PRINCIPAL 2-BLOCKS WITH WREATHED DEFECT GROUPS UP TO SPLENDID MORITA EQUIVALENCE

SHIGEO KOSHITANI, CAROLINE LASSUEUR AND BENJAMIN SAMBALE

To the memory of Professor Atumi Watanabe

Abstract. We classify principal 2-blocks of finite groups G with Sylow 2-subgroups isomorphic to a wreathed 2-group $C_{2^n} \wr C_2$ with $n \geq 2$ up to Morita equivalence and up to splendid Morita equivalence. As a consequence, we obtain that Puig's Finiteness Conjecture holds for such blocks. Furthermore, we obtain a classification of such groups modulo $O_{2'}(G)$, which is a purely group theoretical result and of independent interest. Methods previously applied to blocks of tame representation type are used. They are, however, further developed in order to deal with blocks of wild representation type.

1 Introduction

Let k be an algebraically closed field of positive characteristic p. A splendid Morita equivalence between two block algebras B_1 and B_2 of finite groups G_1 and G_2 of order divisible by p is a Morita equivalence which is induced by a (B_1, B_2)-bimodule (and its dual) which is a p-permutation module when regarded as a one-sided $k(G_1 \times G_2)$-module. Such equivalences play an important role in the modular representation theory of finite groups as they preserve many important invariants such as the defect groups or the generalised decomposition numbers, and encode the structure of the source algebras. In this respect, Puig's finiteness conjecture (see [Bro94, 6.2] or [Lin18, Conjecture 6.4.2]) extends Donovan's conjecture to include the structure of the source algebra of p-blocks and postulates that given a finite p-group D, there are only finitely many interior D-algebras, up to isomorphism, which are source algebras of p-blocks of finite groups with defect group D. This is equivalent to postulating that there are only finitely many splendid Morita equivalence classes of p-blocks of finite groups with defect group D.

In a series of previous articles [KL20a, KL20b, KLS22] the authors classified principal block algebras of tame representation type up to splendid Morita equivalence, that is, in the case in which $p = 2$ and the Sylow 2-subgroups of the groups considered are either dihedral, semi-dihedral, or generalised quaternion 2-groups. As a corollary, the validity of Puig's conjecture is verified for this class of 2-blocks. (The tame domestic case was settled in [CEKL11].) The aim of the present article is to give a first try at applying

Date: June 3, 2024.

2010 *Mathematics Subject Classification.* 20C05, 20C20, 20C15, 20C33,16D90.

Key words and phrases. wreathed 2-group, Morita equivalence, splendid Morita equivalence, Puig's Finiteness Conjecture, principal block, trivial source module, p-permutation module, Scott module, Brauer indecomposability, decomposition matrix.

The first author was partially supported by the Japan Society for Promotion of Science (JSPS), Grant-in-Aid for Scientific Research (C)19K03416, 2019–2021, and also by the Research Institute for Mathematical Sciences (RIMS), an International Joint Usage/Research Center located in Kyoto University. The second author was supported by the DFG SFB/TRR 195.

similar methods in *wild* representation type under good hypotheses: we investigate here groups with Sylow 2-subgroups isomorphic to a wreathed 2-group $C_{2^n} \wr C_2$ with $n \geq 2$. We choose this defect group for its many similarities with the tame cases. In this respect, from the group theory point of view, we strongly rely on the facts that the wreathed 2-groups $C_{2^n} \wr C_2$ have 2-rank 2 and an automorphism group which is a 2-group, whereas from the modular representation theory point of view we rely on the Brauer indecomposability of Scott modules with wreathed vertices proved by the first author and Tuvay in [KT21].

In order to state our main results, we first need to introduce some notation. Given a finite group G and $H \leq G$, we set $\Delta H := \{(h,h) \in G \times G \mid h \in H\}$ and we recall that the *Scott module* of kG with respect to H, denoted by $\mathrm{Sc}(G,H)$, is, up to isomorphism, the unique indecomposable direct summand of the trivial kH-module induced from H to G with the property that the trivial kG-module is a constituent of its head (or equivalently of its socle). Furthermore, given an integer $t \geq 0$ and a power $q = r^f$ of a prime number with $f \geq 1$ an integer, we let

$$\mathrm{SL}_2^t(q) := \{A \in \mathrm{GL}_2(q) \mid \det(A)^{2^t} = 1\} \quad \text{and} \quad \mathrm{SU}_2^t(q) := \{A \in \mathrm{GU}_2(q) \mid \det(A)^{2^t} = 1\}\,.$$

Now, in order to apply the previously developed methods, our first main result provides a classification of the finite groups G with a wreathed Sylow 2-subgroup $C_{2^n} \wr C_2$ ($n \geq 2$) modulo $O_{2'}(G)$, which is of independent interest.

Theorem 1.1. *Let G be a finite group with a Sylow 2-subgroup isomorphic to a wreathed 2-group $C_{2^n} \wr C_2$ for an integer $n \geq 2$ such that $O_{2'}(G) = 1$. Then one of the following holds:*

(WR1) $G \cong C_{2^n} \wr C_2\,,$
(WR2) $G \cong (C_{2^n} \times C_{2^n}) \rtimes \mathfrak{S}_3\,,$
(WR3) $G \cong \mathrm{SL}_2^n(q) \rtimes C_d$ *where* $(q-1)_2 = 2^n$ *and* $d \mid f$ *is odd,*
(WR4) $G \cong \mathrm{SU}_2^n(q) \rtimes C_d$ *where* $(q+1)_2 = 2^n$ *and* $d \mid f$ *is odd,*
(WR5) $G \cong \mathrm{PSL}_3(q).H$ *where* $(q-1)_2 = 2^n$, $H \leq C_{(q-1,3)} \times C_d$ *and* $d \mid f$ *is odd, or*
(WR6) $G \cong \mathrm{PSU}_3(q).H$ *where* $(q+1)_2 = 2^n$, $H \leq C_{(q+1,3)} \times C_d$ *and* $d \mid f$ *is odd,*

where $q = r^f$ denotes a power of a prime number r and $f \geq 1$ an integer.

This theorem, which we prove in Section 3, is a byproduct of Alperin–Brauer–Gorenstein's work [ABG70] on finite groups with quasi-dihedral and wreathed Sylow 2-subgroups.

Our second main result is then a classification of principal blocks with defect groups isomorphic to a wreathed 2-group $C_{2^n} \wr C_2$ with $n \geq 2$.

Theorem 1.2. *Let k be an algebraically closed field of characteristic 2 and let G be a finite group with a Sylow 2-subgroup P isomorphic to a wreathed 2-group $C_{2^n} \wr C_2$ for a fixed integer $n \geq 2$. Then, the following assertions hold.*

(a) *The principal 2-block $B_0(kG)$ of G is splendidly Morita equivalent to the principal 2-block $B_0(kG')$ of a finite group G' belonging to precisely one of the following families of finite groups:*
   (W1(n)) $C_{2^n} \wr C_2\,;$
   (W2(n)) $(C_{2^n} \times C_{2^n}) \rtimes \mathfrak{S}_3\,;$
   (W3(n)) $\mathrm{SL}_2^n(q)$ *where q is a power of a prime number such that* $(q-1)_2 = 2^n$;
   (W4(n)) $\mathrm{SU}_2^n(q)$ *where q is a power of a prime number such that* $(q+1)_2 = 2^n$;
   (W5(n)) $\mathrm{PSL}_3(q)$ *where q is a power of a prime number such that* $(q-1)_2 = 2^n$; *or*
   (W6(n)) $\mathrm{PSU}_3(q)$ *where q is a power of a prime number such that* $(q+1)_2 = 2^n$.

Moreover, in all cases, the splendid Morita equivalence is induced by the Scott module $\mathrm{Sc}(G \times G', \Delta P)$*, where* P *is also seen as a Sylow* 2*-subgroup of* G'*.*

(b) *In* (a)*, more accurately, if* G_1 *and* G_2 *are two finite groups belonging to the same infinite family of finite groups* (Wj(n)) *with* j $\in \{3, 4, 5, 6\}$*, then* $\mathrm{Sc}(G_1 \times G_2, \Delta P)$ *induces a splendid Morita equivalence between* $B_0(kG_1)$ *and* $B_0(kG_2)$*.*

We emphasize that in the case of principal blocks of tame representation type, treated in [KL20a, KL20b, KLS22], a classification of these blocks up to Morita equivalence was known by Erdmann's work on tame algebras [Erd90]. A major difference in the case of wreathed Sylow 2-subgroups lies in the fact that a classification of these blocks up to Morita equivalence was, to our knowledge, not known. However, it follows from our methods, that the classification up to splendid Morita equivalence, which we have obtained, coincides with the classification up to Morita equivalence.

Theorem 1.3. *Let* k *be an algebraically closed field of characteristic* 2 *and let* G *be a finite group with a Sylow* 2*-subgroup isomorphic to a wreathed* 2*-group* $C_{2^n} \wr C_2$ *for a fixed integer* $n \geq 2$*. Then* $B_0(kG)$ *is Morita equivalent to the principal block of precisely one of the families of groups* (W1(n))*,* (W2(n))*,* (W3(n))*,* (W4(n))*,* (W5(n))*, or* (W6(n)) *as in Theorem 1.2(a).*

As an immediate consequence of Theorem 1.2 we also obtain that Puig's Finiteness Conjecture holds if we restrict our attention to principal blocks with a defect group isomorphic to a wreathed 2-group $C_{2^n} \wr C_2$.

Corollary 1.4. *For each integer* $n \geq 2$ *there are only finitely many splendid Morita equivalence classes of principal* 2*-blocks with defect groups isomorphic to a wreathed* 2*-group* $C_{2^n} \wr C_2$*.*

This paper is organised as follows. In Section 2 the notation is introduced. In Section 3 we state and prove the classification of finite groups G with a wreathed Sylow 2-subgroup and $O_{2'}(G) = 1$. In Section 4 we recall, state and prove preliminary results on splendid Morita equivalences and on module theory over finite-dimensional algebras. In Sections 5, 6 and 7 we prove part (b) of Theorem 1.2. Section 8 contains the proof of Theorem 1.2 and Theorem 1.3. Finally, Appendix A fixes a gap in the proof of [KL20b, Proposition 3.3(b)].

2 Notation

Throughout this paper, unless otherwise stated, we adopt the following notation and conventions. We let k be an algebraically closed field of characteristic $p > 0$. All groups considered are finite, all k-algebras are finite-dimensional and all modules over finite-dimensional algebras considered are finitely generated right modules. The symbols G, G', G_1, G_2, $\mathcal{G}_1$ and $\mathcal{G}_2$ always denote finite groups of order divisible by p.

Furthermore, we denote by $\mathrm{Syl}_p(G)$ the set of all Sylow p-subgroups of G, and for $P \in \mathrm{Syl}_p(G)$, we let $\mathcal{F}_P(G)$ be the fusion system of G on P. If $H \leq G$, we let $\Delta H := \{(h, h) \in G \times G \mid h \in H\}$ denote the diagonal embedding of H in $G \times G$. Given an integer $m \geq 2$, we let D_{2^m} denote the dihedral group of order 2^m, C_m denote the cyclic group of order m, and $C_{2^m} \wr C_2$ denote the wreathed product of C_{2^m} by C_2. Given an integer $t \geq 0$ and a positive prime power q, we let

$$\mathrm{SL}_2^t(q) := \{A \in \mathrm{GL}_2(q) \mid \det(A)^{2^t} = 1\} \text{ and } \mathrm{SU}_2^t(q) := \{A \in \mathrm{GU}_2(q) \mid \det(A)^{2^t} = 1\},$$

as already defined in the introduction.

Given a finite-dimensional k-algebra A, we denote by $\mathrm{rad}(A)$ the Jacobson radical of A and by 1_A the unit element of A, respectively. Furthermore, if X is an A-module and $m \geq 0$ is an integer, then we denote by $\mathrm{soc}^m(X) := \{x \in X \mid x \cdot \mathrm{rad}(A)^m = 0\}$ the m-th socle of X, where $\mathrm{soc}(X) := \mathrm{soc}^1(X)$ is the socle of X, and for $1 \leq i \leq \ell$, where ℓ is the Loewy (or radical) length of X, we set

$$S_i(X) := \mathrm{soc}^i(X)/\mathrm{soc}^{i-1}(X) \quad \text{and} \quad L_i(X) := X\,\mathrm{rad}(A)^{i-1}/X\,\mathrm{rad}(A)^i$$

and we write $\mathrm{hd}(X)$ for the head of X. We then talk about the *radical (Loewy) series* and about the *socle series* of X as defined in [Lan83, Chap. I §8]. We describe a uniserial A-module X with simple composition factors $L_i(X) \cong S_i$ for simple A-modules $S_1, \cdots, S_\ell$ via the diagram

$$X = \boxed{\begin{matrix} S_1 \\ \vdots \\ S_\ell \end{matrix}}\,.$$

We denote by $P(X)$ the projective cover of an A-module X and by $\Omega(X)$ the kernel of the canonical morphism $P(X) \twoheadrightarrow X$. Dually, we let $\Omega^{-1}(X) := I(X)/X$ where $I(X)$ is an injective envelope of X, and we denote by X^* the k-dual of X (which is a left A-module). Given a simple A-module S, we denote by $c_X(S)$ the multiplicity of S as a composition factor of X and if $S_1, \cdots, S_n$ are all the pairwise non-isomorphic composition factors of X with multiplicities $m_1, \ldots, m_n$, respectively, then we write $X = m_1 \times S_1 + \cdots + m_n \times S_n$ (as composition factors). If Y is another A-module, then $Y \mid X$ (resp. $Y \nmid X$) means that Y is isomorphic (resp. not isomorphic) to a direct summand of X, (proj) denotes a projective A-module (which we do not need to specify).

