (see [3, 5, 11, 7]). Problem 16.59 in the Kourovka Notebook [18] has asked if  $H$  can be chosen finite when  $G$  is finite. This was answered in 2020 by Cornulier on MathOverflow [6].

**Theorem (CORNULIER).** *Every finite group is the outer automorphism group of some finite group.*

His proof uses Lie algebras and is not easy to follow as it is written backwards. In Section 3 we present a purely group-theoretical proof based on Cornulier’s ideas. As in Pettet’s theorem, the idea is to construct a semidirect product  $P \rtimes Q$ , but this time  $Q$  is abelian and  $P$  is a  $p$ -group of exponent  $p$  with a certain nilpotency class. Our group is slightly smaller compared to Cornulier’s construction.

The construction of our group has already been used by Entin–Tsang [9] to prove that every finite group is isomorphic to the normalizer quotient of a subgroup of a symmetric group.

## 2 Characterizing inner automorphisms

Our first lemma is a prototype of Theorem 4.

**Lemma 1.** *Let  $G$  be a group acting faithfully on a cyclic group  $N$ . Then every automorphism of  $\hat{G} := N \rtimes G$  normalizing  $N$ , centralizes  $\hat{G}/N$ .*

*Proof.* We identify  $N = \langle x \rangle$  and  $G$  with the natural subgroups of  $\hat{G}$ . Let  $\alpha \in \text{Aut}(\hat{G})$  normalizing  $N$ . For a fixed  $g \in G$  there exist  $s, t \in \mathbb{Z}$  with  $gxg^{-1} = x^s$  and  $\alpha(x) = x^t$ . Hence,

$$gx^t g^{-1} = x^{st} = \alpha(x)^s = \alpha(x^s) = \alpha(gxg^{-1}) = \alpha(g)x^t \alpha(g)^{-1}.$$

Since  $G$  acts faithfully on  $N = \langle x^t \rangle$ , we obtain  $\alpha(g) \equiv g \pmod{N}$  as desired.  $\square$

In the following we develop some elementary facts of finitely presented groups. For elements  $x_1, x_2, \dots$  of a group  $G$  we define commutators by  $[x_1, x_2] = x_1 x_2 x_1^{-1} x_2^{-1}$  and  $[x_1, \dots, x_k] := [x_1, [x_2, \dots, x_k]]$  for  $k \geq 3$ . The commutator subgroup of  $G$  is denoted by  $G'$ .

**Lemma 2.** *Let  $p$  be a prime and  $a, b \in \mathbb{Z}$ . Then*

$$P := \langle x, y \mid [x, x, y] = [y, x, y] = 1, x^p = [x, y]^a, y^p = [x, y]^b \rangle$$

*is a non-abelian group of order  $p^3$ .*

*Proof.* The commutator relations show that  $P$  has nilpotency class  $\leq 2$ , i.e.  $P' \leq Z(P)$ . Hence,  $[x, y]^p = [x^p, y] = 1$  and  $|P'| \leq p$  (see [15, Hilfssatz III.1.3]). It follows that  $|P| \leq p^3$ . To complete the proof, we construct a group of order  $p^3$  realizing the given relations. Suppose first that  $p = 2$ . If  $ab$  is even, then  $P \cong D_8$  and otherwise  $P \cong Q_8$ . Thus, let  $p > 2$ . If  $a \equiv b \equiv 0 \pmod{p}$ , then the extraspecial group of exponent  $p$  fulfills the relations as is well-known. Now let  $p \nmid a$  and  $a' \in \mathbb{Z}$  such that  $aa' \equiv -b \pmod{p}$ . For  $y' := x^{a'} y$  we compute

$$(y')^p = x^{pa'} y^p [y, x^{a'}]^{\binom{p}{2}} = x^{pa'} y^p = [x, y]^{aa'+b} = 1$$

and  $[x, y'] = [x, y]$  by [15, Hilfssatz III.1.3]. Hence, replacing  $y$  by  $y'$  leads to  $b = 0$ . This remains true when we replace  $y$  by  $y^{-a}$ . Then  $x^p = [x, y]^{-1} = [y, x]$  and

$$P \cong \langle x, y \mid x^{p^2} = y^p = 1, yxy^{-1} = x^{1+p} \rangle \cong C_{p^2} \rtimes C_p. \quad \square$$

**Lemma 3.** *Let  $F$  be the free group in the free generators  $x_1, \dots, x_n$ . Let  $c_1, \dots, c_n \in F'$ . For a prime  $p$ , let  $P$  be the group generated by  $x_1, \dots, x_n$  subject to the relations  $x_i^p = c_i$  and  $[x_i, x_j, x_k] = 1$  for all  $1 \leq i, j, k \leq n$ . Then  $P/P'$  is an elementary abelian  $p$ -group with basis  $\{x_i P' : i = 1, \dots, n\}$  and  $P'$  is an elementary abelian  $p$ -group with basis  $\{[x_i, x_j] : 1 \leq i < j \leq n\}$ . In particular,  $|P| = p^{\binom{n+1}{2}}$ .*

