

# SPLENDID MORITA EQUIVALENCES FOR PRINCIPAL BLOCKS WITH SEMIDIHEDRAL DEFECT GROUPS

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*Dedicated to Gunter Malle on his 60th Birthday.*

ABSTRACT. We classify principal blocks of finite groups with semidihedral defect groups up to splendid Morita equivalence. This completes the classification of all principal 2-blocks of tame representation type up to splendid Morita equivalence and shows that Puig's Finiteness Conjecture holds for such blocks.

## 1. INTRODUCTION

The present article is motivated by Puig's Finiteness Conjecture (see [Thé95, (38.6) Conjecture]), strengthening Donovan's Conjecture and predicting that for a given prime number  $\ell$  and a finite  $\ell$ -group  $P$  there are only finitely many isomorphism classes of interior  $P$ -algebras arising as source algebras of  $\ell$ -blocks of finite groups with defect groups isomorphic to  $P$ , or equivalently that there are only a finite number of *splendid Morita equivalence* classes of blocks of finite groups with defect groups isomorphic to  $P$ . The cases where  $P$  is either cyclic [Lin96] or a Klein-four group [CEKL11] are the only cases for which this conjecture has been proved to hold in full generality. Else, under additional assumptions, Puig's Finiteness Conjecture has also been proved for several classes of finite groups, as for instance for  $\ell$ -soluble groups [Pui94], for the symmetric groups [Pui94], for the alternating groups and the double covers thereof, for Weyl groups, or for classical groups, see [HK00, HK05] and the references therein.

Our principal aim in this article is to classify principal 2-blocks of finite groups with semidihedral defect groups up to splendid Morita equivalence and deduce that Puig's Finiteness Conjecture holds when letting the blocks vary through the class of all principal 2-blocks of tame representation type. We recall that Erdmann [Erd90] classified blocks of tame representation type up to Morita equivalence by describing their basic algebras by generators and relations making intense use of the Auslander-Reiten quiver. However, given a splitting 2-modular system  $(K, \mathcal{O}, k)$ , her results are not liftable to  $\mathcal{O}$  in general and do not imply that the resulting Morita equivalences are necessarily splendid Morita equivalences. By contrast, if Puig's Finiteness Conjecture holds over  $k$ , then it automatically holds over  $\mathcal{O}$ , since the bimodules inducing splendid Morita equivalences are liftable from  $k$  to  $\mathcal{O}$ .

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To state our main results, we introduce the following notation. For  $m \geq 2$  and a prime power  $q = p^f$  let

$$\begin{aligned}\mathrm{SL}_m^{\pm 1}(q) &:= \{A \in \mathrm{GL}_m(q) \mid \det(A) = \pm 1\}, \\ \mathrm{SU}_m^{\pm 1}(q) &:= \{A \in \mathrm{GU}_m(q) \mid \det(A) = \pm 1\}.\end{aligned}$$

Now let  $q = p^{2f}$  where  $p$  is an odd prime. Then there are exactly three groups  $H$  with

$$\mathrm{PSL}_2(q) < H < \mathrm{P}\Gamma\mathrm{L}_2(q) \quad \text{and} \quad |H : \mathrm{PSL}_2(q)| = 2.$$

One is  $\mathrm{PGL}_2(q)$ , one is contained in  $\mathrm{PSL}_2(q) \rtimes \langle F \rangle$  where  $F$  is the Frobenius automorphism on  $\mathbb{F}_q$ , and the third one is denoted by  $\mathrm{PGL}_2^*(q)$  (see [Go69]). Our main result is as follows:

**Theorem 1.1.** *Let  $G$  be a finite group with a semidihedral Sylow 2-subgroup  $P$  of order  $2^n$  with  $n \geq 4$  fixed and let  $k$  be an algebraically closed field of characteristic 2. Then the following assertions hold.*

(a) *The principal block  $B_0(kG)$  of  $kG$  is splendidly Morita equivalent to the principal block of precisely one of the following groups:*

(bb)  $P$ ;

(ba1)  $\mathrm{SL}_2^{\pm}(p^f)$  where  $4(p^f + 1)_2 = 2^n$ ;

(ba2)  $\mathrm{SU}_2^{\pm}(p^f)$  where  $4(p^f - 1)_2 = 2^n$ ;

(ab)  $\mathrm{PGL}_2^*(p^{2f})$  where  $2(p^{2f} - 1)_2 = 2^n$ ;

(aa1)  $\mathrm{PSL}_3(p^f)$  where  $4(p^f + 1)_2 = 2^n$ ; or

(aa2)  $\mathrm{PSU}_3(p^f)$  where  $4(p^f - 1)_2 = 2^n$ ;

where  $p$  is an odd prime number and  $f \geq 1$ . Moreover, the splendid Morita equivalence is realized by the Scott module  $\mathrm{Sc}(G \times G', \Delta P)$ , where  $G'$  is the group listed in the corresponding case.

(b) *If  $G$  and  $G'$  are two groups such that  $|G|_2 = |G'|_2$  and which are both of type (ba1), both of type (ba2), both of type (ab), both of type (aa1), or both of type (aa2), then  $B_0(kG)$  and  $B_0(kG')$  are splendidly Morita equivalent.*

In part (a) the labeling of the fusion patterns originates from [Ols75, p.231] (see also [CG12, Theorem 5.3]) and we emphasize that  $G'$  is not the derived subgroup  $[G, G]$ .

Finally, as Craven-Eaton-Kessar-Linckelmann proved in [CEKL11] that Puig's Finiteness Conjecture holds for 2-blocks with Klein-four defect groups and the first and the second authors proved in [KL20a, KL20b] that it holds as well for principal 2-blocks with dihedral and generalized quaternion defect groups, Theorem 1.1 yields the following corollary:

**Corollary 1.2.** *Puig's Finiteness Conjecture holds for principal 2-blocks of tame representation type.*

## 2. NOTATION

Throughout this paper, unless otherwise stated we adopt the following notation and conventions. All groups considered are finite and all modules are finitely generated right modules. In particular  $G$  always denotes a finite group. We denote the dihedral group of order  $2^m$  with  $m \geq 2$  by  $D_{2^m}$ , the generalized quaternion group of order  $2^m$  with  $m \geq 3$  by  $Q_{2^m}$ , and the cyclic group of order  $m \geq 1$  by  $C_m$ . We denote by  $SD_{2^n}$  the semidihedral group of order  $2^n$  where  $n \geq 4$  is fixed throughout this paper. We denote by  $\mathcal{F}_P(G)$  the fusion system of  $G$  on a Sylow  $p$ -subgroup  $P$  of  $G$  and by  $\mathrm{Syl}_p(G)$  the set of all Sylow  $p$ -subgroups of  $G$ . We write  $\Delta G := \{(g, g) \mid g \in G\} \leq G \times G$ . Given two subgroups

$N \triangleleft G$  and  $L \leq G$  with  $G = NL$  and  $N \cap L = 1$ ,  $N \rtimes L$  denotes the semi-direct product of  $N$  by  $L$ .