We write $B_0(kG)$ for the principal block of the group algebra kG. Given a block B of kG, we write 1_B for the block idempotent of B and C_B for the Cartan matrix of B. We denote by $\mathrm{Irr}(B)$ and $\mathrm{IBr}(B)$, respectively, the sets of all irreducible ordinary and Brauer characters of G belonging to B. If $D \leq G$ is a defect group of the block B, then the integer d such that $|D| = p^d$ is called the defect of B. Assuming $|G| = p^a m$ with $p \nmid m$, if $\chi \in \mathrm{Irr}(G)$ lies in a block of defect d, then the height of χ, denoted by $\mathrm{ht}(\chi)$, is defined to be the exact power of p dividing the integer $\chi(1)/p^{a-d}$. We write $k(B) := |\mathrm{Irr}(B)|$ and $\ell(B) := |\mathrm{IBr}(B)|$ and $k_i(B) := |\{\chi \in \mathrm{Irr}(B) \mid \mathrm{ht}(\chi) = i\}|$ where $\mathrm{ht}(\chi)$ is the height of χ.

We denote by k_G the trivial kG-module. Given a kG-module M and a p-subgroup $Q \leq G$ we denote by $M(Q)$ the Brauer construction of M with respect to Q. (See e.g. [Thé95, p. 219].) When $H \leq G$, N is a kH-module and M is a kG-module, we write $N{\uparrow}^G$ and $M{\downarrow}_H$ respectively for the induction of N to G and the restriction of M to H. For a subgroup $H \leq G$ we denote by $\mathrm{Sc}(G, H)$ the Scott module of kG with respect to H, which by definition is the unique indecomposable direct summand of $k_H{\uparrow}^G$ (up to isomorphism) that has the trivial module k_G as a constituent of its head (or equivalently of its socle). This is a p-permutation module (see [NT88, Chapter 4, §8.4]).

If B_1 and B_2 are two finite-dimensional k-algebras and M is a (B_1, B_2)-bimodule, we also write ${}_{B_1}M_{B_2}$ to emphasize the (B_1, B_2)-bimodule structure on M. Now, if B_1 and B_2 are blocks of kG_1 and kG_2, respectively, then we can view every (B_1, B_2)-bimodule M as a right $k(G_1 \times G_2)$-module via the right $(G_1 \times G_2)$-action defined by $m \cdot (g_1, g_2) := {g_1}^{-1} m g_2$ for every $m \in M$, $g_1 \in G_1$, $g_2 \in G_2$. Furthermore, the blocks B_1 and B_2 are called *splendidly Morita equivalent* (or *source-algebra equivalent*, or *Puig equivalent*), if there is a Morita equivalence between B_1 and B_2 induced by a (B_1, B_2)-bimodule M which is is a

p-permutation module when viewed as a right $k(G_1 \times G_2)$-module. In this case, we write $B_1 \sim_{SM} B_2$. By a result of Puig and Scott, this definition is equivalent to the condition that B_1 and B_2 have source algebras which are isomorphic as interior P-algebras (see [Lin01, Theorem 4.1]). Also, by a result of Puig (see [Lin18, Proposition 9.7.1]), the defect groups of B_1 and B_2 are isomorphic. Hence we may identify them.

3 Finite groups with wreathed Sylow 2-subgroups

To begin with, we collect essential results about finite groups with wreathed Sylow 2-subgroups. In particular, we classify such groups modulo $O_{2'}(G)$. This classification is a byproduct of the results of Alperin–Brauer–Gorenstein in [ABG70].

Lemma 3.1. *Let $P := C_{2^n} \wr C_2$ with $n \geq 2$. Then the 2-rank of P is 2 and $\mathrm{Aut}(P)$ is a 2-group.*

Proof. See e.g. [CG12, p. 5956]. □

For the benefit of legibility we state again Theorem 1.1 of the introduction, before we prove it.

Theorem 3.2. *Let G be a finite group with a Sylow 2-subgroup isomorphic to a wreathed 2-group $C_{2^n} \wr C_2$ for an integer $n \geq 2$ such that $O_{2'}(G) = 1$. Then one of the following holds:*

(WR1) $G \cong C_{2^n} \wr C_2$,
(WR2) $G \cong (C_{2^n} \times C_{2^n}) \rtimes \mathfrak{S}_3$,
(WR3) $G \cong \mathrm{SL}_2^n(q) \rtimes C_d$ *where* $(q-1)_2 = 2^n$ *and* $d \mid f$ *is odd,*
(WR4) $G \cong \mathrm{SU}_2^n(q) \rtimes C_d$ *where* $(q+1)_2 = 2^n$ *and* $d \mid f$ *is odd,*
(WR5) $G \cong \mathrm{PSL}_3(q).H$ *where* $(q-1)_2 = 2^n$, $H \leq C_{(q-1,3)} \times C_d$ *and* $d \mid f$ *is odd, or*
(WR6) $G \cong \mathrm{PSU}_3(q).H$ *where* $(q+1)_2 = 2^n$, $H \leq C_{(q+1,3)} \times C_d$ *and* $d \mid f$ *is odd,*

where $q = r^f$ denotes a power of a prime number r and $f \geq 1$ an integer.

Proof. If G is 2-nilpotent, then Case (WR1) holds since $O_{2'}(G) = 1$. In all other cases, G is a D-group, a Q-group or a QD-group with the notation of [ABG70, Definition 2.1]. Let G be a D-group. Then there exists $K \trianglelefteq G$ of index 2 such that $P \cap K \cong C_{2^n} \times C_{2^n}$. By [Bra64, Theorem 1], $K \cong (C_{2^n} \times C_{2^n}) \rtimes C_3$ and Case (WR2) holds.

If G is a Q-group, then Case (WR3) or (WR4) occurs by Propositions 3.2 and 3.3 of [ABG70]. Finally, let G be a QD-group. Then by [ABG70, Proposition 2.2], $N := O^{2'}(G)$ is simple and the possible isomorphism types of N are given by the main result of [ABG73]. Since $C_G(N) \cap N = Z(N) = 1$ we have $C_G(N) \leq O_{2'}(G) = 1$. The possibilities for $G/N \leq \mathrm{Out}(N)$ can be deduced from [Atlas]. Since $|G/N|$ is odd, no graph automorphism is involved. Hence, $G/N \leq C_{(3,q-1)} \rtimes C_d$ or $G/N \leq C_{(3,q+1)} \rtimes C_d$. In fact, G/N must be abelian since $|G/N|$ is odd. □

Theorem 3.3. *Let G be as in Theorem 3.2 and let $B := B_0(kG)$. With the same labelling of cases as in Theorem 3.2 the following holds:*

(WR1) $\ell(B) = 1$, $k(B) = 2^{2n-1} + 3 \cdot 2^{n-1}$, $k_0(B) = 2^{n+1}$, $k_1(B) = 2^{2n-1} - 2^{n-1}$;
(WR2) $\ell(B) = 2$, $k(B) = (2^{2n-1} + 9 \cdot 2^{n-1} + 4)/3$, $k_0(B) = 2^{n+1}$,
$k_1(B) = (2^{2n-1} - 3 \cdot 2^{n-1} + 4)/3$;

(WR3,4) $\ell(B) = 2$, $k(B) = 2^{2n-1} + 2^{n+1}$, $k_0(B) = 2^{n+1}$, $k_1(B) = 2^{2n-1} - 2^{n-1}$, $k_n(B) = 2^{n-1}$;

(WR5,6) $\ell(B) = 3$, $k(B) = (2^{2n-1} + 3 \cdot 2^{n+1} + 4)/3$, $k_0(B) = 2^{n+1}$, $k_1(B) = (2^{2n-1} - 3 \cdot 2^{n-1} + 4)/3$, $k_n(B) = 2^{n-1}$.

Proof. Cases (WR1) and (WR2) follow from elementary group theory. If Case (WR3) or Case (WR4) of Theorem 3.2 holds, then the numbers follow from [Kue80, Proposition (7.G)]. Suppose now that Case (WR5) or Case (WR6) holds, then the number $k(B)$ follows from [Bra71, Theorem 1A] – here, Brauer even computed the degrees of the ordinary irreducible characters in B – whereas the number $\ell(B)$ can be obtained with [Kue80, Lemma 7.I] for instance. □

4 Preliminaries

We state below several results which will enable us to construct splendid Morita equivalences induced by Scott modules, but which are not restricted to characteristic 2. Therefore, throughout this section we may assume that k is an algebraically closed field of arbitrary characteristic $p > 0$.

Our first main tool to construct splendid Morita equivalences is given by the following Theorem which is an extended version of a well-known result due to Alperin [Alp76] and Dade [Dad77] restated in terms of splendid Morita equivalences.

Theorem 4.1 (Alperin–Dade). *Let $\widetilde{G}_1$ and $\widetilde{G}_2$ be a finite groups and assume $G_1 \trianglelefteq \widetilde{G}_1$, $G_2 \trianglelefteq \widetilde{G}_2$ are normal subgroups such that $\widetilde{G}_1/G_1$, $\widetilde{G}_2/G_2$ are a p'-groups and having a common Sylow p-subgroup $P \in \mathrm{Syl}_p(G_1) \cap \mathrm{Syl}_p(G_2)$ such that $\widetilde{G}_1 = G_1 C_{\widetilde{G}_1}(P)$ and $\widetilde{G}_2 = G_2 C_{\widetilde{G}_2}(P)$. Then the following assertions hold.*

(a) *If $\tilde{e}$ and e denote the block idempotents of $B_0(k\widetilde{G}_1)$ and $B_0(kG_1)$, respectively, then the map $B_0(kG_1) \longrightarrow B_0(k\widetilde{G}_1), a \mapsto a\tilde{e}$ is an isomorphism of k-algebras. Moreover, the right $k[\widetilde{G}_1 \times G_1]$-module $\mathrm{Sc}(\widetilde{G}_1 \times G_1, \Delta P) = B_0(k\widetilde{G}_1){\downarrow}^{\widetilde{G}_1 \times \widetilde{G}_1}_{\widetilde{G}_1 \times G_1} = \tilde{e}k\widetilde{G}_1 = \tilde{e}k\widetilde{G}_1 e$, induces a splendid Morita equivalence between $B_0(k\widetilde{G}_1)$ and $B_0(kG_1)$.*

(b) *The Scott module $\mathrm{Sc}(\widetilde{G}_1 \times \widetilde{G}_2, \Delta P)$ induces a splendid Morita equivalence between $B_0(k\widetilde{G}_1)$ and $B_0(k\widetilde{G}_2)$ if and only if the Scott module $\mathrm{Sc}(G_1 \times G_2, \Delta P)$ induces a splendid Morita equivalence between $B_0(kG_1)$ and $B_0(kG_2)$.*

Proof. Assertion (a) follows from [Alp76, Dad77]. More precisely, the given map is an isomorphism of k-algebras by [Dad77, Theorem] and [Alp76, Theorems 1 and 2] proves that restriction from $\widetilde{G}_1$ to G_1 induces a splendid Morita equivalence. Assertion (b) is given by [KLS22, Lemma 5.1].

□

Lemma 4.2. *Let $\widetilde{G}_1, \widetilde{G}_2$ be finite groups. Assume that $G_1 \trianglelefteq \widetilde{G}_1$ and $G_2 \trianglelefteq \widetilde{G}_2$ are normal subgroups such that $\widetilde{G}_1/G_1$, $\widetilde{G}_2/G_2$ are p'-groups and assume that G_1 and G_2 have a common Sylow p-subgroup P such that $\mathrm{Aut}(P)$ is a p-group. Then, conclusions* (a) *and* (b) *of Theorem 4.1 hold.*

Proof. It suffices to prove that the hypotheses of Theorem 4.1 are satisfied. So, let $i \in \{1,2\}$. Since $\mathrm{Aut}(P)$ is a p-group we have $N_{\widetilde{G}_i}(P) = PC_{\widetilde{G}_i}(P)$. Moreover, by Frattini's argument $\widetilde{G}_i = G_iN_{\widetilde{G}_i}(P)$, thus $\widetilde{G}_i = G_iC_{\widetilde{G}_i}(P)$, as required. ☐

Next, it is well-known that inflation from the quotient by a normal p'-subgroup induces an isomorphism of blocks as k-algebras. In fact, there is splendid Morita equivalence induced by a Scott module and we have the following stronger result.