*Proof.* It follows from  $x_i^p = w_i \in P'$  that  $P/P'$  is an elementary abelian  $p$ -group generated by  $\{x_i P' : i = 1, \dots, n\}$ . Since  $[x_i, x_j, x_k] = 1$  for all  $i, j, k$ ,  $P$  has nilpotency class  $\leq 2$ . Hence,  $[xy, z] = [x, z][y, z]$  and  $[x, yz] = [x, y][x, z]$  for all  $x, y, z \in P$  (see [15, Hilfssatz III.1.2]). In particular,  $[x_i, x_j]^p = [x_i^p, x_j] = [c_i, x_j] = 1$ . This shows that  $P'$  is an elementary abelian  $p$ -group generated by  $\{[x_i, x_j] : 1 \leq i < j \leq n\}$ .

Suppose that  $x := \prod_{i < j} [x_i, x_j]^{a_{ij}} = 1$  for some integers  $0 \leq a_{ij} \leq p-1$ . We fix  $i < j$  and consider the free group  $F_2$  generated by  $y_i$  and  $y_j$ . Let  $\varphi : F \rightarrow F_2$  be the homomorphism defined by

$$\varphi(x_k) = \begin{cases} y_k & \text{if } k \in \{i, j\}, \\ 1 & \text{otherwise} \end{cases}$$

for  $k = 1, \dots, n$ . Set

$$P_2 := \langle y_i, y_j \mid [y_i, y_i, y_j] = [y_j, y_i, y_j] = 1, y_i^p = \varphi(c_i), y_j^p = \varphi(c_j) \rangle.$$

As elements of  $P_2$ ,  $\varphi(c_i)$  and  $\varphi(c_j)$  are (possibly trivial) powers of  $[y_i, y_j]$ . Thus by Lemma 2,  $P_2$  is a non-abelian group of order  $p^3$ . Since every relation of  $P$  in the  $x_k$  is satisfied by a relation of  $P_2$  in the  $y_k$ ,  $\varphi$  factors through a homomorphism  $\bar{\varphi} : P \rightarrow P_2$ . It follows that  $[y_i, y_j]^{a_{ij}} = \bar{\varphi}(x) = 1$  and  $a_{ij} = 0$ . This shows that the commutators  $[x_i, x_j]$  with  $i < j$  are linearly independent, so they form a basis of  $P'$ . Now let  $x := \prod_{i=1}^n x_i^{a_i} \in P'$  for some  $0 \leq a_i \leq p-1$ . Then

$$1 = [x_1, x] = \prod_{i=2}^n [x_1, x_i]^{a_i}$$

and  $a_i = 0$  for  $i = 2, \dots, n$ . Similarly, we obtain  $a_1 = 0$ . Therefore,  $P/P'$  has rank  $n$  as claimed. Finally,  $|P| = |P'| |P/P'| = p^{\binom{n+1}{2}} = p^{\binom{n+1}{2}}$ .  $\square$

**Theorem 4.** *For every finite group  $G$  there exist primes  $q > p > |G|$  and a finite  $\{p, q\}$ -group  $N$  such that every automorphism of  $\hat{G} := N \rtimes G$  induces an inner automorphism of  $\hat{G}/N \cong G$ .*

*Proof.* Without loss of generality, we assume that  $G \neq 1$ . Let  $p > |G|$  be a prime. Let  $x_1, \dots, x_n \in G$  be a generating set of  $G$  not containing 1. Let  $P$  be the  $p$ -group with generators  $\{v_g : g \in G\}$  and relations

$$[v_g, v_h, v_k] = 1, \quad v_g^p = \prod_{i=1}^n [v_g, v_{gx_i}]^i$$

for all  $g, h, k \in G$ . By Lemma 3,  $P/P'$  is elementary abelian of rank  $|G|$  and  $P'$  is elementary abelian with basis  $[v_g, v_h]$  where  $g < h$  for some fixed total order on  $G$ . For a fixed  $g \in G$ , the elements  $\{v_{gh} : h \in G\}$  fulfill the same relations as the  $v_h$ . Thus, there exists an automorphism  $\varphi_g \in \text{Aut}(P)$  with  $\varphi_g(v_h) = v_{gh}$  for all  $h \in G$ . This gives rise to a regular action  $\varphi : G \rightarrow \text{Aut}(P)$ ,  $g \mapsto \varphi_g$ .