We let  $k$  be an algebraically closed field of characteristic 2. We write  $B_0(kG)$  for the principal block of the group algebra  $kG$ . For a block  $B$  of  $kG$ , we write  $1_B$  for the block idempotent of  $B$ ,  $C_B$  for the Cartan matrix of  $B$ , and  $k(B)$  and  $\ell(B)$ , respectively, are the numbers of irreducible ordinary and Brauer characters of  $G$  belonging to  $B$ . We denote by  $\text{mod-}B$  the category of finitely generated right  $B$ -modules and by  $\underline{\text{mod-}}B$  the associated stable module category.

We denote by  $k_G$  the trivial  $kG$ -module. Given a  $kG$ -module  $M$  and a 2-subgroup  $Q \leq G$  we denote by  $M(Q)$  the Brauer construction of  $M$  with respect to  $Q$ . For  $H \leq G$  we denote by  $\text{Sc}(G, H)$  the Scott  $kG$ -module with respect to  $H$ . By definition  $\text{Sc}(G, H)$  is, up to isomorphism, the unique indecomposable direct summand of the induced module  $k_H \uparrow^G$  which contains  $k_G$  in its head (or equivalently in its socle) and is a 2-permutation module by definition. See [NT88, Chap.4 §8.4]. Equivalently,  $\text{Sc}(G, H)$  is the relative  $H$ -projective cover of  $k_G$  (see [Thé85, Proposition 3.1]).

If  $G$  and  $H$  are finite groups,  $A$  and  $B$  are blocks of  $kG$  and  $kH$  respectively and  $M$  is an  $(A, B)$ -bimodule inducing a Morita equivalence between  $A$  and  $B$ , then we view  $M$  as a right  $k[G \times H]$ -module via the right  $G \times H$ -action defined by  $m \cdot (g, h) := g^{-1}mh$  for every  $m \in M, g \in G, h \in H$ . Furthermore, the algebras  $A$  and  $B$  are called *splendidly Morita equivalent* (or *source-algebra equivalent*, or *Puig equivalent*), if there is a Morita equivalence between  $A$  and  $B$  induced by an  $(A, B)$ -bimodule  $M$  such that  $M$ , viewed as a right  $k[G \times H]$ -module, is a 2-permutation module. In this case, due to a result of Puig (see [Pui99, Corollary 7.4] and [Lin18, Proposition 9.7.1]), the defect groups  $P$  and  $Q$  of  $A$  and  $B$  respectively are isomorphic. Hence from now on we identify  $P$  and  $Q$ . Obviously  $M$  is indecomposable as a  $k(G \times H)$ -module and since  $M$  induces a Morita equivalence,  ${}_A M$  and  $M_B$  are both projective and therefore  $\Delta P \leq G \times H$  is a vertex of  $M$ . By a result of Puig and Scott, this definition is equivalent to the condition that  $A$  and  $B$  have source algebras which are isomorphic as interior  $P$ -algebras (see [Lin01, Theorem 4.1] and [Pui99, Remark 7.5]).

In this paper, in order to produce splendid Morita equivalences between principal blocks of two finite groups  $G$  and  $G'$  with a common defect group  $P$ , we will use 2-permutation modules given by Scott modules of the form  $\text{Sc}(G \times G', \Delta P)$ , which are obviously  $(B_0(kG), B_0(kG'))$ -bimodules. Furthermore, we shall rely on the classification of principal 2-blocks of finite groups with dihedral Sylow 2-subgroups, up to splendid Morita equivalence, obtained in [KL20a], where the result for Klein-four groups is in [CEKL11]. We will use the results of [CEKL11, KL20a, KL20b] without further introduction in this text and refer the reader directly to the relevant material in these articles.

### 3. FINITE GROUPS WITH SEMIDIHEDRAL SYLOW 2-SUBGROUPS

One of the starting points of this project is the following very useful observation due to the third author. In fact, the next theorem is already implicitly in [ABG70].

**Theorem 3.1** (Implicitly in [ABG70]). *Let  $G$  be a finite group with a semidihedral Sylow 2-subgroup  $P$  of order  $2^n$  with  $n \geq 4$ , and assume that  $O_{2'}(G) = 1$ . Then one of the following holds:*

- (bb)  $G = P$ .
- (ba1)  $G = \text{SL}_2^\pm(p^f) \rtimes C_d$  where  $4(p^f + 1)_2 = 2^n$  and  $d$  is an odd divisor of  $f$ .
- (ba2)  $G = \text{SU}_2^\pm(p^f) \rtimes C_d$  where  $4(p^f - 1)_2 = 2^n$  and  $d$  is an odd divisor of  $f$ .

- (ab)  $G = \mathrm{PGL}_2^*(p^{2f}) \rtimes C_d$  where  $2(p^{2f} - 1)_2 = 2^n$  and  $d$  is an odd divisor of  $f$ .
- (aa1)  $G = \mathrm{PSL}_3(p^f).H$  where  $4(p^f + 1)_2 = 2^n$  and  $H \leq C_{(3,p^f-1)} \times C_d$  for an odd divisor  $d$  of  $f$ .
- (aa2)  $G = \mathrm{PSU}_3(p^f).H$  where  $4(p^f - 1)_2 = 2^n$  and  $H \leq C_{(3,p^f+1)} \times C_d$  for an odd divisor  $d$  of  $f$ .
- (aa3)  $G = M_{11}$ .

*Proof.* If  $G$  is 2-nilpotent, then Case (bb) holds since  $O_{2'}(G) = 1$ . In all other cases,  $G$  is a  $D$ -group, a  $Q$ -group or a  $QD$ -group with the notation of [ABG70, Definition 2.1]. If  $G$  is a  $D$ -group, then  $G$  has a normal subgroup  $K$  of index 2 with a dihedral Sylow 2-subgroup. Hence, the structure of  $K$  (and  $G$ ) follows essentially from the classification of Gorenstein–Walter. The precise information can be extracted from Proposition 3.4 of [ABG70] and its proof. We see that Case (ab) holds. If  $G$  is a  $Q$ -group, then Case (ba1) or (ba2) occurs by Propositions 3.2 and 3.3 (and its proof) of [ABG70]. Finally, let  $G$  be a  $QD$ -group. Then by [ABG70, Proposition 2.2],  $N := O^{2'}(G)$  is simple and the possible isomorphism types of  $N$  are given in the third main theorem of [ABG70], namely  $M_{11}$ ,  $\mathrm{PSL}_3(p^f)$  and  $\mathrm{PSU}_3(p^f)$ . Since  $C_G(N) \cap N = Z(N) = 1$  we have  $C_G(N) \leq O_{2'}(G) = 1$ . The possibilities for  $G/N \leq \mathrm{Out}(N)$  can be deduced from the Atlas [Atlas]. In particular, Case (aa1) holds if  $N \cong M_{11}$ . Now let  $N$  be  $\mathrm{PSL}_3(p^f)$  or  $\mathrm{PSU}_3(p^f)$ . Since,  $G/N$  has odd order, it does not induce graph automorphisms on  $N$ . Hence,  $G/N \leq C_{(3,p^f-1)} \rtimes C_f$  or  $G/N \leq C_{(3,p^f+1)} \rtimes C_f$ . Again, since  $G/N$  has odd order,  $G/N$  is abelian.  $\square$