Lemma 4.3. *Let G_1, G_2 be finite groups with a common Sylow p-subgroup P. Let $N_1 \trianglelefteq G_1$ and $N_2 \trianglelefteq G_2$ be normal p'-subgroups and write $^{-} : G_1 \longrightarrow G_1/N_1 =: \overline{G_1}$, respectively $^{-} : G_2 \longrightarrow G_2/N_2 =: \overline{G_2}$, for the quotient homomorphisms, so that, by abuse of notation, we may identify $\overline{P} = PN_1/N_1 \cong P$ with $\overline{P} = PN_2/N_2 \cong P$. Then the following assertions hold:*

- (a) $\mathrm{Sc}(G_1 \times \overline{G_1}, \Delta P)$ *induces a splendid Morita equivalence between $B_0(kG_1)$ and $B_0(k\overline{G_1})$, where ΔP is identified with $\{(u, \bar{u}) \,|\, u \in P\}$;*
- (b) $\mathrm{Sc}(G_1 \times G_2, \Delta P)$ *induces a splendid Morita equivalence between $B_0(kG_1)$ and $B_0(kG_2)$ if and only if* $\mathrm{Sc}(\overline{G_1} \times \overline{G_2}, \Delta \overline{P})$ *induces a splendid Morita equivalence between $B_0(k\overline{G_1})$ and $B_0(k\overline{G_2})$.*

Proof. (a) By the assumption $N_1 \leq O_{p'}(G_1)$, hence N_1 acts trivially on $B_0(kG_1)$. Thus, $B_0(kG_1)$ and its image $B_0(k\overline{G_1})$ in $k\overline{G_1}$ are isomorphic as interior P-algebras. Part (a) follows then immediately from the fact that $\mathrm{Sc}(G_1 \times \overline{G}_1, \Delta P) = {}_{kG_1}B_0(kG_1)_{k\overline{G_1}}$ (seen as a $(kG_1, k\overline{G_1})$-bimodule). Part (b) follows from (a) and the fact that

$$\mathrm{Sc}(G_1 \times \overline{G_1}, \Delta P) \otimes_{B_0(k\overline{G_1})} \mathrm{Sc}(\overline{G_1} \times \overline{G_2}, \Delta \overline{P}) \otimes_{B_0(k\overline{G_2})} \mathrm{Sc}(\overline{G_2} \times G_2, \Delta P) \cong \mathrm{Sc}(G_1 \times G_2, \Delta P)\,.$$

(See e.g. the proof of [KLS22, Lemma 5.1] for a detailed argument proving this isomorphism.) ☐

The following Lemma is also essential to treat central extensions.

Lemma 4.4. *Let G_1 and G_2 be finite groups having a common Sylow p-subgroup $P \in \mathrm{Syl}_p(G_1) \cap \mathrm{Syl}_p(G_2)$. Assume moreover that $Z_1 \leq Z(G_1)$ and $Z_2 \leq Z(G_2)$ are central subgroups such that $P \cap Z_1 = P \cap Z_2$ (after identification of the chosen Sylow p-subgroups of G_1 and G_2). Set $\overline{G_1} := G_1/Z_1$ and $\overline{G_2} := G_2/Z_2$. Then the subgroup $\overline{P} := PZ_1/Z_1 (\cong P/(P \cap Z_1) \cong P/P \cap Z_2) \cong PZ_2/Z_2)$ can be considered as a common Sylow p-subgroup of $\overline{G_1}$ and $\overline{G_2}$. Then,* $\mathrm{Sc}(G_1 \times G_2, \Delta P)$ *induces a splendid Morita equivalence between $B_0(kG_1)$ and $B_0(kG_2)$ if and only if* $\mathrm{Sc}(\overline{G_1} \times \overline{G_2}, \Delta \overline{P})$ *induces a splendid Morita equivalence between $B_0(k\overline{G_1})$ and $B_0(k\overline{G_2})$.*

Proof. Let $i \in \{1,2\}$. Clearly, we have $Z_i = (P \cap Z_i) \times O_{p'}(Z_i)$ and

$$\overline{G_i} = G_i/Z_i \cong (G_i/(P \cap Z_i)) \,/\, (Z_i/(P \cap Z_i)) =: \overline{\overline{G_i}}\,.$$

Write $\widetilde{P}$ for the image of P in the quotients $G_i/(P \cap Z_i)$ and write $\overline{\overline{P}}$ for the image of P in the quotients $\overline{\overline{G_i}}$. Now, on the one hand, by Theorem A.2, the Scott module $\mathrm{Sc}(G_1 \times G_2, \Delta P)$ induces a splendid Morita equivalence between $B_0(kG_1)$ and $B_0(kG_2)$ if and only if $\mathrm{Sc}(G_1/(P \cap Z_1) \times G_2/(P \cap Z_2), \Delta \widetilde{P})$ induces a splendid Morita equivalence between $B_0(k[G_1/(P \cap Z_1)])$ and $B_0(k[G_2/(P \cap Z_2)])$, which by Lemma 4.3(b) happens if and only if $\mathrm{Sc}(\overline{\overline{G_1}} \times \overline{\overline{G_2}}, \Delta \overline{\overline{P}})$ induces a splendid Morita equivalence between $B_0(k\overline{\overline{G_1}})$ and $B_0(k\overline{\overline{G_2}})$. The claim follows. ☐

The next theorem is a standard method, called the "gluing method", which was already applied in [KL20a, KLS22]. It relies on gluing results, allowing us to construct stable equivalences of Morita type, and is a slight variation of different results of the same type due to Broué, Rouquier, Linckelmann and Rickard. See e.g. [Bro94, 6.3.Theorem], [Rou01, Theorem 5.6] and [Lin01, Theorem 3.1].

Theorem 4.5. *Let G_1 and G_2 be finite groups with a common Sylow p-subgroup P satisfying $\mathcal{F}_P(G_1) = \mathcal{F}_P(G_2)$. Then, $M := \mathrm{Sc}(G_1 \times G_2, \Delta P)$ induces a splendid Morita equivalence between $B_0(kG_1)$ and $B_0(kG_2)$ provided the following two conditions are satisfied:*

(I) *for every subgroup $Q \leq P$ of order p, the bimodule $M(\Delta Q)$ induces a Morita equivalence between $B_0(k\,C_{G_1}(Q))$ and $B_0(k\,C_{G_2}(Q))$; and*

(II) *for every simple $B_0(kG_1)$-module S_1, the $B_0(kG_2)$-module $S_1 \otimes_{B_0(kG_1)} M$ is again simple.*

Proof. By [KL20a, Lemma 4.1], Condition (I) is equivalent to the fact that M induces a stable equivalence of Morita type between $B_0(kG_1)$ and $B_0(kG_2)$. Therefore, applying [Lin96, Theorem 2.1], Condition (II) now implies that M induces a Morita equivalence between $B_0(kG_1)$ and $B_0(kG_2)$. This equivalence is necessarily splendid since M is a p-permutation module by definition. □

Lemma 4.6. *Let A be a finite-dimensional k-algebra. Let X be an A-module and let Y be an A-submodule such that X/Y and $\mathrm{soc}(Y)$ are both simple. If Y is not a direct summand of X, then $\mathrm{soc}(X) = \mathrm{soc}(Y)$, and hence X is indecomposable.*

Proof. Since $\mathrm{soc}(X)$ and $\mathrm{soc}(Y)$ are semisimple, we have $\mathrm{soc}(X) \cap Y = \mathrm{soc}(Y)$ and $\mathrm{soc}(X) = \mathrm{soc}(Y) \oplus S$ where S is a submodule of X. Thus $S \cap Y = S \cap \mathrm{soc}(X) \cap Y = S \cap \mathrm{soc}(Y) = 0$. Hence, either $S = 0$ and $\mathrm{soc}(X) = \mathrm{soc}(Y)$, or Y is a submodule of $S \oplus Y$ and so $S \oplus Y = X$. □

Finally, the next lemma is often called the "stripping-off method". It will be used to verify Condition (II) of Theorem 4.5 in concrete cases.

Lemma 4.7 ([KMN11, Lemma A.1])**.** *Let A and B be self-injective finite-dimensional k-algebras. Let $F : \text{mod-}A \longrightarrow \text{mod-}B$ be a covariant functor satisfying the following conditions:*

(C1) *F is exact;*

(C2) *if X is a projective A-module, then $F(X)$ is a projective B-module;*

(C3) *F realises a stable equivalence from mod-A to mod-B.*

Then, the following assertions hold.

(a) (Stripping-off method, case of socles.) *Let X be a projective-free A-module, and write $F(X) = Y \oplus (\mathrm{proj})$ where Y is a projective-free B-module. Let S be a simple A-submodule of X and set $T := F(S)$. If T is a simple non-projective B-module, then there exists a B-submodule W of $F(X)$ such that $W \cong Y$, $T \subseteq W$ and*
$$F(X/S) \cong W/T \oplus (\mathrm{proj})\,.$$

(b) (Stripping-off method, case of radicals.) *Let X be a projective-free A-module, and write $F(X) = Y \oplus R$ where Y is a projective-free B-module and R is a projective B-module. Let X' be an A-submodule of X such that X/X' is simple and let $\pi : X \longrightarrow X/X'$ be the quotient homomorphism. If $T := F(X/X')$ is a simple B-module, then there exists a B-submodule R' of $F(X)$ such that $R' \cong R$,*

$R' \subseteq \ker(F(\pi))$, $F(X) = Y \oplus R'$ *and*

$$\ker\left(F(X) \overset{F(\pi)}{\twoheadrightarrow} F(X/X')\right) = \ker\left(Y \overset{F(\pi)|_Y}{\twoheadrightarrow} F(X/X')\right) \oplus (\mathrm{proj})\,.$$

5 Groups of type (W3(n)) and (W4(n))

Hypothesis 5.1. From now on and until the end of this manuscript we assume that the algebraically closed field k has characteristic $p = 2$. Furthermore, G, G_1, G_2, $\mathcal{G}_1$, $\mathcal{G}_2$, G, G_1 and G_2 always denote finite groups with a common Sylow 2-subgroup $P \cong C_{2^n} \wr C_2$, where $n \geq 2$ is a fixed integer. In other words, we choose a Sylow 2-subgroup of each of these groups and we identify them for simplicity. Moreover, q, q_1 and q_2 are (possibly different) positive powers of odd prime numbers.

In this section and the next two ones, we prove Theorem 1.2(b) through a case-by-case analysis. We start with the groups of types (W3(n)) and (W4(n)), for which we reduce the problem to the classification of principal blocks with dihedral defect groups up to splendid Morita equivalence obtained in [KL20a, Theorem 1.1]. The group theory setting to keep in mind is described in the following remark.

Remark 5.2. For any positive power q of an odd prime number [ABG70, p.4] shows that we have the following inclusions of normal subgroups with the given indices:

$$\begin{array}{ccc}
\mathrm{GL}_2(q) & & \mathrm{GU}_2(q) \\
{\scriptstyle (q-1)/2^n}\Big| & & \Big|{\scriptstyle (q+1)/2^n} \\
\mathrm{SL}_2^n(q) & & \mathrm{SU}_2^n(q) \\
{\scriptstyle 2^n}\Big| & & \Big|{\scriptstyle 2^n} \\
\mathrm{SL}_2(q) & \cong & \mathrm{SU}_2(q)
\end{array}$$

Proposition 5.3. *For each $i \in \{1, 2\}$ let $G_i := \mathrm{SL}_2^n(q_i)$, $\mathcal{G}_i := \mathrm{GL}_2(q_i)$ and assume that $(q_i - 1)_2 = 2^n$. Then, the following assertions hold:*

(a) $\mathrm{Sc}(\mathcal{G}_1 \times \mathcal{G}_2, \Delta P)$ *induces a splendid Morita equivalence between $B_0(k\mathcal{G}_1)$ and $B_0(k\mathcal{G}_2)$;*

(b) $\mathrm{Sc}(G_1 \times G_2, \Delta P)$ *induces a splendid Morita equivalence between $B_0(kG_1)$ and $B_0(kG_2)$.*

Proof. Elementary calculations yield $G_i \lhd \mathcal{G}_i$ and $|\mathcal{G}_i/G_i| = (q_i - 1)/2^n$ for each $i \in \{1, 2\}$ (see Remark 5.2). In particular both indices are odd. Hence, by Lemma 3.1 and Lemma 4.2, assertion (b) follows from assertion (a), so it suffices to prove (a).

Now, $P \cap Z(\mathcal{G}_1) = P \cap Z(\mathcal{G}_2) = Z(P)$, so $\overline{P} := (PZ(\mathcal{G}_1))/Z(\mathcal{G}_1) \cong (PZ(\mathcal{G}_2))/Z(\mathcal{G}_2)$, and hence, up to identification, we can consider that $\overline{P} \in \mathrm{Syl}_2(\mathcal{G}_1/Z(\mathcal{G}_1)) \cap \mathrm{Syl}_2(\mathcal{G}_2/Z(\mathcal{G}_2))$. Moreover, we have

$$\overline{P} \cong P/Z(P) \cong D_{2^{n+1}}\,,$$

see e.g. [Kue80, (2.A) Lemma (iii)]. Since $\mathcal{G}_i/Z(\mathcal{G}_i) \cong \mathrm{PGL}_2(q_i)$ for each $i \in \{1, 2\}$, assertion (a) now follows directly from Lemma 4.4 and [KL20a, Theorem 1.1]. □

Proposition 5.4. *For each $i \in \{1, 2\}$ let $G_i := \mathrm{SU}_2^n(q_i)$, $\mathcal{G}_i := \mathrm{GU}_2(q_i)$ and assume that $(q_i + 1)_2 = 2^n$. Then, the following assertions hold:*

(a) $\mathrm{Sc}(\mathcal{G}_1\times\mathcal{G}_2,\Delta P)$ *induces a splendid Morita equivalence between* $B_0(k\mathcal{G}_1)$ *and* $B_0(k\mathcal{G}_2)$*;*
(b) $\mathrm{Sc}(G_1\times G_2,\Delta P)$ *induces a splendid Morita equivalence between* $B_0(kG_1)$ *and* $B_0(kG_2)$*.*

Proof. In this case $G_i \lhd \mathcal{G}_i$ and $|\mathcal{G}_i/G_i| = (q_i+1)/2^n$ for each $i\in\{1,2\}$ (See Remark 5.2). Thus both indices are odd. Again by Lemma 3.1 and Lemma 4.2, it suffices to prove (a).