By an elementary special case of Dirichlet's theorem (see [10, Theorem 3.1.12]), there exists a prime  $q$  such that  $p \mid q-1$ . Let  $Q$  be the elementary abelian  $q$ -group with basis  $\{w_g : g \in G\}$ . Let  $w_g \mapsto w_g^\zeta$  be an automorphism of order  $p$  of  $\langle w_g \rangle$ . For  $g \in G$ , define  $\gamma_g \in \text{Aut}(Q)$  by

$$\gamma_g(w_h) := \begin{cases} w_g^\zeta & \text{if } h = g, \\ w_h & \text{if } h \neq g. \end{cases} \quad (h \in G)$$

Let  $\gamma : P \rightarrow \text{Aut}(Q)$ ,  $v_g \mapsto \gamma_g$  be the homomorphism with kernel  $P'$ . This gives rise to the semidirect product  $N := Q \rtimes P$  with  $Z(N) = P'$ . As usual, we identify  $P$  and  $Q$  with the natural subgroups of  $N$ . Then  $v_g w_h v_g^{-1} = \gamma_g(w_h)$  for all  $g, h \in G$ . Again, we have a regular action  $\psi : G \rightarrow \text{Aut}(Q)$ ,  $g \mapsto \psi_g$  with  $\psi_g(w_h) = w_{gh}$ . Moreover,  $\varphi$  and  $\psi$  are compatible in the sense that

$$\varphi_g(v_h) \psi_g(w_k) \varphi_g(v_h)^{-1} = v_{gh} w_{gk} v_{gh}^{-1} = \gamma_{gh}(w_{gk}) = \psi_g(\gamma_h(w_k)) = \psi_g(v_h w_k v_h^{-1})$$

for all  $g, h, k \in G$ . In this way,  $G$  acts faithfully on  $N$ . As before, we identify  $G$  and  $N$  with subgroups of  $\hat{G} := N \rtimes G$ . Then  $g v_h g^{-1} = v_{gh}$  and  $g w_h g^{-1} = w_{gh}$  for  $g, h \in G$ .

Now let  $\alpha \in \text{Aut}(\hat{G})$ . Since  $q > p > |G|$ ,  $\alpha$  normalizes the normal Sylow  $q$ -subgroup  $Q$ , the normal Hall subgroup  $N$ , and in turn  $Z(N) = P'$ . By the Schur–Zassenhaus theorem, there exists some  $y \in N$  such that  $\alpha(G) = y G y^{-1}$  (we do not require the Feit–Thompson theorem, because  $N$  is solvable). Since  $y$  centralizes  $\hat{G}/N$ , we may compose  $\alpha$  with the inner automorphism induced by  $y^{-1}$ . Then  $\alpha$  normalizes  $G$ . Next, we consider the action of  $\alpha$  on

$$N/P' \cong \bigtimes_{g \in G} \langle w_g, v_g \rangle \cong (C_q \rtimes C_p)^{|G|}.$$

It follows from

$$|C_{N/P'}(\alpha(v_1)P')| = |C_{N/P'}(v_1P')| = p^{|G|}q^{|G|-1}$$

that  $\alpha(v_1) \in \langle v_g, w_g \rangle P'$  for some  $g \in G$ . Composing  $\alpha$  with the inner automorphism induced by  $g^{-1}$ , we may assume that  $g = 1$ . Then  $\alpha$  induces an automorphism of  $\langle v_1, w_1 \rangle P' / P' \cong C_q \times C_p$ . By Lemma 1, there exists  $t_1 \in P'$  such that  $\alpha(v_1) \equiv v_1 t_1 \pmod{Q}$ . Moreover, for every  $g \in G$  there exists  $t_g \in P'$  with

$$\alpha(v_g) = \alpha(gv_1g^{-1}) \equiv \alpha(g)v_1t_1\alpha(g)^{-1} \equiv v_{\alpha(g)}t_g \pmod{Q}.$$

Consequently,

$$\prod_{i=1}^n [v_1, v_{x_i}]^i = v_1^p = (v_1 t_1)^p \equiv \alpha(v_1)^p \equiv \prod_{i=1}^n [v_1 t_1, v_{\alpha(x_i)} t_{x_i}]^i \equiv \prod_{i=1}^n [v_1, v_{\alpha(x_i)}]^i \pmod{Q}.$$

This is a relation in the linearly independent generators  $[v_1, v_g]$  of the elementary abelian group  $P'$ . Notice that  $\alpha(x_i) \neq 1$  and  $n < |G| < p$ . Comparing exponents reveals  $\alpha(x_i) = x_i$  for  $i = 1, \dots, n$ . Since  $G = \langle x_1, \dots, x_n \rangle$ ,  $\alpha$  induces the identity automorphism on  $G$ .  $\square$