#### 4. THE PRINCIPAL 2-BLOCKS OF $M_{11}$ AND $\mathrm{PSL}_3(3)$

Benson and Carlson [BC87, (14.1)] observed that the principal 2-blocks of the groups  $\mathrm{PSL}_3(3)$  and  $M_{11}$  are Morita equivalent by comparing their basic algebras. In this section, we prove that their result can be refined to a splendid Morita equivalence. More precisely, we prove that this Morita equivalence is induced by a Scott module using the gluing method developed by the first and the second author in [KL20a, Section 3 and Section 4].

**Lemma 4.1.** *Set  $G := \mathrm{PSL}_3(3)$ ,  $G' := M_{11}$  and let  $P \in \mathrm{Syl}_2(G) \cap \mathrm{Syl}_2(G')$ , so that  $P \cong SD_{16}$ . Then  $\mathrm{Sc}(G \times G', \Delta P)$  induces a stable equivalence of Morita type between  $B_0(kG)$  and  $B_0(kG')$ .*

*Proof.* Set  $P := \langle s, t \mid s^8 = t^2 = 1, tst = s^3 \rangle \cong SD_{16}$ ,  $z := s^4$  and  $Z := \langle z \rangle = Z(P) \cong C_2$  and observe that  $\mathcal{F}_P(G) = \mathcal{F}_P(G')$  by [CG12, Theorem 5.3]. We read from the Atlas [Atlas, p.13 and p.18] that  $C_G(z) \cong \mathrm{GL}_2(3) \cong C_{G'}(z)$ . Obviously,  $k\mathrm{GL}_2(3)$  has only one 2-block, namely the principal block. Thus,  $\mathrm{Sc}(C_G(z) \times C_{G'}(z), \Delta P)$  realizes a (splendid) Morita equivalence between  $B_0(kC_G(z))$  and  $B_0(kC_{G'}(z))$  because  $\mathrm{Sc}(C_G(z) \times C_{G'}(z), \Delta P) = kC_G(z)$  seen as  $(kC_G(z), kC_{G'}(z))$ -bimodule. On the other hand,  $\mathrm{Sc}(C_G(z) \times C_{G'}(z), \Delta P) \mid M(\Delta Z)$  by [KL20a, Lemma 3.2]. Hence, as  $\mathrm{Sc}(C_G(z) \times C_{G'}(z), \Delta P)$  is Brauer indecomposable by [KT19, Theorem 1.2], we have in fact  $\mathrm{Sc}(C_G(z) \times C_{G'}(z), \Delta P) \cong M(\Delta Z)$ . Thus,  $M(\Delta Z)$  induces a Morita equivalence between  $B_0(kC_G(z))$  and  $B_0(kC_{G'}(z))$ . Therefore, as all involutions in  $G$  are  $G$ -conjugate and  $\mathcal{F}_P(G) = \mathcal{F}_P(G')$ , [KL20a, proof of Case 1 of Proposition 4.6] yields that for every involution  $t \in P$

$$\mathrm{Sc}(C_G(t) \times C_{G'}(t), \Delta P) = M(\Delta \langle t \rangle)$$

and induces a Morita equivalence between  $B_0(kC_G(t))$  and  $B_0(kC_{G'}(t))$ . Therefore the assertion follows from [KL20a, Lemma 4.1].  $\square$

**Proposition 4.2.** *With the notation of Lemma 4.1,  $\text{Sc}(G \times G', \Delta P)$  induces a splendid Morita equivalence between  $B_0(kG)$  and  $B_0(kG')$ .*

*Proof.* Set  $M := \text{Sc}(G \times G', \Delta P)$ ,  $B := B_0(kG)$  and  $B' := B_0(kG')$ . The block  $B$  has three simple  $kG$ -modules:  $k_G$  and two modules 12 and 26 of  $k$ -dimension 12 and 26 respectively. Similarly the block  $B' := B_0(kG')$  has three simple  $kG'$ -modules:  $k_{G'}$  and two modules 10 and 44 of  $k$ -dimension 12 and 26 respectively. (See [ModAtl]).

To start with, we claim that these six simple modules are all trivial source modules. First, the trivial modules  $k_G$  and  $k_{G'}$  are obviously trivial source modules with vertex  $P$ , and for  $G' := M_{11}$ , the module 10 is a trivial source module with vertex  $Q_8$  by [Sch83, Lemma 2.1(a) and (d)], whereas the module 44 is a trivial source module with vertex  $C_2 \times C_2$  by [Sch83, Lemma 2.2(a) and (c)]. Next, consider  $G := \text{PSL}_3(3)$  and its maximal subgroup  $\mathfrak{M} := 3^2 \rtimes 2\mathfrak{S}_4 = 3^2 \rtimes \text{GL}_2(3)$  where  $\mathfrak{S}_4$  is the symmetric group of degree 4 (see [Atlas, p.13]). Using the Atlas [Atlas, p.13] and the 2-decomposition matrix of  $B$  given in [ModAtl] we easily compute that  $k_{\mathfrak{M}} \uparrow^G = k_G + 12$  as composition factors. Then, as  $k_G$  and 12 are self-dual, we must have  $k_{\mathfrak{M}} \uparrow^G = k_G \oplus 12$ . Hence 12 is a trivial source module. Moreover, the module 12 is liftable and affords the ordinary character  $\chi_2$  (in the Atlas notation [Atlas, p.13]). Therefore, it follows from [Lan83, II Lemma 12.6(ii)] and the character values of  $\chi_2$  at 2-elements that 12 has vertex  $C_2 \times C_2$ . To prove that 26 is a trivial source module, we consider  $SD_{16} = P < \text{GL}_2(3) =: \tilde{G} < \mathfrak{M} < G$ . We easily compute that  $1_P \uparrow^{\tilde{G}} = 1_{\tilde{G}} + \tilde{\chi}_{2a}$  where  $\tilde{\chi}_{2a}$  is the unique 2-rational irreducible ordinary character of  $\tilde{G}$  of degree 2. Hence, as above by self-duality,  $k_P \uparrow^{\tilde{G}} = k_{\tilde{G}} \oplus \tilde{2}$  where  $\tilde{2}$  is the unique simple  $k\tilde{G}$ -module in  $B_0(k\tilde{G})$ , so that the simple module  $\tilde{2}$  is a trivial source  $k\tilde{G}$ -module. Again, we read from the character table of  $\tilde{G}$  and [Lan83, II Lemma 12.6(ii)] that  $\tilde{2}$  has vertex  $Q_8$ . Moreover, by the character tables of  $G$  and  $\tilde{G}$ , we have  $\tilde{\chi}_{2a} \uparrow^G = \chi_8$ , so that  $\tilde{2} \uparrow^G = 26$ . Hence 26 is also a trivial source  $kG$ -module with vertex  $Q_8$ .