Now, $P\cap Z(\mathcal{G}_1) = P\cap Z(\mathcal{G}_2) = Z(P)$. Thus $\overline{P} := (PZ(\mathcal{G}_1))/Z(\mathcal{G}_1) \cong (PZ(\mathcal{G}_2))/Z(\mathcal{G}_2)$ and we can consider that $\overline{P}\in \mathrm{Syl}_2(\mathcal{G}_1/Z(\mathcal{G}_1))\cap\mathrm{Syl}_2(\mathcal{G}_2/Z(\mathcal{G}_2))$. As in the previous proof,

$$\overline{P}\cong P/Z(P)\cong D_{2^{n+1}}\,.$$

Next, for each for $i\in\{1,2\}$ we have an isomorphism $\mathrm{SU}_2(q_i)\cong\mathrm{SL}_2(q_i)$, and hence $\mathrm{PSU}_2(q_i)\cong\mathrm{PSL}_2(q_i)$. Furthermore, since q_i is odd, $\mathrm{PGL}_2(q_i)=\mathrm{PSL}_2(q_i).2$ (where 2 denotes the cyclic group of order 2 generated by the diagonal automorphism of $\mathrm{PSL}_2(q_i)$) by Steinberg's result (see [Suz86, Chap.6 (8.8), p. 511 and Theorem 8.11]). In other words, we have

$$\mathcal{G}_i/Z(\mathcal{G}_i)=\mathrm{PGU}_2(q_i)\cong\mathrm{PGL}_2(q_i)\,. \tag{1}$$

Therefore, assertion (a) follows immediately from Lemma 4.4 and [KL20a, Theorem 1.1], proving the Proposition. □

6 Groups of type (W5(n))

We now turn to the groups of type (W5(n)). We continue using Hypothesis 5.1.

Notation 6.1. Throughout this section we let $i\in\{1,2\}$ be arbitrary and set $G_i := \mathrm{PSL}_3(q_i)$, $\mathsf{G}_i := \mathrm{SL}_3(q_i)$ and $\widetilde{G}_i := \mathrm{GL}_3(q_i)$ where we assume that $(q_i-1)_2 = 2^n$. After identification, we may assume that G_1, G_2, G_1 and G_2 have a common Sylow 2-subgroup P isomorphic to $C_{2^n}\wr C_2$. Then,

$$B_0(kG_i)\sim_{SM} B_0(k\mathsf{G}_i) \tag{2}$$

where the splendid Morita equivalence is induced by inflation (as $Z(\mathrm{SL}_3(q_i))\cong C_{(3,q_i-1)}$ is a $2'$-group). Using [GM20, Proposition 4.3.1 and Remark 4.2.1] we know that $B_0(k\mathsf{G}_i)$ contains three unipotent characters, namely

$$1_{\mathsf{G}_i},\chi_{q_i^2+q_i},\chi_{q_i^3}\,,$$

where we use the convention that the indices denote the degrees, whereas those lying in $B_0(k\widetilde{G}_i)$ can be written as

$$1_{\widetilde{G}_i},\widetilde{\chi}_{q_i^2+q_i},\widetilde{\chi}_{q_i^3}$$

and satisfy $1_{\widetilde{G}_i}{\downarrow_{\mathsf{G}_i}} = 1_{\mathsf{G}_i}$, $\widetilde{\chi}_{q_i^2+q_i}{\downarrow_{\mathsf{G}_i}} = \chi_{q_i^2+q_i}$ and $\widetilde{\chi}_{q_i^3}{\downarrow_{\mathsf{G}_i}} = \chi_{q_i^3}$. (We also refer to [Ste51], [Jam86, 7.19. Theorem(i)], that first described these characters and their degrees.)
We obtain from [Jam90, §4] and the above that $3=\ell(B_0(k\widetilde{G}_i))=\ell(B_0(k\mathsf{G}_i))$ and we may write

$$\mathrm{Irr}_k(B_0(k\mathsf{G}_i)) =: \{k_{\mathsf{G}_i},S_i,T_i\}\quad\text{ and }\quad \mathrm{Irr}_k(B_0(k\widetilde{G}_i)) =: \{k_{\widetilde{G}_i},\widetilde{S}_i,\widetilde{T}_i\}\,,$$

where $S_i=\widetilde{S}_i{\downarrow_{\mathsf{G}_i}}$ and $T_i=\widetilde{T}_i{\downarrow_{\mathsf{G}_i}}$. Moreover, by [Jam90, p. 253], the part of the 2-decomposition matrix of $B_0(k\widetilde{G}_i)$ whose rows are labelled by the unipotent characters is as follows:

	$k_{\widetilde{G}_i}$	$\widetilde{S}_i$	$\widetilde{T}_i$
$1_{\widetilde{G}_i}$	1	.	.
$\widetilde{\chi}_{q_i^2+q_i}$	.	1	.
$\widetilde{\chi}_{q_i^3}$	1	.	1

(This is the case Δ_3 with $n = 3$, $e = 2$ and $p \geq 2$.)

We start by describing some trivial source modules belonging to the principal 2-block of $\mathrm{SL}_3(q_i)$ which we will use in the sequel.

Lemma 6.2. *The principal block $B_0(k\mathsf{G}_i)$ contains, amongst others, the following trivial source modules:*

(a) *the trivial module k_{G_i}, with vertex P and affording the trivial character 1_{G_i};*
(b) *the simple module S_i, having $Q := C_{2^n} \times C_{2^n} \leq P$ as a vertex, and affording the character $\chi_{q_i^2+q_i}$;*
(c) *the Scott module $Sc(\mathsf{G}_i, Q)$ with vertex Q, satisfying $Sc(\mathsf{G}_i, Q) \ncong S_i$;*
(d) *the Scott module $Sc(\mathsf{G}_i, \mathbb{B}_i)$ on a Borel subgroup $\mathbb{B}_i$ of G_i, which is uniserial with composition series*

$$\boxed{\begin{matrix} k_{\mathsf{G}_i} \\ T_i \\ k_{\mathsf{G}_i} \end{matrix}}$$

and affords the character $1_{\mathsf{G}_i} + \chi_{q_i^3}$.

Proof. First we note that it is clear that all the given modules belong to the principal block as at least one of their constituents obviously does.
(a) It is clear that the trivial module is a trivial source module with vertex P affording the trivial character.
(b) As the restriction of a trivial source module is always a trivial source module, to prove that S_i is a trivial source module affording $\chi_{q_i^2+q_i}$, it is enough to prove that the $k\widetilde{G}_i$-module $\widetilde{S}_i$ is a trivial source module affording $\widetilde{\chi}_{q_i^2+q_i}$. (See e.g. [Las23, §4] for these properties.) Now, [Ste51, pp. 228–229] shows that $1_{\widetilde{G}_i} + \widetilde{\chi}_{q^2+q}$ is a permutation character. More precisely there exists a subgroup $\widetilde{H}_i \leq \widetilde{G}_i$ such that $\widetilde{H}_i \cong (C_{q_i} \times C_{q_i}) \rtimes \mathrm{GL}_2(q_i)$, $|\widetilde{G}_i : \widetilde{H}_i| = 1 + q_i + q_i^2$ and $1_{\widetilde{H}_i}\uparrow^{\widetilde{G}_i} = 1_{\widetilde{G}_i} + \widetilde{\chi}_{q_i^2+q_i}$. Thus, setting $X_i := k_{\widetilde{H}_i}\uparrow^{\widetilde{G}_i}$, the decomposition matrix given in Notation 6.1 implies that

$$X_i = k_{\widetilde{G}_i} + \widetilde{S}_i \text{ (as composition factors)}\,.$$

Then $X_i = k_{\widetilde{G}_i} \oplus \widetilde{S}_i$ as k_{G_i} must occur as a composition factor of the socle and of the head, proving that $\widetilde{S}_i$ is a trivial source module affording the character $\widetilde{\chi}_{q_i^2+q_i}$. Finally, using [Lan83, II Lemma 12.6(iii)] and the character table of $\mathrm{SL}_3(q_i)$ in [SF73] we can read from the values of the character $\chi_{q_i^2+q_i}$ at non-trivial 2-elements that $Q = C_{2^n} \times C_{2^n} \leq P$ is a vertex of S_i.
(c) The Scott module $\mathrm{Sc}(\mathsf{G}_i, Q)$ is a trivial source module with vertex Q and clearly $S_i \ncong \mathrm{Sc}(\mathsf{G}_i, Q)$, as a Scott module always has a trivial constituent in its head by definition.
(d) [Ste51, pp. 228–229] also shows that $1_{\widetilde{G}_i} + 2\widetilde{\chi}_{q_i^2+q_i} + \widetilde{\chi}_{q_i^3}$ is a permutation character. More precisely, there is a Borel subgroup $\widetilde{\mathbb{B}}_i \leq \widetilde{G}_i$ such that $1_{\widetilde{\mathbb{B}}_i}\uparrow^{\widetilde{G}} = 1_{\widetilde{G}_i} + 2\widetilde{\chi}_{q_i^2+q_i} + \widetilde{\chi}_{q_i^3}$.

Setting $Y_i := k_{\widetilde{\mathbb{B}}_i}\uparrow^{\widetilde{G}_i}$ we obtain from the decomposition matrix in Notation 6.1 that

$$Y_i = 2 \times k_{\widetilde{G}_i} + 2 \times \widetilde{S}_i + \widetilde{T}_i \quad \text{(as composition factors)}.$$

As both Y_i and $\widetilde{S}_i$ are trivial source modules, we have

$$\dim_k \operatorname{Hom}_{k\widetilde{G}_i}(Y_i, \widetilde{S}_i) = \dim_k \operatorname{Hom}_{k\widetilde{G}_i}(\widetilde{S}_i, Y_i) = \langle 1_{\widetilde{G}_i} + 2\widetilde{\chi}_{q_i^2+q_i} + \widetilde{\chi}_{q_i^3}, \widetilde{\chi}_{q_i^2+q_i}\rangle_{\widetilde{G}_i} = 2$$

(see [Lan83, II Theorem 12.4(iii)]), implying that $\widetilde{S}_i \oplus \widetilde{S}_i \mid \operatorname{soc}(Y_i)$ and $\widetilde{S}_i \oplus \widetilde{S}_i \mid \operatorname{hd}(Y_i)$. Thus, there exists a submodule U_i of Y_i such that $Y_i \cong \widetilde{S}_i \oplus \widetilde{S}_i \oplus U_i$ and hence U_i is a trivial source module with composition factors $2 \times k_{\widetilde{G}_i} + T_i$ and U_i affords the ordinary character $1_{\widetilde{G}_i} + \widetilde{\chi}_{q_i^3}$. Applying [Lan83, II Theorem 12.4(iii)] again, we get

$$\dim_k \operatorname{Hom}_{k\widetilde{G}}(U_i, U_i) = \langle 1_{\widetilde{G}_i} + \widetilde{\chi}_{q_i^3}, 1_{\widetilde{G}_i} + \widetilde{\chi}_{q_i^3}\rangle_{\widetilde{G}_i} = 2$$

and

$$\dim_k \operatorname{Hom}_{k\widetilde{G}_i}(k_{\widetilde{G}_i}, U_i) = \langle 1_{\widetilde{G}_i}, 1_{\widetilde{G}_i} + \widetilde{\chi}_{q_i^3}\rangle_{\widetilde{G}_i} = 1 = \dim_k \operatorname{Hom}_{k\widetilde{G}}(U_i, k_{\widetilde{G}_i})\,.$$

It follows that

$$U_i = \boxed{\begin{matrix} k_{\widetilde{G}_i} \\ \widetilde{T} \\ k_{\widetilde{G}_i} \end{matrix}} = \operatorname{Sc}(\widetilde{G}_i, \widetilde{\mathbb{B}}_i)$$

and setting $\mathbb{B}_i := \widetilde{\mathbb{B}}_i \cap \mathsf{G}_i$ yields assertion (d). □

We can now prove Theorem 1.2(b) for the groups of types (W5(n)).

Proposition 6.3. *The Scott module* $\operatorname{Sc}(G_1 \times G_2, \Delta P)$ *induces a splendid Morita equivalence between* $B_0(kG_1)$ *and* $B_0(kG_2)$.

Proof. Below $i \in \{1, 2\}$. First, we observe that by Lemma 4.4, $\operatorname{Sc}(G_1 \times G_2, \Delta P)$ induces a splendid Morita equivalence between the principal blocks $B_0(kG_1)$ and $B_0(kG_2)$ if and only if $\operatorname{Sc}(\mathsf{G}_1 \times \mathsf{G}_2, \Delta P) =: M$ induces a splendid Morita equivalence between $B_1 := B_0(k\mathsf{G}_1)$ and $B_2 := B_0(k\mathsf{G}_2)$. Thus, we may work with G_i instead of G_i (for $i \in \{1, 2\}$). Now, observe that $\mathcal{F}_P(\mathsf{G}_1) = \mathcal{F}_P(\mathsf{G}_2)$ and all involutions in G_i are G_i-conjugate (see e.g. [CG12, Theorem 5.3] and [ABG70, Proposition 2 on p.11]). Thus, it follows that it now suffices to prove that Conditions (I) and (II) of Theorem 4.5 hold.