How does the group  $P$  in the proof above relate to Heineken–Liebeck’s construction? Recall that Frucht’s theorem states that every finite group  $G$  is the automorphism group of some finite graph  $\mathcal{G}$ . Frucht’s graph is based on the Cayley color graph  $\mathcal{C}$ , which depends on a generating set  $x_1, \dots, x_n$  of  $G$ . More precisely, the vertex set of  $\mathcal{C}$  is  $\{v_g : g \in G\}$  and there is an arrow  $v_g \rightarrow v_h$  of color  $i$  if and only if  $h = gx_i$ . Our proof of Theorem 4 rests on the elementary fact that the color-preserving automorphism group of  $\mathcal{C}$  is isomorphic to  $G$ . The purpose of the group  $Q$  is to enforce automorphisms to permute the generators of  $P$ .

*Proof of Pettet’s theorem.* The implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are obvious. Conversely, if  $\alpha$  fulfills (2) or (3), then  $\alpha$  extends/lifts to the group  $\hat{G}$  constructed in Theorem 4. The theorem implies that  $\alpha$  is inner.  $\square$

If  $G$  is solvable, the proof above shows that (2) and (3) of Pettet’s theorem can be restricted to solvable extension groups  $\hat{G}$ .

### 3 Realizing outer automorphisms

We start by proving [6, Lemma 2]. Let  $n \in \mathbb{N}$  and  $p > n$  be a prime. Let  $C_p \rtimes C_{p-1}$  be the holomorph of  $C_p$ , i. e.  $C_{p-1}$  acts faithfully on  $C_p$ . Let  $S_n$  be the symmetric group of degree  $n$ .

**Lemma 5.** *We have  $\text{Out}((C_p \rtimes C_{p-1})^n) \cong S_n$ .*

*Proof.* The argument is similar as for the group  $(C_q \rtimes C_p)^n$  in the proof of Theorem 4, although the statement requires the factor  $p - 1$ . Let  $P = \langle x_1, \dots, x_n \rangle \cong C_p^n$  and  $Q := \langle y_1, \dots, y_n \rangle \cong C_{p-1}^n$  such that each  $y_i$  induces an automorphism of  $\langle x_i \rangle$  of order  $p - 1$  and centralizes  $x_j$  for  $j \neq i$ . It is obvious that every permutation  $\pi \in S_n$  induces an automorphism  $\alpha_\pi$  of  $G := P \rtimes Q$  by permuting the factors  $\langle x_i, y_i \rangle$ . If  $\pi \neq \text{id}$ , then  $\alpha_\pi$  is not inner, because it acts non-trivially on the abelian quotient  $G/P \cong Q$ . Hence,  $S_n$  induces a subgroup of  $\text{Out}(G)$ .

Conversely, let  $\alpha \in \text{Aut}(G)$  be arbitrary. Then  $\alpha$  normalizes the normal Sylow  $p$ -subgroup  $P$  of  $G$ . By the Schur–Zassenhaus theorem, we can further assume that  $\alpha(Q) = Q$ . Since

$$|C_G(\alpha(x_i))| = |C_G(x_i)| = p^n(p-1)^{n-1},$$

there exists some  $\pi \in S_n$  with  $\alpha(x_i) \in \langle x_{\pi(i)} \rangle$  for  $i = 1, \dots, n$ . By composing  $\alpha$  with  $\alpha_\pi^{-1}$ , we may assume that  $\pi = \text{id}$ . A similar argument yields  $\alpha(y_i) \in \langle y_i \rangle$ . By Lemma 1 applied to  $\langle x_i \rangle \rtimes \langle y_i \rangle$ , we have  $\alpha(y_i) = y_i$ . Since  $\text{Aut}(\langle x_i \rangle) \cong \langle y_i \rangle$ , the action of  $\alpha$  on  $\langle x_i \rangle$  is induced by conjugation with some power of  $y_i$ . Composing  $\alpha$  with the corresponding inner automorphism, gives  $\alpha(x_i) = x_i$  (this will not affect the action of  $\alpha$  on  $\langle x_j, y_j \rangle$  for  $j \neq i$ ). Doing this for  $i = 1, \dots, n$ , leads to  $\alpha = \text{id}$ .  $\square$