Next, we recall that there is a bijection between the set of isomorphism classes of indecomposable trivial source  $kG'$ -modules (resp.  $kG$ -modules) with vertex  $X \leq P$  and the set of isomorphism classes of indecomposable projective  $k[N_{G'}(X)/X]$ -modules (resp.  $k[N_G(X)/X]$ -modules). (See [NT88, Chap.4, Problem 10]). Now consider  $Q \leq P$  with  $Q \cong Q_8$  and  $K \leq P$  with  $K \cong C_2 \times C_2$ . It is easy to compute (e.g. using [GAP]) that  $N_{G'}(P)/P = 1$  and  $N_{G'}(Q)/Q \cong N_{G'}(K)/K \cong \mathfrak{S}_3$  and it is well-known that  $k\mathfrak{S}_3$  has two PIMs. Hence there are precisely two non-isomorphic indecomposable trivial source  $kG'$ -modules with vertex  $Q$ . One of them is 10 by the above, and the other one has to be  $\text{Sc}(G', Q)$ , since  $\text{Sc}(G', Q) \not\cong 10$  as it must contain a copy of the trivial module in its socle. Namely,

$$(1) \quad \{\text{iso. classes of indec. trivial source } B'\text{-modules with vertex } Q\} = \{\text{Sc}(G', Q), 10\}$$

and similarly

$$(2) \quad \{\text{iso. classes of indec. trivial source } B'\text{-modules with vertex } K\} = \{\text{Sc}(G', K), 44\}.$$

For  $G$  we also have  $N_G(P)/P = 1$  and  $N_G(Q)/Q \cong N_G(K)/K \cong \mathfrak{S}_3$  (e.g. using [GAP]). Thus, the same arguments as above yield:

$$(3) \quad \{\text{iso. classes of indec. trivial source } B\text{-modules with vertex } Q\} = \{\text{Sc}(G, Q), 26\}$$

and

$$(4) \quad \{\text{iso. classes of indec. trivial source } B\text{-modules with vertex } K\} = \{\text{Sc}(G, K), 12\}.$$

Now, let us consider the functor

$$F : \text{mod-}B \rightarrow \text{mod-}B', \quad X_B \mapsto (X \otimes_B M)_{B'}.$$

By Lemma 4.1,  $F$  is a functor realizing a stable equivalence of Morita type, hence an additive category equivalence between  $\underline{\text{mod-}}B$  and  $\underline{\text{mod-}}B'$ . Therefore, as  $\mathcal{F}_P(G) = \mathcal{F}_P(G')$  (see [CG12, Theorem 5.3]), first by [KL20a, Lemma 3.4(a)] we have

$$F(k_G) = k_{G'},$$

and by [KL20a, Theorem 2.1(a)],  $F(26)$  and  $F(12)$  are both indecomposable  $kG'$ -modules in  $B'$ . Next, we prove that  $F(26) = 10$ . It follows from [KL20a, Lemma 3.4(b)] that

$$F(26) \in \{\text{Sc}(G', Q), 10\}.$$

If  $F(26) = \text{Sc}(G', Q)$ , then

$$\begin{aligned} 0 &\neq \text{Hom}_{kG'}(F(26), k_{G'}) = \text{Hom}_{kG'}(26 \otimes_{kG} M, k_{G'}) \\ &= \text{Hom}_{kG}(26, k_{G'} \otimes_{kG'} M^*) \quad \text{by adjointness} \\ &= \text{Hom}_{kG}(26, k_G) \quad \text{by [KL20a, Lemma 3.4(a)]} \\ &= 0, \end{aligned}$$

a contradiction, so that we have  $F(26) = 10$ . A similar argument using (2) and (4) yields  $F(12) = 44$ . Therefore, by [Lin18, Theorem 4.14.10], the pair  $(M, M^*)$  induces a Morita equivalence between  $B$  and  $B'$  where  $M^*$  is the  $k$ -dual of  $M$ , because all simple  $B$ -modules are mapped to simple  $B'$ -modules.  $\square$

## 5. PROOF OF THEOREM 1.1(b)

First of all, we give a lemma which is a direct consequence of a well-known result due to Alperin and Dade, see [Alp76] and [Dad77], restated in terms of splendid Morita equivalences in [KL20a, Theorem 2.2].

**Lemma 5.1.** *Assume  $k$  is an algebraically closed field of arbitrary prime characteristic  $\ell$ . Let  $G$  and  $G'$  be finite groups with a common Sylow  $\ell$ -subgroup  $P \in \text{Syl}_\ell(G) \cap \text{Syl}_\ell(G')$ . Assume further that there are finite groups  $\tilde{G}$  and  $\tilde{G}'$  such that  $\tilde{G} \triangleright G$  and  $\tilde{G}' \triangleright G'$ ,  $\tilde{G}/G$  and  $\tilde{G}'/G'$  are  $\ell'$ -groups, and  $\tilde{G} = C_G(P)G$ ,  $\tilde{G}' = C_{G'}(P)G'$ . If  $\text{Sc}(G \times G', \Delta P)$  realizes a (splendid) Morita equivalence between  $B_0(kG)$  and  $B_0(kG')$ , then  $\text{Sc}(\tilde{G} \times \tilde{G}', \Delta P)$  realizes a (splendid) Morita equivalence between  $B_0(k\tilde{G})$  and  $B_0(k\tilde{G}')$ .*