Condition (I). By the above we only need to consider one involution in P, so we choose an involution $z \in Z(P)$, and set $C_i := C_{\mathsf{G}_i}(z)$. Clearly, $C_i \cong \mathrm{GL}_2(q_i)$ and again, up to identification, we see $P \in \mathrm{Syl}_2(\mathcal{G}_1) \cap \mathrm{Syl}_2(\mathcal{G}_2)$ (see Remark 5.2). We have to prove that $M(\Delta\langle z\rangle)$ induces a Morita equivalence between $B_0(kC_1)$ and $B_0(kC_2)$. Now, recall that $M_z := \operatorname{Sc}(C_1 \times C_2, \Delta P)$ induces a splendid Morita equivalence between $B_0(kC_1)$ and $B_0(kC_2)$ by Proposition 5.3(a). Moreover, obviously, it is always true that $M_z \mid M(\Delta\langle z\rangle)$, and we obtain that equality $M_z = M(\Delta\langle z\rangle)$ holds by the Brauer indecomposability of M proved in [KT21, Theorem 1.1]. Thus Condition (I) is verified.

Condition (II). We have to prove that the functor $-\otimes_{B_1} M$ maps the simple B_1-modules to the simple B_2-modules. First, we have $k_{\mathsf{G}_1} \otimes_{B_1} M \cong k_{\mathsf{G}_2}$ by [KL20a, Lemma 3.4(a)]. Next, as $N_{\mathsf{G}_i}(Q)/Q \cong \mathfrak{S}_3$, there are precisely $|\mathfrak{S}_3|_2 = 2$ non-isomorphic trivial source $k\mathsf{G}_i$-modules (see e.g. [Las23, Theorem 4.6(c)]), namely the modules $\operatorname{Sc}(\mathsf{G}_i, Q)$ and S_i, both belonging to the principal block by Lemma 6.2. Now, on the one hand, we know from [KL20a, Theorem 2.1(a)] that $S_1 \otimes_{B_1} M =: V$ is indecomposable and non-projective, and on the other hand we know from [KL20a, Lemma 3.4(b)] that V is a trivial source module with vertex Q. Thus V is either $\mathrm{Sc}(\mathsf{G}_2, Q)$ or S_2. However, $\mathrm{Sc}(\mathsf{G}_1, Q) \otimes_{B_1} M \cong \mathrm{Sc}(\mathsf{G}_2, Q) \oplus (\mathsf{proj})$ by [KL20a, Lemma 3.4(c)]. Hence, it follows immediately that

$$S_1 \otimes_{B_1} M \cong S_2 \,.$$

It remains to treat T_1. By our assumption, $(q-1)_2 = (q_2-1)_2 = 2^n$, so the Sylow 2-subgroups of $\mathbb{B}_1$ and $\mathbb{B}_2$ are isomorphic, meaning that the Scott modules $\mathrm{Sc}(\mathsf{G}_1, \mathbb{B}_1)$ and $\mathrm{Sc}(\mathsf{G}_2, \mathbb{B}_2)$ have isomorphic vertices (see e.g. [NT88, Corollary 4.8.5]). Therefore, [KL20a, Lemma 3.4(c)] together with Lemma 6.2(d) yield

$$\boxed{\begin{matrix} k_{\mathsf{G}_1} \\ T_1 \\ k_{\mathsf{G}_1} \end{matrix}} \otimes_{B_1} M = \mathrm{Sc}(\mathsf{G}_1, \mathbb{B}_1) \otimes_{B_1} M \cong \mathrm{Sc}(\mathsf{G}_2, \mathbb{B}_2) \oplus (\mathsf{proj}) = \boxed{\begin{matrix} k_{\mathsf{G}_2} \\ T_2 \\ k_{\mathsf{G}_2} \end{matrix}} \oplus (\mathsf{proj})$$

and Lemma 4.7 implies that $T_1 \otimes_{B_1} M \cong T_2 \oplus (\mathsf{proj})$. However, again [KL20a, Theorem 2.1(a)] tells us that $T_1 \otimes_{B_1} M$ is indecomposable non-projective, proving that

$$T_1 \otimes_{B_1} M \cong T_2 \,.$$

Thus, Condition (II) is verified and the proposition is proved. □

7 Groups of type (W6(n))

Finally, we examine the groups of type (W6(n)), and we continue using Hypothesis 5.1. Our aim is to prove Theorem 1.2(b) for such groups. However, in order to reach this aim, first we start by collecting some information about the principal 2-block of $\mathrm{PGU}_3(q)$ and about some of its modules.

Notation 7.1. Throughout this section, given a positive power q of prime number satisfying $(q+1)_2 = 2^n$, we set the following notation. The 3-dimensional projective unitary group is

$$\mathrm{GU}_3(q) = \{(a_{rs}) \in \mathrm{GL}_3(q^2) \mid (a_{sr}) w_0 (a_{rs}^q) = w_0\} \qquad \text{with} \qquad w_0 := \left(\begin{smallmatrix} 0&0&1 \\ 0&1&0 \\ 1&0&0 \end{smallmatrix}\right),$$

$\mathsf{G} := \mathsf{G}(q) := \mathrm{PGU}_3(q) = \mathrm{GU}_3(q)/Z(\mathrm{GU}_3(q))$ where $Z(\mathrm{GU}_3(q))$ consists of the scalar matrices in $\mathrm{GU}_3(q)$, and $\mathrm{PSU}_3(q) =: G(q)$ is the commutator subgroup of $\mathrm{PGU}_3(q)$, which is a normal subgroup of index $(3, q+1)$. Furthermore, we let $\mathbb{B} := \mathbb{B}(q)$ denote the Borel subgroup of $\mathrm{GU}_3(q)$ defined by $\mathbb{B}(q) = \mathbb{T}(q)\mathbb{U}(q)$, with $\mathbb{T}(q) := \{\mathrm{diag}(\zeta^{-1}, 1, \zeta^q) \mid \zeta \in \mathbb{F}_{q^2}^\times\}$ and

$$\mathbb{U}(q) := \left\{ \left(\begin{smallmatrix} 1&0&0 \\ \alpha&1&0 \\ \beta&-\alpha^q&1 \end{smallmatrix}\right) \in \mathrm{GU}_3(q) \mid \alpha, \beta \in \mathbb{F}_{q^2} \text{ and } \alpha^{q+1} + \beta^q + \beta = 1 \right\}.$$

It is clear that $\mathbb{B} \cap Z(\mathrm{GU}_3(q)) = 1$, thus we may, and we do, identify $\mathbb{B}$ with a subgroup of $\mathrm{PGU}_3(q)$.
Next, we observe that [Bra71, Theorem 1A] gives us the number of ordinary characters in the principal 2-block of G and their degrees. Moreover, using [SF73] and [Gec90, Table 1.1 and Table 3.1], or CHEVIE [GHL+96] it is easy to compute central characters and we have that $B_0(k\mathsf{G})$ contains the following ordinary irreducible characters, in the notation

of [Gec90]:

	condition	number of characters
1_{G}		1
$\chi_{q(q-1)}$		1
χ_{q^3}		1
$\chi^{(u)}_{q^2-q+1}$	$u \equiv 0 \pmod{(q+1)_{2'}}$	$2^n - 1$
$\chi^{(u)}_{q(q^2-q+1)}$	$u \equiv 0 \pmod{(q+1)_{2'}}$	$2^n - 1$
$\chi^{(u,v)}_{(q-1)(q^2-q+1)}$	$u, v \equiv 0 \pmod{(q+1)_{2'}}$	$(2^n - 1)(2^{n-1} - 1)/3$
$\chi^{(u)}_{q^3+1}$	$u \equiv 0 \pmod{(q+1)_{2'}}$	2^{n-1}

where the subscripts denote the degrees. Finally, the principal block of $k\mathsf{G}$ contains precisely three pairwise non-isomorphic simple modules and we write

$$\mathrm{Irr}_k(B_0(k\mathsf{G})) = \{k_{\mathsf{G}}, \varphi, \theta\}$$

as in [His04, Theorem 4.1] where the simples and their Brauer characters are identified for simplicity.

Lemma 7.2. *With the notation of Notation 7.1, the decomposition matrix of the principal 2-block of* $\mathsf{G} = \mathrm{PGU}_3(q)$ *is as follows:*

	k_{G}	φ	θ	number of characters
1_{G}	1	.	.	1
$\chi_{q(q-1)}$	.	1	.	1
χ_{q^3}	1	2	1	1
$\chi^{(u)}_{q^2-q+1}$	1	1	.	$2^n - 1$
$\chi^{(u)}_{q(q^2-q+1)}$	1	1	1	$2^n - 1$
$\chi^{(u,v)}_{(q-1)(q^2-q+1)}$	.	.	1	$(2^n - 1)(2^{n-1} - 1)/3$
$\chi^{(u)}_{q^3+1}$	2	2	1	2^{n-1}

Proof. To start with, [His04, Appendix] gives us the unipotent part of the decomposition matrix. (See also [GJ11, Table 4.5].) Then, direct computations using [Gec90, Table 1.1 and Table 3.1] (see also [SF73]) or CHEVIE [GHL+96] yield the remaining entries. In particular, it follows easily from the character table that any two irreducible characters of the same degree have the same reduction modulo 2. □

Corollary 7.3. *The* $B_0(k\mathsf{G})$*-simple modules* φ *and* θ *are not trivial source modules.*

Proof. It follows from the decomposition matrix of $B_0(k\mathsf{G})$ in Lemma 7.2 that φ and θ are liftable modules. Moreover, any lift of φ to an $\mathcal{O}\mathsf{G}$-lattice affords the unipotent character $\chi_{q(q-1)}$, and any lift of θ to an $\mathcal{O}\mathsf{G}$-lattice affords one of the characters $\chi^{(u,v)}_{(q-1)(q^2-q+1)}$ of degree $(q-1)(q^2-q+1)$. However, it follows from [Lan83, II Theorem 12.4(iii)] that neither $\chi_{q(q-1)}$ nor the characters $\chi^{(u,v)}_{(q-1)(q^2-q+1)}$ can be the characters of trivial source modules, because it is easily checked from the character table that these characters take strictly negative values at some 2-elements. (See e.g. [Gec90, Table 3.1].) □

Next we collect useful information about the permutation module $k_{\mathbb{B}}\!\uparrow^{\mathsf{G}}$ and the 2nd Heller translate $\Omega^2(k_{\mathsf{G}})$, based on ideas of [OW02, pp. 259–260 and p. 263] and which complements the information provided in [His04, pp. 227–228].

Lemma 7.4. *Assume* $\mathsf{G} = \mathrm{PGU}_3(q)$ *and set* $X := \Omega^2(k_{\mathsf{G}})$*. Then, the following assertions hold:*

(a) *the permutation module* $k_{\mathbb{B}}\!\uparrow^{\mathsf{G}}$ *is a trivial source module affording the ordinary character* $1_{\mathbb{B}}\!\uparrow^{\mathsf{G}} = 1_{\mathsf{G}} + \chi_{q^3}$ *and satisfying*

$$k_{\mathbb{B}}\!\uparrow^{\mathsf{G}} \;=\; \begin{array}{|c|}\hline k_{\mathsf{G}} \\ \varphi \\ \theta \\ \varphi \\ k_{\mathsf{G}} \\ \hline\end{array} \;=\; \mathrm{Sc}(\mathsf{G},\mathbb{B}) \;=\; \mathrm{Sc}(\mathsf{G},Q)$$

where $Q \in \mathrm{Syl}_2(\mathbb{B})$ *is such that* $Q \cong C_{2^{n+1}}$ *and we may assume that* $Q \leq P$*;*

(b) *no indecomposable direct summand* U *of* $\varphi{\downarrow}_{\mathbb{B}}$ *or* $\theta{\downarrow}_{\mathbb{B}}$ *belongs to* $B_0(k\mathbb{B})$*;*

(c) $\mathrm{Ext}^1_{k\mathsf{G}}(k_{\mathsf{G}}, k_{\mathsf{G}}) = 0$*;*

(d) $\dim_k \mathrm{Ext}^1_{k\mathsf{G}}(k_{\mathsf{G}}, \varphi) = \dim_k \mathrm{Ext}^1_{k\mathsf{G}}(\varphi, k_{\mathsf{G}}) = 1$*;*

(e) $\mathrm{Ext}^1_{k\mathsf{G}}(k_{\mathsf{G}}, \theta) = \mathrm{Ext}^1_{k\mathsf{G}}(\theta, k_{\mathsf{G}}) = 0$*;*

(f) $\mathrm{hd}(\Omega(k_{\mathsf{G}})) = \varphi$ *and so there exists a surjective* $k\mathsf{G}$*-homomorphism* $P(\varphi) \twoheadrightarrow \Omega(k_{\mathsf{G}})$*;*

(g) X *lifts to an* $\mathcal{O}\mathsf{G}$*-lattice which affords the character* $\chi_{q(q-1)} + \chi_{q^3}$ *and as composition factor* $X = k_{\mathsf{G}} + 3 \times \varphi + \theta$ *;*

(h) $\mathrm{soc}(X) \cong \varphi$ *and* $\varphi \mid \mathrm{hd}(X)$ *;*

(i) $\dim_k \mathrm{Hom}_{k\mathsf{G}}(X, k_{\mathbb{B}}\!\uparrow^{\mathsf{G}}) = \dim_k \mathrm{Hom}_{k\mathsf{G}}(k_{\mathbb{B}}\!\uparrow^{\mathsf{G}}, X) = 1$*;*

(j) $k_{\mathsf{G}} \nmid \mathrm{soc}^2(X)$*;*

(k) X *has a uniserial* $k\mathsf{G}$*-submodule* $Z \cong k_{\mathbb{B}}\!\uparrow^{\mathsf{G}}/\mathrm{soc}(k_{\mathbb{B}}\!\uparrow^{\mathsf{G}})$ *of the form*

$$\begin{array}{|c|}\hline k_{\mathsf{G}} \\ \varphi \\ \theta \\ \varphi \\ \hline\end{array}$$

and hence if $Y := \mathrm{rad}(Z) = \begin{array}{|c|}\hline \varphi \\ \theta \\ \varphi \\ \hline\end{array}$ *then* X/Y *is of the form* $\begin{array}{|c|}\hline \varphi \\ k_{\mathsf{G}} \\ \hline\end{array}$ *or of the form* $k_{\mathsf{G}} \oplus \varphi$ *.*

Proof. (a) The claim about the structure of Q is clear from the structure of $\mathbb{B}$. Hence, it is clear that $\mathrm{Sc}(\mathsf{G},\mathbb{B}) = \mathrm{Sc}(\mathsf{G},Q)$ (see e.g. [NT88, Corollary 4.8.5]). The claim about $k_{\mathbb{B}}\!\uparrow^{\mathsf{G}}$ being uniserial with the given composition series and the given ordinary character is given by [His04, Theorem 4.1(c) and Appendix (pp. 238–241)]. Then, as $k_{\mathbb{B}}\!\uparrow^{\mathsf{G}}$ is indecomposable, and $\mathrm{Sc}(\mathsf{G},\mathbb{B})$ is an indecomposable direct summand of $k_{\mathbb{B}}\!\uparrow^{\mathsf{G}}$ by definition, certainly $k_{\mathbb{B}}\!\uparrow^{\mathsf{G}} = \mathrm{Sc}(\mathsf{G},\mathbb{B})$.