Let  $F$  be the free group in the free generators  $x_1, \dots, x_n$ . Let  $F^p = \langle x^p : x \in F \rangle^F \trianglelefteq F$  be the normal closure of the set of all  $p$ -powers in  $F$ . Then  $\overline{F} := F/F^p$  is the free group of rank  $n$  and exponent  $p$ . We will identify  $x_i$  with its image in  $\overline{F}$ . Define the lower central series by  $\overline{F}^{[1]} := \overline{F}$  and

$$\overline{F}^{[k+1]} := [\overline{F}, \overline{F}^{[k]}] = \langle [x, y] : x \in \overline{F}, y \in \overline{F}^{[k]} \rangle$$

for  $k \geq 1$  as usual. We call  $\overline{F}_c := \overline{F}/\overline{F}^{[c+1]}$  the free group of rank  $n$ , exponent  $p$  and nilpotency class  $c$  (we will see in Lemma 7 that the class of  $\overline{F}_c$  cannot be smaller than  $c$ ). The adjective “free” is justified by the following universal property: If  $G$  is a nilpotent group of exponent  $p$  and class  $\leq c$ , and if  $y_1, \dots, y_n \in G$ , then there exists a (unique) homomorphism  $\overline{F}_c \rightarrow G$ ,  $x_i \mapsto y_i$  for  $i = 1, \dots, n$ . We will use this principle frequently in order to verify certain commutator relations in  $\overline{F}_c$ .

It is easy to show that  $\overline{F}^{[k]}/\overline{F}^{[k+1]}$  is generated by the cosets of the  $k$ -fold commutators  $[x_{i_1}, \dots, x_{i_k}]$  where  $1 \leq i_1, \dots, i_k \leq n$  (see [15, Hilfssatz III.1.11]). It follows that  $\overline{F}^{[k]}/\overline{F}^{[k+1]}$  is a finite elementary abelian  $p$ -group. In particular,  $\overline{F}_c$  is a finite group. The following lemma is certainly known, but I was unable to find a proper reference.

**Lemma 6.** *For every finite group  $G$  of exponent  $p$  and  $k \in \mathbb{N}$ , the map*

$$\begin{aligned} \Phi_k : (G/G')^k &\rightarrow G/G^{[k+1]}, \\ (g_1 G', \dots, g_k G') &\mapsto [g_1, \dots, g_k] G^{[k+1]} \end{aligned}$$

*is well-defined and multilinear over  $\mathbb{F}_p$ , i.e. for  $1 \leq i \leq k$ ,  $h \in G$  and  $h' \in G'$ ,*

$$[g_1, \dots, g_{i-1}, g_i h h', g_{i+1}, \dots, g_k] \equiv [g_1, \dots, g_k][g_1, \dots, h, \dots, g_k] \pmod{G^{[k+1]}}.$$

*Proof.* If we interpret  $[x_1]$  as  $x_1$ , then  $\Phi_1$  becomes the identity map. Now let  $k \geq 2$ . Recall that  $[G^{[s]}, G^{[t]}] \leq G^{[s+t]}$  by [15, Hauptsatz III.2.11]. A direct computation shows  $[xy, z] = [x, y, z][y, z][x, z]$  and  $[z, xy] = [xy, z]^{-1} = [z, x][z, y][x, y, z]^{-1}$  for every  $x, y, z \in G$ . If  $i = 1$ , we put  $z := [g_2, \dots, g_k] \in G^{[k-1]}$  and obtain

$$\begin{aligned} [g_1 h h', g_2, \dots, g_k] &= [g_1 h, h', z][h', z][g_1 h, z] \equiv [g_1 h, z] \equiv [g_1, h, z][h, z][g_1, z] \\ &\equiv [g_1, z][h, z] \equiv [g_1, \dots, g_k][h, g_2, \dots, g_k] \pmod{G^{[k+1]}}. \end{aligned}$$

It remains to consider the component  $i \geq 2$ . Let  $z := [g_{i+1}, \dots, g_k]$ . By induction on  $k$ , we derive

$$[g_2, \dots, g_i h h', \dots, g_k] \equiv [g_2, \dots, g_i, z][g_2, \dots, h, z] \pmod{G^{[k]}}$$

and

$$[g_1, \dots, g_i h h', \dots, g_k] \equiv [g_1, \dots, g_i, z][g_1, \dots, h, z] \pmod{G^{[k+1]}}. \quad \square$$

The following result resembles [6, Lemma 5].<sup>1</sup>

**Lemma 7.** *Let  $n \geq 3$ ,  $\pi \in S_{n-1}$  and  $0 \leq a \leq p-1$  such that*

$$[x_1, \dots, x_{n-1}, x_1] \equiv [x_{\pi(1)}, \dots, x_{\pi(n-1)}, x_{\pi(1)}]^a \pmod{\overline{F}^{[n+1]}}. \quad (3.1)$$