*Proof.* Set  $B := B_0(kG)$ ,  $\tilde{B} := B_0(k\tilde{G})$ ,  $B' := B_0(kG')$  and  $\tilde{B}' := B_0(k\tilde{G}')$ . By [KL20a, Theorem 2.2],  $\tilde{B}$  and  $B$  are splendidly Morita equivalent via  $1_{\tilde{B}}k\tilde{G}1_B = \text{Sc}(\tilde{G} \times G, \Delta P)$ , and  $B'$  and  $\tilde{B}'$  are splendidly Morita equivalent via  $1_{B'}k\tilde{G}'1_{\tilde{B}'} = \text{Sc}(G' \times \tilde{G}', \Delta P)$ . Furthermore  $\text{Sc}(G \times G', \Delta P)$  induces a splendid Morita equivalence between  $B$  and  $B'$  by assumption. Hence composing these three splendid Morita equivalences, we have that

$$\text{Sc}(\tilde{G} \times G, \Delta P) \otimes_B \text{Sc}(G \times G', \Delta P) \otimes_{B'} \text{Sc}(G' \times \tilde{G}', \Delta P) =: \mathfrak{M}$$

induces a splendid Morita equivalence between  $\tilde{B}$  and  $\tilde{B}'$ . It remains to see that  $\mathfrak{M} = \text{Sc}(\tilde{G} \times \tilde{G}', \Delta P)$ . Indeed, by the above

$$\begin{aligned} \mathfrak{M} &= 1_{\tilde{B}}k\tilde{G}1_B \otimes_B \text{Sc}(G \times G', \Delta P) \otimes_{B'} 1_{B'}k\tilde{G}'1_{\tilde{B}'} \Big| k\tilde{G} \otimes_{kG} (kG \otimes_{kP} kG') \otimes_{kG'} k\tilde{G}' \\ &= k\tilde{G} \otimes_{kP} k\tilde{G}' = k_{\Delta P} \uparrow^{\tilde{G} \times \tilde{G}'} \end{aligned}$$

and hence the definition of the Scott module yields  $\text{Sc}(\tilde{G} \times \tilde{G}', \Delta P) \mid \mathfrak{M}$ . However, as  $\mathfrak{M}$  induces a Morita equivalence between  $\tilde{B}$  and  $\tilde{B}'$ , which are both indecomposable as bimodules,  ${}_{\tilde{B}}\mathfrak{M}_{\tilde{B}'}$  must be indecomposable and it follows that  $\mathfrak{M} = \text{Sc}(\tilde{G} \times \tilde{G}', \Delta P)$ .  $\square$

We now prove Theorem 1.1(b) through a case-by-case analysis. For the remainder of this section, we let  $p, p'$  be prime numbers,  $f, f' \geq 1$  and  $n \geq 4$  be integers, and we use, without further mention, the fact that if a Scott module induces Morita equivalence between two blocks, then this equivalence is automatically splendid.

**Proposition 5.2.** *Let  $G := \text{SL}_2^\pm(p^f)$  and  $G' := \text{SL}_2^\pm(p'^{f'})$  with  $4(p^f + 1)_2 = 4(p'^{f'} + 1)_2 = 2^n$  and let  $P \in \text{Syl}_2(G) \cap \text{Syl}_2(G')$ . Then,  $\text{Sc}(G \times G', \Delta P)$  induces a splendid Morita equivalence between  $B_0(kG)$  and  $B_0(kG')$ .*

*Proof.* The groups  $G$  and  $G'$  have a common central subgroup  $Z \leq P$  of order 2 such that  $\bar{G} := G/Z \cong \text{PGL}_2(p^f)$  and  $\bar{G}' := G'/Z \cong \text{PGL}_2(p'^{f'})$  have a common Sylow 2-subgroup  $\bar{P} := P/Z$  isomorphic to  $D_{2^{n-1}}$  (see [ABG70, p.4]). Hence it follows from [KL20a, Theorem 1.1(6)] that  $\text{Sc}(\bar{G} \times \bar{G}', \Delta \bar{P})$  induces a splendid Morita equivalence between  $B_0(k\bar{G})$  and  $B_0(k\bar{G}')$ . Therefore  $\text{Sc}(G \times G', \Delta P)$  induces a Morita equivalence between  $B_0(kG)$  and  $B_0(kG')$  by [KL20b, Proposition 3.3(b)]. The claim follows.  $\square$

**Proposition 5.3.** *Let  $G := \text{SU}_2^\pm(p^f)$  and  $G' := \text{SU}_2^\pm(p'^{f'})$  with  $4(p^f - 1)_2 = 4(p'^{f'} - 1)_2 = 2^n$  and let  $P \in \text{Syl}_2(G) \cap \text{Syl}_2(G')$ . Then,  $\text{Sc}(G \times G', \Delta P)$  induces a splendid Morita equivalence between  $B_0(kG)$  and  $B_0(kG')$ .*

*Proof.* Again, the groups  $G$  and  $G'$  have a common central subgroup  $Z \leq P$  of order 2 such that  $\bar{G} := G/Z \cong \text{PGL}_2(p^f)$  and  $\bar{G}' := G'/Z \cong \text{PGL}_2(p'^{f'})$  have a common Sylow 2-subgroup  $\bar{P} := P/Z$  isomorphic to  $D_{2^{n-1}}$  (see [ABG70, p.4]). Hence the assertion follows from the same argument as in the proof of Proposition 5.2, where [KL20a, Theorem 1.1(6)] is replaced by [KL20a, Theorem 1.1(5)].  $\square$

**Proposition 5.4.** *Let  $G := \text{PSL}_3(p^f)$  and  $G' := \text{PSL}_3(p'^{f'})$  with  $4(p^f + 1)_2 = 4(p'^{f'} + 1)_2 = 2^n$  and let  $P \in \text{Syl}_2(G) \cap \text{Syl}_2(G')$ . Then,  $\text{Sc}(G \times G', \Delta P)$  induces a splendid Morita equivalence between  $B_0(kG)$  and  $B_0(kG')$ .*

*Proof.* First, we claim that  $M := \text{Sc}(G \times G', \Delta P)$  induces a stable equivalence of Morita type between  $B_0(kG)$  and  $B_0(kG')$ . Let  $z$  be the unique involution in  $Z := Z(P)$ , and set  $C := C_G(z)$ ,  $C' := C_{G'}(z)$ ,  $\bar{C} := C/O_{2'}(C)$ ,  $\bar{C}' := C'/O_{2'}(C')$  and  $\bar{P} := PO_{2'}(C)/O_{2'}(C) \cong P \cong PO_{2'}(C')/O_{2'}(C')$  (and we identify the two groups). Then, by [ABG70, Proposition 4(iii), p.21] and Theorem 3.1, we obtain that

$$\begin{aligned} \bar{C} &\cong \text{SL}_2^\pm(p^f) \rtimes C_d && \text{for an odd } d \text{ with } d \mid f \text{ and} \\ \bar{C}' &\cong \text{SL}_2^\pm(p'^{f'}) \rtimes C_{d'} && \text{for an odd } d' \text{ with } d' \mid f'. \end{aligned}$$