(b) Suppose that $U \mid \varphi{\downarrow}_{\mathbb{B}}$ and U lies in $B_0(k\mathbb{B})$. Since $\mathbb{B}$ is 2-nilpotent, its principal block is nilpotent and so $\mathrm{Irr}_k(B_0(k\mathbb{B})) = \{k_{\mathbb{B}}\}$. All the composition factors of U are isomorphic to $k_{\mathbb{B}}$ as they must lie in $B_0(k\mathbb{B})$. Thus, $0 \neq \mathrm{Hom}_{k\mathbb{B}}(U, k_{\mathbb{B}})$ and Frobenius reciprocity yields

$$0 \neq \mathrm{Hom}_{k\mathbb{B}}(\varphi{\downarrow}_{\mathbb{B}}, k_{\mathbb{B}}) \cong \mathrm{Hom}_{kG}(\varphi, k_{\mathbb{B}}\!\uparrow^{\mathsf{G}}),$$

proving that φ is a constituent of the socle of $k_{\mathbb{B}}\!\uparrow^{G}$. This contradicts (a) and so the first claim follows. The claim about θ is proved analogously.

(c) By [Lan83, I Corollary 10.13], $\mathrm{Ext}^1_{k\mathsf{G}}(k_{\mathsf{G}}, k_{\mathsf{G}}) = 0$ as $O^2(\mathsf{G}) = \mathsf{G}$.

(d) First, it is immediate from (a) that $\dim_k \operatorname{Ext}^1_{k\mathsf{G}}(k_{\mathsf{G}},\varphi) \geq 1$. Now, suppose that $\dim_k \operatorname{Ext}^1_{k\mathsf{G}}(k_{\mathsf{G}},\varphi) \geq 2$. Then, there exists a non-split short exact sequence

$$0 \to \varphi \to V \to k_{\mathbb{B}}{\uparrow}^{\mathsf{G}} \to 0$$

of $k\mathsf{G}$-modules, i.e. $\operatorname{Ext}^1_{k\mathsf{G}}(k_{\mathbb{B}}{\uparrow}^{\mathsf{G}},\varphi) \neq 0$. However, by the Eckmann–Shapiro Lemma,

$$\operatorname{Ext}^1_{k\mathsf{G}}(k_{\mathbb{B}}{\uparrow}^{\mathsf{G}},\varphi) \cong \operatorname{Ext}^1_{k\mathbb{B}}(k_{\mathbb{B}},\varphi{\downarrow}_{\mathbb{B}}),$$

which is zero by (b). This is a contradiction and so it follows that $\dim_k \operatorname{Ext}^1_{k\mathsf{G}}(k_{\mathsf{G}},\varphi) = 1$. Moreover, $\dim_k \operatorname{Ext}^1_{k\mathsf{G}}(k_{\mathsf{G}},\varphi) = 1$ as well by the self-duality of k_{G} and φ.

(e) Suppose that $\operatorname{Ext}^1_{k\mathsf{G}}(k_{\mathsf{G}},\theta) \neq 0$. Then, with arguments similar to those used in the proof of (d), we obtain that $\operatorname{Ext}^1_{k\mathsf{G}}(k_{\mathbb{B}}{\uparrow}^{\mathsf{G}},\theta) \neq 0$, which contradicts (b). Again, as k_{G} and θ are self-dual, it follows that $\operatorname{Ext}^1_{k\mathsf{G}}(k_{\mathsf{G}},\theta) = 0$ as well.

(f) Since $\operatorname{Irr}_k(B_0(k\mathsf{G})) = \{k_{\mathsf{G}},\varphi,\theta\}$, it follows from (c), (d) and (e) that the second Loewy layer of $P(k_{\mathsf{G}})$ consists just of the simple module φ, with multiplicity 1. Thus, the claim follows from the fact that $\Omega(k_{\mathsf{G}}) = P(k_{\mathsf{G}})\cdot\operatorname{rad}(kG)$.

(g) First, it is well-known that X lifts to an $\mathcal{O}\mathsf{G}$-lattice (see e.g. [Las23, §7.3]). Moreover, by (f) we have that $\Omega^2(k_{\mathsf{G}})$ is the kernel of a short exact sequence of $k\mathsf{G}$-modules of the form

$$0 \to \Omega^2(k_{\mathsf{G}}) \to P(\varphi) \to \Omega(k_{\mathsf{G}}) \to 0\,.$$

Thus, in the Grothendieck ring of kG, we have

$$\Omega^2(k_{\mathsf{G}}) = P(\varphi) - \Omega(k_{\mathsf{G}}) = P(\varphi) - P(k_{\mathsf{G}})\cdot\operatorname{rad}(k\mathsf{G}) = P(\varphi) - (P(k_{\mathsf{G}}) - k_{\mathsf{G}})\,.$$

Using the decomposition matrix of $B_0(k\mathsf{G})$ given in Lemma 7.2, we obtain that the character afforded by $\Omega^2(k_{\mathsf{G}})$ is $\chi_{q(q-1)} + \chi_{q^3}$, and the composition factors of X are as claimed.

(h) It is clear that $\operatorname{soc}(X) \cong \varphi$ as $\Omega^2(k_{\mathsf{G}})$ is a submodule of $P(\varphi)$ by the proof of assertion (g). Now, by Lemma 7.2, any lift of φ affords the character $\chi_{q(q-1)}$. Thus, by [Lan83, I Theorem 17.3], X has a pure submodule Y affording the Steinberg character χ_{q^3}. Then, $X/Y \cong \varphi$, proving the claim.

(i) It follows from Frobenius reciprocity that

$$\operatorname{Hom}_{k\mathsf{G}}(k_{\mathbb{B}}{\uparrow}^{\mathsf{G}}, X) \cong \operatorname{Hom}_{k\mathbb{B}}(k_{\mathbb{B}}, X{\downarrow}_{\mathbb{B}}).$$

Now, as $\operatorname{Irr}_k(B_0(k\mathsf{G})) = \{k_{\mathsf{G}},\varphi,\theta\}$ and by assertion (g) we have that k_{G} has multiplicity one as a composition factor of X, it follows from (b) that

$$\operatorname{Hom}_{k\mathbb{B}}(k_{\mathbb{B}}, X{\downarrow}_{\mathbb{B}}) \cong \operatorname{Hom}_{k\mathbb{B}}(k_{\mathbb{B}}, k_{\mathsf{G}}{\downarrow}_{\mathbb{B}}) \cong k$$

as k-vector spaces. The second equality is obtained analogously.

(j) Consider the Auslander–Reiten sequence $(\mathcal{E}) : 0 \to X \xrightarrow{g} E \xrightarrow{\pi} k_{\mathsf{G}} \to 0$ starting at $X = \Omega^2(k_{\mathsf{G}})$ (and hence ending at k_{G}). (See e.g. [Thé95, §34] for this notion.) By (d) there exists a uniserial module of length 2 of the form

$$\boxed{\begin{matrix} k_{\mathsf{G}} \\ \varphi \end{matrix}} =: Y\,.$$

Consider the quotient homomorphism $\rho : Y \to Y/\varphi \cong k_{\mathsf{G}}$, which is obviously not a split-epi. Hence there exists a $k\mathsf{G}$-homomorphism $\alpha : Y \to E$ with $\pi \circ \alpha = \rho$. Next we claim that $\ker(\alpha) \neq \operatorname{soc}(Y)$. So assume $\ker(\alpha) = \operatorname{soc}(Y)$. Then, $E \geq \operatorname{im}(\alpha) \cong k_{\mathsf{G}}$, proving that $\operatorname{im}(\alpha) \leq \operatorname{soc}(E)$ (as it is simple). On the other hand, by (h), $\operatorname{soc}(X) = \varphi$, implying that $\operatorname{im}(\alpha) \cap \operatorname{soc}(X) = 0$. Thus, identifying X with its image in E, we get that $\operatorname{im}(\alpha) \cap X = 0$ as $\operatorname{im}(\alpha)$ is simple. (Use here the same argument as in the last five lines of the proof of Lemma 4.6.) Hence, E has a submodule of the form $\mathrm{im}(\alpha) \oplus X$, which implies that $E = \mathrm{im}(\alpha) \oplus X$ as we can read from the s.e.s. $(\mathcal{E})$ that they have the same k-dimension. Thus, it follows from [Car96, Lemma 6.12] that the sequence $(\mathcal{E})$ splits, which is a contradiction and the claim follows. Next, since $\alpha \neq 0$, it follows that $\ker(\alpha) = 0$, that is, α is injective. Hence, $\mathrm{im}(\alpha) \cong Y$. Now, suppose that $k_{\mathsf{G}} \,|\, \mathrm{soc}^2(X)$. Set $W := \mathrm{soc}^2(X) + \mathrm{im}(\alpha) \leq E$. Note that $\mathrm{im}(\alpha) \not\leq X$ since $X = \ker(\pi)$, so that $\mathrm{im}(\alpha) \not\leq \mathrm{soc}^2(X)$. Hence $\mathrm{soc}^2(X) + \mathrm{im}(\alpha)$ has the following socle series

$$\begin{bmatrix} k_{\mathsf{G}} \ k_{\mathsf{G}} \\ \varphi \end{bmatrix},$$

since by Lemma 4.6 we have $\mathrm{soc}(E) = \mathrm{soc}(X) \cong \varphi$, where the last isomorphism holds by (h). This is a contradiction to (d), and so the claim follows.

(k) It follows from assertions (i) and (a) that

$$1 = \dim_k \mathrm{Hom}_{k\mathsf{G}}(k_{\mathbb{B}}{\uparrow}^{\mathsf{G}}, X) = \dim_k \mathrm{Hom}_{k\mathsf{G}}\left(\begin{array}{|c|}\hline k_{\mathsf{G}} \\ \varphi \\ \theta \\ \varphi \\ k_{\mathsf{G}} \\ \hline\end{array}, X\right) = \dim_k \mathrm{Hom}_{k\mathsf{G}}\left(\begin{array}{|c|}\hline k_{\mathsf{G}} \\ \varphi \\ \theta \\ \varphi \\ \hline\end{array}, X\right),$$

where $\begin{array}{|c|}\hline k_{\mathsf{G}} \\ \varphi \\ \theta \\ \varphi \\ \hline\end{array} := k_{\mathbb{B}}{\uparrow}^{\mathsf{G}}/\mathrm{soc}(k_{\mathbb{B}}{\uparrow}^{\mathsf{G}})$ and the last equality holds because $c_X(k_{\mathsf{G}}) = 1$ by (g). Therefore, there exists a non-zero $k\mathsf{G}$-homomorphism

$$\gamma : \begin{array}{|c|}\hline k_{\mathsf{G}} \\ \varphi \\ \theta \\ \varphi \\ \hline\end{array} \longrightarrow X$$

Now, either γ is injective and we are done, or $\ker(\gamma) \neq 0$. In the latter case, by (h) we have $\ker(\gamma) = \begin{array}{|c|}\hline \theta \\ \varphi \\ \hline\end{array}$ and so there is an injective homomorphism

$$j : \begin{array}{|c|}\hline k_{\mathsf{G}} \\ \varphi \\ \hline\end{array} \hookrightarrow X\,.$$

which contradicts assertion (j). The claim follows. ∎

We can now prove the main result of this section.