*Then  $\pi = \text{id}$  and  $a = 1$ .*

*Proof.* By the universal property, it suffices to prove the claim for any elements  $x_1, \dots, x_{n-1}$  of a group  $P$  with exponent  $p$  and nilpotency class  $\leq n$ . Let  $P \leq \text{GL}(n+1, p)$  be the group of upper unitriangular matrices. For  $x \in P$  we have

$$x^p - 1 = (x - 1)^p = (x - 1)^{n+1}(x - 1)^{p-n-1} = 0$$

since  $p > n$ . Hence,  $P$  has exponent  $p$ . For  $i < j$ , let  $E_{ij} \in P$  be the unitriangular matrix with 1 on position  $(i, j)$  and zero elsewhere off the diagonal. A direct calculation shows that

$$[E_{ij}, E_{kl}] = E_{il}^{\delta_{jk}} E_{kj}^{-\delta_{il}}, \quad (3.2)$$

where  $\delta_{jk}\delta_{il} = 0$  since  $i < j$  and  $k < l$ . An induction shows that  $P^{[k]}$  is generated by the matrices  $E_{ij}$  with  $|j - i| \geq k$ . In particular,  $P^{[n]} = \langle E_{1,n+1} \rangle \cong C_p$  and  $P^{[n+1]} = 1$  (see [15, Satz III.16.3]). So  $P$  has indeed nilpotency class  $n$ . We define  $x_1 := E_{12}E_{n,n+1}$  and  $x_i := E_{i,i+1}$  for  $i = 2, \dots, n-1$ . Then the right hand side of (3.1) is

$$[x_1, \dots, x_{n-1}, x_1] = \begin{cases} [x_1, \dots, x_{n-2}, E_{n-1,n+1}] = \dots = [x_1, E_{2,n+1}] = E_{1,n+1} & \text{if } n \geq 4, \\ [x_1, E_{24}E_{13}^{-1}] = [E_{12}E_{34}, E_{24}][E_{12}E_{34}, E_{13}]^{-1} = E_{14} & \text{if } n = 3. \end{cases}$$

Suppose first that  $\pi(1) \neq 1$ . Then  $x_1$  appears only once in  $c := [x_{\pi(1)}, \dots, x_{\pi(n-1)}, x_{\pi(1)}]$ . By Lemma 6,  $c \equiv c_1 c_2 \pmod{\overline{F}^{[n+1]}}$ , where  $c_1$  and  $c_2$  are  $n$ -fold commutators in the elements  $E_{i,i+1}$ . Both  $c_i$  contain  $x_{\pi(1)}$  twice, so they must miss some  $E_{r,r+1}$ . But now each  $c_i$  lives inside a direct product of the form

$$Q := \left\{ \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} : Q_1 \leq \text{GL}(r, p), Q_2 \leq \text{GL}(n+1-r, p) \right\}.$$

Since  $Q$  has nilpotency class  $< n$ , we derive the contradiction  $c \equiv c_1 c_2 \equiv 1 \pmod{\overline{F}^{[n+1]}}$ .

Therefore,  $\pi(1) = 1$ . Now  $[x_{\pi(n-1)}, x_1] \neq 1$  implies  $\pi(n-1) \in \{2, n-1\}$  by (3.2). Assume first that  $\pi(n-1) = n-1$ . Then  $[x_{\pi(n-2)}, x_{n-1}, x_1] = [x_{\pi(n-2)}, E_{n-1,n+1}] \neq 1$  implies  $\pi(n-2) = n-2$ . Inductively, one obtains  $\pi = \text{id}$ ,  $c = E_{1,n+1}$  and  $a = 1$  in this case. Now suppose that  $\pi(n-1) = 2$  and without loss of generality,  $n \geq 4$ . Here we use a different realization of  $\overline{F}_n$  inside  $P$ . More precisely, we reassign  $x_i := E_{12}E_{23} \dots E_{n-1,n}$  for  $i = 1, \dots, n-2$  and  $x_{n-1} := E_{n,n+1}$ . Then clearly,  $c = [\dots, [x_1, x_1]] = 1$ . On the other hand, the right hand side of (3.1) becomes

$$[x_1, \dots, x_{n-1}, x_1] = [x_1, \dots, x_1, E_{n-1,n+1}]^{-1} = \dots = [x_1, E_{2,n+1}]^{-1} = E_{1,n+1}^{-1}.$$

Contradiction. □

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<sup>1</sup>Since the proof of [6, Lemma 5] takes place in the non-nilpotent Lie algebra  $\mathfrak{gl}_n$ , it is not clear to me that the obtained result can actually be used to prove the main theorem.