We can consider that  $B_0(kC) = B_0(k\bar{C})$ ,  $B_0(kC') = B_0(k\bar{C}')$  and  $\bar{P} \in \text{Syl}_2(\bar{C}) \cap \text{Syl}_2(\bar{C}')$ . Hence it follows from Proposition 5.2 and Lemma 5.1 that  $\text{Sc}(\bar{C} \times \bar{C}', \Delta \bar{P})$  induces a Morita equivalence between  $B_0(k\bar{C})$  and  $B_0(k\bar{C}')$ . Thus,  $M_Z := \text{Sc}(C \times C', \Delta P)$  induces a Morita equivalence between  $B_0(kC)$  and  $B_0(kC')$  by [KL20b, Proposition 3.3(b)]. On the other hand,  $\mathcal{F}_P(G) = \mathcal{F}_P(G')$  by [CG12, Theorem 5.3]. Hence it follows from [KL20a, Lemma 3.2] that  $M_Z \mid M(\Delta Z)$  and therefore by [KT19, Theorem 1.2],  $M_Z = M(\Delta Z)$ . Thus, again the gluing method of [KL20a, Lemma 4.1] implies the claim.

Next we claim that the stable equivalence of Morita type between  $B_0(kG)$  and  $B_0(kG')$  induced by  $M$  is actually a Morita equivalence. Since  $\text{Aut}(P)$  is a 2-group,  $N_G(P) = P \times O_{2'}(C_G(P))$ , so that  $N_G(P) \leq C$  and we can consider the Green correspondences  $f := f_{(G,P,C)}$  and  $f' := f_{(G',P,C')}$ . Then, it follows from [Erd79, (3.4)] that we can consider

$$B_0(kC) = B_0(kC') \text{ and } f(S) = f'(S')$$

for all three simple  $kG$ -modules  $S$  and  $S'$  in  $B_0(kG)$  and  $B_0(kG')$ , respectively, where  $S$  corresponds to  $S'$ . Thus, [Lin18, Theorem 4.14.10] yields that  $M$  induces a Morita equivalence between  $B_0(kG)$  and  $B_0(kG')$ , which is automatically splendid.  $\square$

**Proposition 5.5.** *Let  $G := \text{PSU}_3(p^f)$  and  $G' := \text{PSU}_3(p'^{f'})$  with  $4(p^f - 1)_2 = 4(p'^{f'} - 1)_2 = 2^n$  and let  $P \in \text{Syl}_2(G) \cap \text{Syl}_2(G')$ . Then,  $\text{Sc}(G \times G', \Delta P)$  induces a splendid Morita equivalence between  $B_0(kG)$  and  $B_0(kG')$ .*

*Proof.* The same arguments as in the proof of Proposition 5.4 yield the result. More precisely, in this case  $\overline{C} \cong \text{SU}_2^\pm(p^f)$  and  $\overline{C}' \cong \text{SU}_2^\pm(p'^{f'})$  so that it follows from Proposition 5.3 that the Scott module  $\overline{M_Z} := \text{Sc}(\overline{C} \times \overline{C}', \Delta \overline{P})$  induces a Morita equivalence between  $B_0(k\overline{C})$  and  $B_0(k\overline{C}')$ , and [Erd79, (3.4)] is replaced by [Erd79, (4.10)].  $\square$

Finally we deal with the groups of type (ab), that is of the form  $\text{PGL}_2^*(p^{2f})$ . This case requires more involved arguments. However, the proof of [KL20a, Proposition 5.4] – showing that the principal blocks of  $\text{PGL}_2(q)$  and  $\text{PGL}_2(q')$  with a common dihedral Sylow 2-subgroup and  $q \equiv q' \equiv 1 \pmod{4}$  are splendidly Morita equivalent – can be imitated because the vast majority of the arguments rely on the fact that  $\text{PGL}_2(q)$  is an extension of degree two of  $\text{PSL}_2(q)$ .

**Proposition 5.6.** *Let  $G := \text{PGL}_2^*(p^{2f})$ ,  $G' := \text{PGL}_2^*(p'^{2f'})$  with  $2(p^{2f} - 1)_2 = 2(p'^{2f'} - 1)_2 = 2^n$  and let  $P \in \text{Syl}_2(G) \cap \text{Syl}_2(G')$ . Then,  $\text{Sc}(G \times G', \Delta P)$  induces a splendid Morita equivalence between  $B_0(kG)$  and  $B(kG')$ .*

*Proof.* Set  $B := B_0(kG)$ ,  $B' := B_0(kG')$  and  $M := \text{Sc}(G \times G', \Delta P)$ . By the definition of  $G$  and  $G'$  in Section 1, there are normal subgroups  $N \triangleleft G$  and  $N' \triangleleft G'$  with  $G$  has a normal subgroup  $N \mid G/N = 2$ ,  $|G'/N'| = 2$  and  $N \cong \text{PSL}_2(p^{2f})$ ,  $N' \cong \text{PSL}_2(p'^{2f'})$ . Hence there is  $Q \in \text{Syl}_2(N) \cap \text{Syl}_2(N')$  such that  $Q \cong D_{2^{n-1}}$  (see [ABG70]). Recall that  $\mathcal{F}_P(G) = \mathcal{F}_P(G')$  by [CG12, Theorem 5.3].

First, we claim that

(5)  $M$  realizes a stable equivalence of Morita type between  $B$  and  $B'$ .

Let  $z$  be the unique involution in  $Z(P)$ . Set  $C := C_G(z)$  and  $C' := C_{G'}(z)$ . We know that  $z \in Q \in \text{Syl}_2(N) \cap \text{Syl}_2(N')$ . Now recall that  $C_N(z)$  and  $C_{N'}(z)$  are both 2-nilpotent by [Bra66, Lemma (7A)]. Hence, as  $|G/N| = 2 = |G'/N'|$ ,  $C$  and  $C'$  are also 2-nilpotent. Set  $\overline{C} := C/O_{2'}(C)$ ,  $\overline{C}' := C'/O_{2'}(C')$ ,  $\overline{P} := [P O_{2'}(C)]/O_{2'}(C) \cong [P O_{2'}(C')]/O_{2'}(C')$ . Obviously,  $\overline{C} \cong \overline{C}' \cong \overline{P} \cong P$ . Hence, [KL20a, Lemma 3.1] implies that  $\text{Sc}(C \times C', \Delta P)$  induces a Morita equivalence between  $B_0(kC)$  and  $B_0(kC')$  and [KL20a, Lemma 3.2] yields

$$\text{Sc}(C \times C', \Delta P) \mid M(\Delta\langle z \rangle)$$

However, as  $M$  is Brauer indecomposable by [KT19], we have  $\text{Sc}(C \times C', \Delta P) = M(\Delta\langle z \rangle)$ . Since, by [ABG70, Proposition 1.1(iii), p.10], all involutions in  $G$  are  $G$ -conjugate and all involutions in  $G'$  are  $G'$ -conjugate, (5) follows from [KL20a, Lemma 4.1] as we have already seen in the proof of Lemma 4.1.