Proposition 7.5. *For each $i \in \{1, 2\}$ let $\mathsf{G}_i := \mathsf{G}(q_i) = \mathrm{PGU}_3(q_i)$ and $G_i := G(q_i) = \mathrm{PSU}_3(q_i)$, where we assume that $(q_i + 1)_2 = 2^n$. Then, the following assertions hold:*

(a) $\mathrm{Sc}(\mathsf{G}_1 \times \mathsf{G}_2, \Delta P)$ *induces a splendid Morita equivalence between $B_0(k\mathsf{G}_1)$ and $B_0(k\mathsf{G}_2)$;*

(b) $\mathrm{Sc}(G_1 \times G_2, \Delta P)$ *induces a splendid Morita equivalence between $B_0(kG_1)$ and $B_0(kG_2)$.*

Proof. Again, as $G_i \lhd \mathsf{G}_i$ and $|\mathsf{G}_i/G_i| = (3, q+1)$ is odd for each $i \in \{1, 2\}$, by Lemma 3.1 and Lemma 4.2, assertion (b) follows from assertion (a), so it suffices to prove (a).

Set $M := \mathrm{Sc}(\mathsf{G}_1 \times \mathsf{G}_2, \Delta P)$. For each $i \in \{1,2\}$ write $B_i := B_0(k\mathsf{G}_i)$. Write $\mathrm{Irr}_k(B_i) = \{k_{\mathsf{G}_i}, \varphi_i, \theta_i\}$ with $\dim_k \varphi_i = q_i(q_i-1)$ and $\dim_k \theta_i = (q_i-1)(q_i{}^2-q_i+1)$, and set $\mathbb{B}_i := \mathbb{B}_i(q_i)$ as in Notation 7.1. Moreover, let $Q_i \in \mathrm{Syl}_2(\mathbb{B}_i)$ such that $Q_i \leq P$ and let $X_i := \Omega^2(k_{\mathsf{G}_i})$ as in Lemma 7.4. Furthermore, observe that $\mathcal{F}_P(\mathsf{G}_1) = \mathcal{F}_P(\mathsf{G}_2)$ and all involutions in G_i are G_i-conjugate (see e.g. [CG12, Theorem 5.3] and/or [ABG70, Proposition 2 on p.11]). It follows that it suffices to prove that Conditions (I) and (II) of Theorem 4.5 hold.

Condition (I). A similar argument to the one used in the proof of Proposition 6.3 (Condition I) can be used. In the present case, if z is an involution in the centre of P, then $C_{\mathsf{G}_i}(z) =: C_i$ is a quotient of $\mathrm{GU}_2(q)$ by a normal subgroup of odd index by [ABG70, Proposition 4(iii)]. Hence, we obtain from Lemma 4.3 and Proposition 5.4 that $M_z := \mathrm{Sc}(C_1 \times C_2, \Delta P)$ induces a splendid Morita equivalence between $B_0(kC_1)$ and $B_0(kC_2)$ by Proposition 5.3(a). Moreover, $M_z = M(\Delta\langle z\rangle)$ by the Brauer indecomposability of M proved in [KT21, Theorem 1.1], proving that Condition (I) is verified.

Condition (II). Again, we have to prove that the functor $- \otimes_{B_1} M$ maps the simple B_1-modules to the simple B_2-modules, and again, we have $k_{\mathsf{G}_1} \otimes_{B_1} M \cong k_{\mathsf{G}_2}$ by [KL20a, Lemma 3.4(a)]. Thus, it remains to prove that $\varphi_1 \otimes_{B_1} M \cong \varphi_2$ and $\theta_1 \otimes_{B_1} M \cong \theta_2$.

First recall from Lemma 7.4(a) that for each $i \in \{1,2\}$ we have

$$\mathrm{Sc}(\mathsf{G}_i, Q_i) = \boxed{\begin{matrix} k_{\mathsf{G}_i} \\ \varphi_i \\ \theta_i \\ \varphi_i \\ k_{\mathsf{G}_i} \end{matrix}}\,. \tag{3}$$

and moreover by [KL20a, Lemma 3.4(c)] we have

$$\mathrm{Sc}(\mathsf{G}_1, Q_1) \otimes_{B_1} M \cong \mathrm{Sc}(\mathsf{G}_2, Q_2) \oplus (\mathrm{proj})\,.$$

Thus, because we already know that

$$\mathrm{soc}(\mathrm{Sc}(\mathsf{G}_1, Q_1)) \otimes_{B_1} M = k_{\mathsf{G}_1} \otimes_{B_1} M \cong k_{\mathsf{G}_2} = \mathrm{soc}(\mathrm{Sc}(\mathsf{G}_2, Q_2))\,,$$

the stripping-off method (see Lemma 4.7(a)) yields

$$\boxed{\begin{matrix} k_{\mathsf{G}_1} \\ \varphi_1 \\ \theta_1 \\ \varphi_1 \end{matrix}} \otimes_{B_1} M \;\cong\; \boxed{\begin{matrix} k_{\mathsf{G}_2} \\ \varphi_2 \\ \theta_2 \\ \varphi_2 \end{matrix}} \oplus (\mathrm{proj}) \tag{4}$$

where for each $i \in \{1,2\}$, the latter uniserial module of length 4 is defined to be

$$\boxed{\begin{matrix} k_{\mathsf{G}_i} \\ \varphi_i \\ \theta_i \\ \varphi_i \end{matrix}} := \mathrm{Sc}(\mathsf{G}_i, Q_i)/\mathrm{soc}(\mathrm{Sc}(\mathsf{G}_i, Q_i)) =: Z_i$$

as in Lemma 7.4(k). Then, applying again the stripping-off method (Lemma 4.7(b) this time) to equation (4) and $\mathrm{hd}\, Z_i \cong k_{\mathsf{G}_i}$ $(i \in \{1,2\})$, we obtain that

$$\boxed{\begin{matrix} \varphi_1 \\ \theta_1 \\ \varphi_1 \end{matrix}} \otimes_{B_1} M \;=\; \boxed{\begin{matrix} \varphi_2 \\ \theta_2 \\ \varphi_2 \end{matrix}} \oplus (\mathrm{proj}) \tag{5}$$

where for each $i \in \{1,2\}$, the latter uniserial module of length 3 is defined to be

$$\boxed{\begin{matrix}\varphi_i\\ \theta_i\\ \varphi_i\end{matrix}} := \mathrm{rad}(Z_i) =: Y_i\,,$$

again as in Lemma 7.4(k). Now, by the proof of Lemma 7.4(k), we also know that Y_i is (up to identification) a submodule of X_i for each $i \in \{1,2\}$, and

$$X_1 \otimes_{B_1} M \cong X_2 \oplus (\mathrm{proj})$$

by [KL20a, Lemma 3.4(d)]. Because of the way, we have defined X_i and Y_i ($i \in \{1,2\}$) via the stripping-off method, it follows from the exactness of the functor $- \otimes_{B_1} M$ that

$$X_1/Y_1 \otimes_{B_1} M \cong (X_1 \otimes_{B_1} M)/(Y_1 \otimes_{B_1} M) \cong X_2/Y_2 \oplus (\mathrm{proj})\,.$$

Lemma 7.4(k) gives, up to isomorphism, two possibilities for X_1/Y_1 and two possibilities for X_2/Y_2, namely,

$$\boxed{\begin{matrix}\varphi_1\\ k_{\mathsf{G}_1}\end{matrix}} \text{ or } k_{\mathsf{G}_1} \oplus \varphi_1, \text{ and } \boxed{\begin{matrix}\varphi_2\\ k_{\mathsf{G}_2}\end{matrix}} \text{ or } k_{\mathsf{G}_2} \oplus \varphi_2, \text{ respectively,}$$

but in any configuration we can apply the stripping-off method again (Lemma 4.7(a)) to strip off the trivial socle summand of X_1/Y_1 and X_2/Y_2 and we obtain that

$$\varphi_1 \otimes_{B_1} M \cong \varphi_2 \oplus (\mathrm{proj})\,.$$

However, as φ_1 is simple, $\varphi_1 \otimes_{B_1} M$ must be indecomposable by [KL20a, Theorem 2.1(a)], proving that $\varphi_1 \otimes_{B_1} M \cong \varphi_2$. Then, we can apply yet again the stripping-off method twice (once Lemma 4.7(a) and once Lemma 4.7(b)) to equation (5) and $\mathrm{soc}(Y_i)$, respectively $\mathrm{hd}(Y_i)$, ($i \in \{1,2\}$) to obtain that

$$\theta_1 \otimes_{B_1} M \cong \theta_2 \oplus (\mathrm{proj})\,.$$

However, again, as θ_1 is simple, $\theta_1 \otimes_{B_1} M$ must be indecomposable by [KL20a, Theorem 2.1(a)], eventually proving that $\theta_1 \otimes_{B_1} M \cong \theta_2$. □

8 Proofs of Theorem 1.2 and Theorem 1.3.

We can now prove our main results, that is, Theorem 1.2 and Theorem 1.3. We recall that G is a finite group with a fixed Sylow 2-subgroup $P \cong C_{2^n} \wr C_2$, where $n \geq 2$ is a fixed integer.

Proof of Theorem 1.2. (a) To start with, by Lemma 4.3, we may assume that $O_{2'}(G) = 1$ and therefore that G is one of the groups listed in Theorem 1.1. Furthermore, by Lemma 3.1 and Lemma 4.2, we may also assume that $O^{2'}(G) = G$. Hence, Theorem 1.1, applied a second time, implies that G belongs to family $(\mathsf{Wj(n)})$ for some $\mathsf{j} \in \{1, \cdots, 6\}$.

It remains to prove that j is uniquely determined. So, suppose that $G =: G_1$ is a finite group belonging to family $(\mathsf{Wj_1(n)})$ for some $\mathsf{j_1} \in \{1, \cdots, 6\}$ and assume that the following hypothesis is satisfied:

(∗) $B_0(kG_1)$ is splendidly Morita equivalent to the principal block $B_0(kG_2)$ of a finite group G_2 belonging to family $(\mathsf{Wj_2(n)})$ for some $\mathsf{j_2} \in \{1, \cdots, 6\}$.

For $i \in \{1,2\}$ set $B_i := B_0(kG_i)$, and notice that $(*)$ implies that $\ell(B_1) = \ell(B_2)$ and $k(B_1) = k(B_2)$ because these numbers are invariant under Morita equivalences.

Now, first assume that $\mathsf{j}_1 = 1$. Then, it follows from Theorem 3.3 that $\ell(B_1) = 1$ and $\ell(B_2) > 1$ if $\mathsf{j}_2 > 1$, contradicting $(*)$. Hence, we have $\mathsf{j}_2 = 1$ and $G_2 \cong G_1$.

Assume then that $\mathsf{j}_1 = 2$. Then, by Theorem 3.3, we have $\ell(B_1) = 2$ and by $(*)$ we may also assume that $\mathsf{j}_2 \in \{2,3,4\}$. If $\mathsf{j}_2 \neq 2$, then, as $n \geq 2$, Theorem 3.3 yields

$$k(B_1) = (2^{2n-1} + 9{\cdot}2^{n-1} + 4)/3 \neq 2^{2n-1} + 2^{n+1} = k(B_2)\,,$$

also contradicting $(*)$, so that $\mathsf{j}_2 = 2$ and $G_2 \cong G_1$.

Suppose next that $\mathsf{j}_1 = 3$. Then, again by Theorem 3.3, we have $\ell(B_1) = 2$ and by $(*)$ we may assume that $\mathsf{j}_2 \in \{2,3,4\}$. Moreover, by the previous case, we have $\mathsf{j}_2 \neq 2$. So, let us assume that $\mathsf{j}_2 = 4$. We can consider that $B_1 = B_0(k\,\mathrm{SL}_2^n(q_1))$ and $B_2 = B_0(k\,\mathrm{SU}_2^n(q_2))$ for prime powers q_1, q_2 such that $(q_1-1)_2 = 2^n = (q_2+1)_2$. Then, again, as $\mathrm{SL}_2^n(q_1) \lhd \mathrm{GL}_2(q_1)$ and $\mathrm{SU}_2^n(q_2) \lhd \mathrm{GU}_2(q_2)$ are normal subgroups of odd index, it follows from Lemma 3.1 and Lemma 4.2 that $B_0(k\,\mathrm{GL}_2(q_1))$ and $B_0(k\,\mathrm{GU}_2(q_2))$ are splendidly Morita equivalent, and so Lemma 4.4 implies that $B_0(k\,\mathrm{PGL}_2(q_1))$ and $B_0(k\,\mathrm{PGU}_2(q_2))$ are splendidly Morita equivalent. Now, as $\mathrm{PGL}_2(q_1) \cong \mathrm{PGU}_2(q_1)$, we have that $\mathcal{B}_1 := B_0(k\,\mathrm{PGL}_2(q_1))$ and $\mathcal{B}_2 := B_0(k\,\mathrm{PGL}_2(q_2))$ are splendidly Morita equivalent, where $D_{2^{n+1}} \in \mathrm{Syl}_2(\mathrm{PGL}_2(q_1)) \cap \mathrm{Syl}_2(\mathrm{PGL}_2(q_2))$ by the proofs of Proposition 5.3 and Proposition 5.4. However, the conditions on q_1 and q_2 imply that $\mathcal{B}_1$ and $\mathcal{B}_2$ are in (5) and (6), respectively, in the list of [KL20a, Theorem 1.1], hence cannot be splendidly Morita equivalent. Thus, we have a contradiction, proving that $\mathsf{j}_2 = 3$ if $\mathsf{j}_1 = 3$. Moreover, swapping the roles of j_1 and j_2 in the previous argument, we obtain that $\mathsf{j}_2 = 4$ if $\mathsf{j}_1 = 4$.