We have duplicated  $x_1$  in the commutator in Lemma 7 to avoid relations of the form

$$[* , \dots , * , x , y] \equiv [* , \dots , * , y , x]^{-1} \pmod{\overline{F}^{[n+1]}}.$$

To prove Cornulier's theorem, let  $G = \{g_1, \dots, g_n\}$  be a finite group of order  $n$ . We construct a finite group  $H$  with  $\text{Out}(H) \cong G$ . Since  $\text{Out}(1) = 1$  and  $\text{Out}(C_3) \cong C_2$ , we may assume that  $n \geq 3$  (as in Lemma 7). We identify the generators  $x_i$  of  $\overline{F}$  with  $x_{g_i}$  and define

$$N := \langle [x_{hg_1}, \dots, x_{hg_{n-1}}, x_{hg_1}] : h \in G \rangle \overline{F}^{[n+1]} \leq \overline{F}^{[n]}.$$

Since  $\overline{F}^{[n]} / \overline{F}^{[n+1]} \leq Z(\overline{F}_n)$ , it follows that  $N \leq \overline{F}$ . Let  $P := \overline{F} / N \cong \overline{F}_n / (N / \overline{F}^{[n+1]})$ . Notice that  $P$  has exponent  $p$  and nilpotency class  $\leq n$ . Moreover,  $P / P' \cong \overline{F} / \overline{F}' \cong C_p^n$ . Again we will identify the  $x_i$  with their images in  $P$ .

Let  $\mathbb{F}_p^\times = \langle \zeta \rangle$ . For  $1 \leq i \leq n$ , the map  $x_j \mapsto x_j^{\zeta^{\delta_{ij}}}$  can be extended to an automorphism  $q_i$  of  $\overline{F}$ . By Lemma 6,

$$q_i([x_{j_1}, \dots, x_{j_n}]) = [q_i(x_{j_1}), \dots, q_i(x_{j_n})] \equiv [x_{j_1}, \dots, x_{j_n}]^\gamma \pmod{\overline{F}^{[n+1]}}$$

for some  $\gamma \in \mathbb{Z}$ . In particular,  $q_i(N) = N$  and  $q_i$  extends to an automorphism of  $P$ . Moreover, the group  $Q := \langle q_1, \dots, q_n \rangle \leq \text{Aut}(P)$  is isomorphic to  $C_{p-1}^n$ . Finally, we define  $H := P \rtimes Q$ . As usual, we regard  $P$  and  $Q$  as subgroups of  $H$ . Then  $q_i x_j q_i^{-1} = x_j^{\zeta^{\delta_{ij}}}$  for  $1 \leq i, j \leq n$ . Note that  $H / P' \cong (C_p \rtimes C_{p-1})^n$ .

The following result implies Cornulier's theorem.

**Theorem 8.** *With the notation above,  $\text{Out}(H) \cong G$ .*

*Proof.* For  $h \in G$ , the map  $x_i \mapsto x_{hg_i}$  ( $i = 1, \dots, n$ ) can be extended to an automorphism  $\alpha_h$  of  $\overline{F}$ . By the definition of  $N$ , we have  $\alpha_h(N) = N$ . Therefore, we consider  $\alpha_h$  as an automorphism of  $P$ . There is a similar automorphism  $\beta_h \in \text{Aut}(Q)$  with  $\beta_h(q_i) = q_{hg_i}$ , where  $q_i$  is identified with  $q_{g_i}$ . Since

$$\alpha_h(q_i x_j q_i^{-1}) = \alpha_h(x_j)^{\zeta^{\delta_{ij}}} = x_{hg_j}^{\zeta^{\delta_{ij}}} = q_{hg_j} x_{hg_i} q_{hg_j}^{-1} = \beta_h(q_j) \alpha_h(x_i) \beta_h(q_j)^{-1},$$

the actions are compatible. This gives rise to a regular action  $\alpha : G \rightarrow \text{Aut}(H)$ . Since  $g \neq 1$  acts non-trivially on  $H / P \cong Q$ ,  $\alpha(G) \cap \text{Inn}(H) = 1$ . Thus, it suffices to show that  $\text{Aut}(H) = \alpha(G) \text{Inn}(H)$ .

To this end, let  $\gamma \in \text{Aut}(H)$  be arbitrary. Then  $\gamma$  normalizes the normal Sylow  $p$ -subgroup  $P$  and  $P'$ . By Lemma 5, we may assume that  $\gamma$  permutes the factors of  $H / P'$ . So there exists a permutation  $\pi \in S_n$  such that  $\gamma(q_i) = q_{\pi(i)} \pmod{P'}$  and  $\gamma(x_i) \equiv x_{\pi(i)} \pmod{P'}$  for  $i = 1, \dots, n$ . This implies

$$[x_{\pi(1)}, \dots, x_{\pi(n-1)}, x_{\pi(1)}] = \gamma([x_1, \dots, x_{n-1}, x_1]) = \gamma(1) = 1$$

by Lemma 6. This yields an equation inside  $N / \overline{F}^{[n+1]}$ :

$$[x_{\pi(1)}, \dots, x_{\pi(n-1)}, x_{\pi(1)}] \equiv \prod_{h \in G} [x_{hg_1}, \dots, x_{hg_{n-1}}, x_{hg_1}]^{a_h} \pmod{\overline{F}^{[n+1]}}$$

for some  $0 \leq a_h \leq p-1$ . By the universal property, this equation remains true when we set  $x_{\pi(n)} = 1$ . For the unique  $h \in G$  with  $x_{hg_n} = x_{\pi(n)}$  we deduce

$$[x_{\pi(1)}, \dots, x_{\pi(n-1)}, x_{\pi(1)}] \equiv [x_{hg_1}, \dots, x_{hg_{n-1}}, x_{hg_1}]^{a_h} \pmod{\overline{F}^{[n+1]}}.$$