Second, in order to prove that the stable equivalence realized by  $M$  is in fact a Morita equivalence, it suffices by [Lin18, Theorem 4.14.10] to prove that

(6) for every simple  $B$ -module  $S$ ,  $F(S)$  is a simple  $B'$ -module where  $F(S) = S \otimes_B M$ .

Recall that in the proof of the case  $q \equiv 1 \pmod{4}$  of [KL20a, Lemma 5.3],  $\mathrm{PSL}_2(q)$  is normal of index 2 in  $\mathrm{PGL}_2(q)$  and  $\ell(B_0(k\mathrm{PGL}_2(q))) = 2$ . The same thing happens here for  $G$  and  $G'$ . Therefore, it is enough to note that in the statement of [KL20a, Proposition 5.4] the groups  $\mathrm{PGL}_2(q)$  and  $\mathrm{PGL}_2(q')$ , respectively, can be replaced with  $G = \mathrm{PGL}_2^*(p^{2f})$  and  $G' = \mathrm{PGL}_2^*(p'^{2f'})$ . Then the proof of [KL20a, Proposition 5.4] can be replaced by the proof of the case  $q \equiv 1 \pmod{4}$  of [KL20a, Lemma 5.3], and (6) follows.  $\square$

## 6. PROOF OF THEOREM 1.1

We can now prove Theorem 1.1.

**Proof of Theorem 1.1.** Part (b) is given by the case-by-case analysis of Section 5. Hence it remains to prove (a).

To start with,  $B_0(kG) = \mathrm{Sc}(G \times [G/O_{2'}(G)], \Delta P)$  (seen as a  $(kG, k[G/O_{2'}(G)])$ -bimodule) induces a splendid Morita equivalence between  $B_0(kG)$  and  $B_0(k[G/O_{2'}(G)]) =: \bar{B}$ , because  $O_{2'}(G)$  acts trivially on the principal block. Furthermore, if  $G'$  denotes one of the groups listed in Theorem 1.1(a) and there is a splendid Morita equivalence between  $B_0(k[G/O_{2'}(G)])$  and  $B_0(kG')$  realized by the Scott module  $\mathrm{Sc}([G/O_{2'}(G)] \times G', \Delta P)$ , then composing both equivalences, we obtain a splendid Morita equivalence between  $B_0(kG)$  and  $B_0(kG')$  realized by

$$\mathrm{Sc}(G \times [G/O_{2'}(G)], \Delta P) \otimes_{\bar{B}} \mathrm{Sc}([G/O_{2'}(G)] \times G', \Delta P) \cong \mathrm{Sc}(G \times G', \Delta P).$$

Therefore, we may assume that  $O_{2'}(G) = 1$  and so  $G$  must be of type (x), where (x) denotes one of the seven families of groups (bb), (ba1), (ba2), (ab), (aa1), (aa2), (aa3) of Theorem 3.1.

**Claim 1:**  $B := B_0(kG)$  is splendidly Morita equivalent to the principal block  $B' := B_0(kG')$  for some group  $G'$  of type (bb), (ba1), (ba2), (ab), (aa1), (aa2) listed in Theorem 1.1(a) with  $P \in \mathrm{Syl}_2(G) \cap \mathrm{Syl}_2(G')$  and the splendid Morita equivalence is realized by  $\mathrm{Sc}(G \times G', \Delta P)$ .

Here we emphasize that the lists of groups in Theorem 3.1 and in the statement of Theorem 1.1(a) are not the same, hence we use different fonts to distinguish them. We prove Claim 1 through a case-by-case analysis as follows.

- Suppose that  $G$  is of type (bb). Then  $G = P$  by Theorem 3.1(bb). Then, we may take  $G' := P$ , that is  $G'$  of type (bb). Obviously  $B = B' = kP$  and  $\mathrm{Sc}(P \times P, \Delta P) = {}_{kP}kP_{kP}$  induces a splendid Morita equivalence between  $B$  and  $B'$ , as required.

- Suppose that  $G$  is of type (ba1). Then  $G = \mathrm{SL}_2^\pm(p^f) \rtimes C_d$  where  $4(p^f + 1)_2 = 2^n$  and  $d$  is an odd divisor of  $f$ . We take  $G' := \mathrm{SL}_2^\pm(p'^{f'})$ , that is of type (ba1), and we may assume that we have chosen  $P$  such that  $P \in \mathrm{Syl}_2(G) \cap \mathrm{Syl}_2(G')$ . Set  $H := \mathrm{SL}_2^\pm(p^f)$ ,  $H' := \mathrm{SL}_2^\pm(p'^{f'})$ ,  $b := B_0(kH)$  and  $b' := B_0(kH')$ ,  $\mathfrak{M} := \mathrm{Sc}(G \times H, \Delta P)$ ,  $\mathfrak{N} := \mathrm{Sc}(H \times H', \Delta P)$  and  $\mathfrak{M}' := \mathrm{Sc}(H' \times G', \Delta P)$ . By the Frattini argument and the fact that  $\mathrm{Aut}(P)$  is a 2-group,  $G = C_G(P)H$  (resp.  $G' = C_{G'}(P)H'$ ), so that  $\mathfrak{M}$  (resp.  $\mathfrak{M}'$ ) induces a Morita equivalence between  $B$  and  $b$  (resp.  $b'$  and  $B'$ ) by Lemma 5.1. Moreover,  $\mathfrak{N}$  realizes a Morita equivalence between  $b$  and  $b'$  by Proposition 5.2. Thus,  $\mathfrak{L} := \mathfrak{M} \otimes_b \mathfrak{N} \otimes_{b'} \mathfrak{M}'$  induces a Morita equivalence between  $B$  and  $B'$ . Now note that  $\mathfrak{M} = \mathrm{Sc}(G \times H, \Delta H)$

by [NT88, Corollary 4.8.5], and hence  $\mathfrak{M} \mid_{kG} kG_{kH}$ . Similarly, we have  $\mathfrak{M}' \mid_{kH'} kG'_{kG'}$ , and obviously,  $\mathfrak{N} \mid_{kH} kH \otimes_{kP} kH'_{kH'}$ . Thus,  $\text{Sc}(G \times G', \Delta P) \mid \mathfrak{L}$ . Finally, since  $B$  and  $B'$  are indecomposable as  $k$ -algebras,  $\mathfrak{L}$  has to be indecomposable as a  $(B, B')$ -bimodule, so that  $\mathfrak{L} = \text{Sc}(G \times G', \Delta P)$ .