By Lemma 7,  $x_{\pi(i)} = x_{hg_i}$  for  $i = 1, \dots, n-1$ . Therefore, after composing  $\gamma$  with  $\alpha(h)^{-1}$ , we may assume that  $\pi = 1$ . Since  $|P'|$  is coprime to  $|Q|$  and  $\gamma(Q) \leq P'Q$ , there exists a  $y \in P'$  with  $\gamma(Q) = yQy^{-1}$  by the Schur–Zassenhaus theorem. Since conjugation with  $y$  does not affect  $H/P'$ , we may assume that  $\gamma(Q) = Q$ . In particular,  $\gamma$  centralizes  $Q$ .

Each quotient  $P^{[k]}/P^{[k+1]}$  has a basis (as an elementary abelian group) consisting of some  $k$ -fold commutators in the  $x_i$ . By concatenation we obtain a basis  $c_1, \dots, c_s$  of  $\times_{k=1}^n P^{[k]}/P^{[k+1]}$ . For  $c_i \in P^{[k]} \setminus P^{[k+1]}$ , we have  $q_j c_i q_j^{-1} \equiv c_i^{\zeta^l} \pmod{P^{[k+1]}}$  by Lemma 6, where  $l$  is the multiplicity of  $x_j$  as a component of  $c_i$ . Clearly,  $l \leq k-1 \leq n-1 < p-1$ . Hence,  $\zeta^l \equiv 1 \pmod{p}$  can only hold if  $l = 0$ . This shows that  $q_j$  centralizes  $c_i$  if and only if  $x_j$  does not appear in  $c_i$ . Since the  $c_i$  form a basis, it follows that  $C_P(q_j) = \langle x_i : i \neq j \rangle$ . On the other hand,  $q_j \gamma(x_i) q_j^{-1} = \gamma(q_j x_i q_j^{-1}) = \gamma(x_i)$  for  $j \neq i$  shows that

$$\gamma(x_i) \in \bigcap_{j \neq i} C_P(q_j) = \langle x_i \rangle.$$

Since we already know that  $\gamma(x_i) \equiv x_i \pmod{P'}$ , we conclude  $\gamma(x_i) = x_i$  for  $i = 1, \dots, n$  and  $\gamma = \text{id}$ , as desired.  $\square$

In order to estimate  $|H|$  in terms of  $n$ , we consider the free nilpotent Lie algebra  $L$  over  $\mathbb{Q}$  of rank  $n$  and class  $n$ . By Witt's formula,

$$\dim L = \sum_{k=1}^n \frac{1}{k} \sum_{d|k} \mu(d) n^{k/d},$$

where  $\mu$  is the Möbius function. This number grows roughly as  $n^{n-1}$  (see [2, Lemma 20.7]). Notice that  $\mathbb{F}_p \otimes L$  is the corresponding free nilpotent Lie algebra over  $\mathbb{F}_p$ . Since  $p > n$ , the Lazard correspondence turns  $\mathbb{F}_p \otimes L$  into  $\overline{F}_n$  (see [17, Example 10.24]). In particular,  $|\overline{F}_n| = p^{\dim L}$ . Moreover, an application of Lemma 7 reveals that  $|N/\overline{F}^{[n+1]}| = p^n \sim (p-1)^n = |Q|$ . Altogether, the order of magnitude of  $|H|$  is  $p^{n^{n-1}}$  (the estimate  $n^n$  in [6] is unjustified).

As a concrete example, consider  $G \cong C_3$ . Here we can take  $p = 5$ . Then  $|\overline{F}_3| = 5^{14}$ ,  $|P| = 5^{11}$  and  $|H| = 2^6 5^{11}$ . Cornulier's construction yields a group of order  $2^6 5^{29}$ , as he remarked at the end of [6].

One may ask if the group  $H$  can also be used to prove Pettet's theorem. This does not seem to be easy, since it is not clear whether every automorphism of  $H \rtimes G$  normalizes  $H$ . Conversely, the group  $N$  in Theorem 4 has outer automorphisms, which do not come from  $G$ .

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