- Suppose that  $G$  is of type (ba2). Then  $G = \text{SU}_2^\pm(p^f) \rtimes C_d$  where  $4(p^f - 1)_2 = 2^n$  and  $d$  is an odd divisor of  $f$ . We take  $G' := \text{SU}_2^\pm(p'^{f'})$ , that is of type (ba2), and we may assume that we have chosen  $P$  such that  $P \in \text{Syl}_2(G) \cap \text{Syl}_2(G')$ . Then the same arguments as in case (ba1) yield the claim.
- Suppose that  $G$  is of type (ab). Then  $G = \text{PGL}_2^*(p^{2f}) \rtimes C_d$  where  $2(p^{2f} - 1)_2 = 2^n$  and  $d$  is an odd divisor of  $f$ . We take  $G' := \text{PGL}_2^*(p'^{2f'})$ , that is of type (ab), and we may assume that we have chosen  $P$  such that  $P \in \text{Syl}_2(G) \cap \text{Syl}_2(G')$ . Then the same arguments as in case (ba1) yield the claim.
- Suppose that  $G$  is of type (aa1). Then  $G = \text{PSL}_3(p^f).H$  where  $4(p^f + 1)_2 = 2^n$  and  $H \leq C_{(3,p^f-1)} \times C_d$  for an odd divisor  $d$  of  $f$ . We take  $G' := \text{PSL}_3(p'^{f'})$ , that is of type (aa1), and we may assume that we have chosen  $P$  such that  $P \in \text{Syl}_2(G) \cap \text{Syl}_2(G')$ . Then the same arguments as in case (ba1) yield the claim, where  $C_d$  is replaced by  $H$ .
- Suppose that  $G$  is of type (aa2). Then  $G = \text{PSU}_3(p^f).H$  where  $4(p^f - 1)_2 = 2^n$  and  $H \leq C_{(3,p^f+1)} \times C_d$  for an odd divisor  $d$  of  $f$ . We take  $G' := \text{PSU}_3(p'^{f'})$ , that is of type (aa2), and we may assume that we have chosen  $P$  such that  $P \in \text{Syl}_2(G) \cap \text{Syl}_2(G')$ . Then the same arguments as in case (ba1) yield the claim, where  $C_d$  is replaced by  $H$ .
- Suppose that  $G$  is of type (aa3). Then  $G = M_{11}$  by Theorem 3.1(aa3) and  $n = 4$ . We take  $G' := \text{PSL}_3(3)$ , so that  $P \in \text{Syl}_2(G) \cap \text{Syl}_2(G')$  and  $G'$  is of type (aa1). Moreover, by Proposition 4.2,  $\text{Sc}(G \times G', \Delta P)$  induces a splendid Morita equivalence between  $B$  and  $B'$ , as required.

Furthermore, the fact that the group  $G'$  in Claim 1 is independent of the choice of  $p$  and  $f$  for types (ba1), (ba2), (ab), (aa1), (aa2) follows directly from Part (b). Hence it only remains to prove the following claim.

**Claim 2.** The principal blocks of the groups listed in the different cases of Theorem 1.1(a) are mutually not splendidly Morita equivalent.

So let  $B := B_0(kG)$  for  $G$  of type (x) with  $(x) \in \{(\text{bb}), (\text{ba1}), (\text{ba2}), (\text{ab}), (\text{aa1}), (\text{aa2})\}$  as in Theorem 1.1(a). It is enough to show that  $B$  is not Morita equivalent to  $B' := B_0(kG')$  for  $G'$  of type (y)  $\neq$  (x) and  $(y) \in \{(\text{bb}), (\text{ba1}), (\text{ba2}), (\text{ab}), (\text{aa1}), (\text{aa2})\}$ .

Now, type (bb) is the unique case in which  $\ell(B) = 1$ , so we can assume that  $G$  is not of type (bb). Next, assume that  $\ell(B) = 3$ . Then, by [Ols75, table on p. 231],  $G$  is of type (aa1) or (aa2). However, the principal blocks of groups of type (aa1) and (aa2) are never Morita equivalent because their 2-decomposition matrices are different by [Erd90, SD(2B)<sub>2</sub>, p.299 and SD(2A)<sub>1</sub>, p.298]. Therefore, it only remains to consider the case  $\ell(B) = 2$ . Then, by [Ols75, table on p. 231],  $G$  is of type (ab) or (ba) (i.e. (ba1) or (ba2)). Then, by looking at  $k(B)$ s (see [Ols75, the table on p.231]), we obtain that the principal blocks of groups of type (ab) and (ba) are never Morita equivalent. Hence we can also assume that  $G$  is not of type (ab), so that we may assume that  $G$  is of type (ba1) and  $G'$  is of type (ba2), that is

$$\begin{aligned} G &= \text{SL}_2^\pm(p^f) & \text{with} & & 4(p^f + 1)_2 &= 2^n \\ G' &= \text{SU}_2^\pm(p'^{f'}) & \text{with} & & 4(p'^{f'} - 1)_2 &= 2^n \end{aligned}$$

and we may identify a Sylow 2-subgroup  $P$  of  $G$  and  $G'$ . Since  $G$  has a central involution, say  $z$ , set  $Z := \langle z \rangle$ ,  $\overline{G} := G/Z \cong \mathrm{PGL}_2(p^f)$ ,  $\overline{B} := B_0(k\overline{G})$ ,  $\overline{P} := P/Z$  and note that  $\overline{P} \cong D_{2^{n-1}}$ . We also have  $Z \leq G'$ , hence we can set  $\overline{G}' := G'/Z \cong \mathrm{PGL}_2(p^{f'})$  and  $\overline{B}' := B_0(k\overline{G}')$ . Then, it follows from the condition on  $p^f$  and [KL20a, Corollary 8.1(f)] (see also [Erd90, D(2 $\mathcal{B}$ ), p.295]) and from the condition on  $p^{f'}$  and [KL20a, Corollary 8.1(e)] (see [Erd90, D(2 $\mathcal{A}$ ), p.294]), respectively, that

$$C_{\overline{B}} = \begin{pmatrix} 4 & 2 \\ 2 & 2^{n-3} + 1 \end{pmatrix} \quad \text{and} \quad C_{\overline{B}'} = \begin{pmatrix} 2^{n-1} & 2^{n-2} \\ 2^{n-2} & 2^{n-3} + 1 \end{pmatrix}.$$

Now suppose that  $B$  and  $B'$  are Morita equivalent. Then,  $C_B = C_{B'}$ . Since  $Z$  is a central subgroup of  $G$  and  $G'$  of order 2, [NT88, Theorem 5.8.11] implies that  $C_B = 2C_{\overline{B}}$  and  $C_{B'} = 2C_{\overline{B}'}$ . Thus  $C_{\overline{B}} = C_{\overline{B}'}$ , a contradiction. Claim 2 follows.  $\square$

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