Suppose next that $\mathsf{j}_1 = 5$. Then, as above, Theorem 3.3 and $(*)$ imply that $\ell(B_1) = 3$ and $\mathsf{j}_2 \in \{5,6\}$. So, assume that $\mathsf{j}_2 = 6$. Hence, we can consider that $B_1 = B_0(k\,\mathrm{PSL}_3(q_1))$ and $B_2 = B_0(k\,\mathrm{PSU}_3(q_2))$ for prime powers q_1 and q_2 such that $(q_1-1)_2 = 2^n = (q_2+1)_2$. However, $B_0(k\,\mathrm{PSL}_3(q_1))$ and $B_0(k\,\mathrm{PSU}_3(q_2))$ cannot be splendidly Morita equivalent by Lemma 6.2 and Corollary 7.3, because such an equivalence maps simple modules to simple modules and also trivial source modules to trivial source modules. It follows that $\mathsf{j}_2 = 5$ if $\mathsf{j}_1 = 5$. Again, swapping the roles of j_1 and j_2 in the previous argument, we obtain that $\mathsf{j}_2 = 6$ if $\mathsf{j}_1 = 6$. Finally, we observe that the claim about the Scott module is immediate by construction.

(b) Assume G_1 and G_2 both belong to family $(\mathsf{Wj(n)})$ for a $\mathsf{j} \in \{3,4,5,6\}$. Then, $\mathrm{Sc}(G_1 \times G_2, \Delta P)$ induces a splendid Morita equivalence between $B_0(kG_1)$ and $B_0(kG_2)$ by Propositions 5.3, 5.4, 6.3 and 7.5 for $\mathsf{j} = 3,4,5$ and 6 respectively. □

Proof of Theorem 1.3. It is clear from the definitions that any splendid Morita equivalence is in particular a Morita equivalence. Thus, it only remains to prove that two distinct splendid Morita equivalence classes of principal blocks in Theorem 1.2 do not merge into one Morita equivalence class. In fact, from the numbers $\ell(B)$ and $k(B)$ in Theorem 3.3, it suffices to argue that the splendid Morita equivalence classes of principal blocks of groups of type $(\mathsf{Wj(3)})$ and $(\mathsf{Wj(4)})$, respectively of type $(\mathsf{Wj(5)})$ and $(\mathsf{Wj(6)})$, do not merge into one Morita equivalence class. In the former case, this is clear from the proof of Theorem 1.2, because else $B_0(k\,\mathrm{PGL}_2(q_1))$ and $B_0(k\,\mathrm{PGL}_2(q_2))$ with $(q_1-1)_2 = 2^n = (q_2+1)_2$ would be Morita equivalent, which would contradict Erdmann's classification of tame blocks in [Erd90]. In the latter case, it follows from the decomposition matrices of $B_0(k\,\mathrm{PSL}_3(q_1))$ and $B_0(k\,\mathrm{PSU}_3(q_2))$ with $(q_1-1)_2 = 2^n = (q_2+1)_2$ given in [Sch15, Proposition 6.12] and Lemma 7.2, respectively, that these blocks are not Morita equivalent. The claim follows. □

A Appendix. On [KL20b, Proposition 3.3(b)]

The purpose of this appendix is to fix a problem in the proof of [KL20b, Proposition 3.3(b)], which was incomplete as written in [KL20b]. See Remark A.3.

The next corollary is the most essential in this appendix, namely, for the proof of (ii) ⇒ (i) in Theorem A.2. This is already implicitly explained in the proof of [Lin18, Theorem 9.7.4].

Corollary A.1 (See [Lin18, Theorem 9.7.4]). *The notation here is the same as that in* [Lin18] *except that $\mathcal{O}$ is replaced by k. Let G, H be finite groups, let b, c be blocks of kG, kH, respectively, such that kGb and kHc have a common defect group P. Let i and j be P-source idempotents of kGb and kHc, respectively (and hence $i \in (kGb)^{\Delta P}$ and $j \in (kHc)^{\Delta P}$).*

Now, suppose that there is an indecomposable direct summand M of the (kGb, kHc)-bimodule $kGi \otimes_{kP} jkH$ such that the pair (M, M^*) induces a Morita equivalence between kGb and kHc. Furthermore let $\varphi := \varphi_M$ be the unitary interior P-algebra isomorphism $\varphi : i\,kGi \overset{\approx}{\to} jkHj$ induced by M as in [Lin18, Theorem 9.7.4]*. Then, for any indecomposable right kGb-module X,*

$$Xi \;\cong\; (X \otimes_{kGb} Mj)_\varphi \quad \textit{as right } i\,kGi\textit{-modules}$$

where $(X \otimes_{kGb} Mj)_\varphi = X \otimes_{kGb} Mj$ as k-vector spaces and the right action of $i\,kGi$ is defined using of φ.

Proof. The notation here is the same as in the proof of [Lin18, (i) ⇒ (ii) in Theorem 9.7.4] except that $\mathcal{O}$ is replaced by k. First we know already $Mj \cong kGi$ as (kG, kP)-bimodules via α there. In the following the endomorphism ring of a left R-module X for a ring R is denoted by $\mathrm{End}_R(X)$. Then, since Mj can be considered as a right not only kP-module but also $\mathrm{End}_{kGb}(Mj)$-module. Since $Mj \cong kGi$ as left kGb-modules from the above, it follows that $\mathrm{End}_{kGb}(Mj) \cong \mathrm{End}_{kGb}(kGi) \cong (i\,kGi)^{\mathrm{op}}$ as k-algebras. On the other hand, $\mathrm{End}_{kGb}(Mj) \cong \mathrm{End}_{kGb}(M \otimes_{kHc} kHj) \cong \mathrm{End}_{kHc}(kHj) \cong (j\,kHj)^{\mathrm{op}}$ as k-algebras (the second isomorphism comes from the fact that M realises a Morita equivalence between kGb and kHc). Since the isomorphism $\varphi = \varphi_M$ is defined by using these isomorphisms (see the final several lines in the proof of [Lin18, (i) ⇒ (ii) in Theorem 9.7.4]), we eventually obtain that $(X \otimes_{kGb} Mj)_\varphi \cong X \otimes_{kGb} kGi \cong Xi$ as right $i\,kGi$-modules. □

Theorem A.2 (See Proposition 3.3(b) in [KL20b]). *Suppose that G_1 and G_2 are finite groups with a common Sylow p-subgroup P, and assume that Z is a subgroup of P such that $Z \leq Z(G_1) \cap Z(G_2)$. Write $\overline{G_1} := G_1/Z$, $\overline{G_2} := G_2/Z$ and $\overline{P} := P/Z$. Then, the following assertions are equivalent:*

(i) $\mathrm{Sc}(G_1 \times G_2, \Delta P)$ *induces a Morita equivalence between $B_0(kG_1)$ and $B_0(kG_2)$;*
(ii) $\mathrm{Sc}(\overline{G_1} \times \overline{G_2}, \Delta \overline{P})$ *induces a Morita equivalence between $B_0(k\overline{G_1})$ and $B_0(k\overline{G_2})$.*

Proof. Let $i \in \{1, 2\}$. Write $B_i := B_0(kG_i)$ and let $\overline{B_i}$ be the image of B_i under the k-algebra epimorphism $kG_i \twoheadrightarrow k\overline{G_i}$ induced by the quotient group homomorphism $G_i \twoheadrightarrow \overline{G_i}$. Then [NT88, Chap. 5 Theorem 8.11] says that $\overline{B_i} = B_0(k\overline{G_i})$. Furthermore $\overline{B_i} \cong k\overline{G_i} \otimes_{kG_i} B_i \otimes_{kG_i} k\overline{G_i}$ as $(k\overline{G_i}, k\overline{G_i})$-bimodules. Write $M := \mathrm{Sc}(G_1 \times G_2, \Delta P)$ and $N := M^* = \mathrm{Sc}(G_2 \times G_1, \Delta P)$. Set $\overline{M} := k\overline{G_1} \otimes_{kG_1} M \otimes_{kG_2} k\overline{G_2}$. Then, the following holds:

$$\begin{aligned}\overline{M}\ \Big|\ & k\overline{G_1}\otimes_{kG_1}\left(\mathrm{Ind}_{\Delta P}^{G_1\times G_2}(k_{\Delta P})\right)\otimes_{kG_2} k\overline{G_2}\\ &= k\overline{G_1}\otimes_{kG_1}(kG_1\otimes_{kP} kG_2)\otimes_{kG_2} k\overline{G_2}\\ &\cong k\overline{G_1}\otimes_{kP} k\overline{G_2}\\ &\cong k\overline{G_1}\otimes_{k\overline{P}} k\overline{G_2}\\ &\cong \mathrm{Ind}_{\Delta\overline{P}}^{\overline{G_1}\times\overline{G_2}}(k_{\Delta\overline{P}}).\end{aligned}$$

Note furthermore that $\overline{M}$ obviously has the trivial $k(\overline{G_1}\times\overline{G_2})$-module $k_{\overline{G_1}\times\overline{G_2}}$ as an epimorphic image. Set $\mathfrak{M} := \mathrm{Sc}(\overline{G_1}\times\overline{G_2},\,\Delta\overline{P})$. Then

$$\mathfrak{M}\ \Big|\ \overline{M}\qquad\text{(equality does not necessarily hold).}\tag{6}$$

(i) $\Rightarrow$ (ii): Set $\overline{N} := k\overline{G_2}\otimes_{kG_2} N\otimes_{kG_1} k\overline{G_1}$. Then,

$$\begin{aligned}\overline{M}\otimes_{\overline{B_2}}\overline{N} &\cong \overline{M}\otimes_{k\overline{G_2}}\overline{N}\\ &\cong (k\overline{G_1}\otimes_{kG_1} M\otimes_{kG_2} k\overline{G_2})\otimes_{k\overline{G_2}}(k\overline{G_2}\otimes_{kG_2} N\otimes_{kG_1} k\overline{G_1})\\ &\cong k\overline{G_1}\otimes_{kG_1}(M\otimes_{kG_2} k\overline{G_2})\otimes_{kG_2} N\otimes_{kG_1} k\overline{G_1}\\ &\cong k\overline{G_1}\otimes_{kG_1}(k\overline{G_1}\otimes_{kG_1} M)\otimes_{kG_2} N\otimes_{kG_1} k\overline{G_1}\\ &\qquad\text{since } M\otimes_{kG_2} k\overline{G_2}\cong k\overline{G_1}\otimes_{kG_1} M\quad\text{as } (kG_1,kG_2)\text{-bimodules}\\ &\cong (k\overline{G_1}\otimes_{kG_1} k\overline{G_1})\otimes_{kG_1} M\otimes_{kG_2} N\otimes_{kG_1} k\overline{G_1}\\ &\cong (k\overline{G_1}\otimes_{k\overline{G_1}} k\overline{G_1})\otimes_{kG_1} M\otimes_{kG_2} N\otimes_{kG_1} k\overline{G_1}\\ &\cong k\overline{G_1}\otimes_{kG_1}(M\otimes_{kG_2} N)\otimes_{kG_1} k\overline{G_1}\\ &\cong k\overline{G_1}\otimes_{kG_1} B_1\otimes_{kG_1} k\overline{G_1}\quad\text{by (i)}\\ &\cong \overline{B_1}.\end{aligned}$$

Since $\overline{B_i}$ is a symmetric k-algebra for $i=1,2$, the above already shows that the pair $(\overline{M},\overline{N})$ induces a Morita equivalence between $\overline{B_1}$ and $\overline{B_2}$, and hence $\overline{M}$ is indecomposable as a right $k(\overline{G}_1\times\overline{G}_2)$-module, which implies that $\mathfrak{M}\cong\overline{M}$ from (6).

(ii) $\Rightarrow$ (i): As in [Lin01, p.822] there exist P-source idempotents j_i of B_i for $i=1,2$ with $M\,|\,(kG_1\,j_1\otimes_{kP} j_2\,kG_2)$. Then, for $i=1,2$, the image $\overline{j_i}$ of j_i via the canonical k-algebra epimorphism $kG_i\twoheadrightarrow k\overline{G_i}$ is a $\overline{P}$-source idempotent of $\overline{B_i}$ (see [NT88, Chap.5 Theorem 8.11], [KP90, §3] and [Kaw03, Lemma 4.1]). Hence,

$$\mathfrak{M}\,|\,(k\overline{G_1}\,\overline{j_1}\otimes_{k\overline{P}}\overline{j_2}\,k\overline{G_2})\tag{7}$$

from (6). Now, we apply [Lin18, (i) $\Rightarrow$ (ii) in Theorem 9.7.4] to the blocks $\overline{B_1}$ and $\overline{B_2}$. Namely, the existence of such an $\mathfrak{M}$ induces a unitary interior $\overline{P}$-algebra isomorphism

$$\Phi := \varphi_{\mathfrak{M}}\ :\ \overline{j_1}\,k\overline{G_1}\,\overline{j_1}\ \overset{\approx}{\to}\ \overline{j_2}\,k\overline{G_2}\,\overline{j_2}.$$

Thanks to [Pui01, Corollary 1.12] (see [Kue01]), Φ lifts to a unitary interior P-algebra isomorphism $\varphi : j_1kG_1\,j_1\overset{\approx}{\to} j_2kG_2\,j_2$, that is, $\Phi(\overline{a})=\overline{\varphi(a)}$ for every $a\in j_1\,kG_1\,j_1$. Then, by making use of φ it follows from [Lin18, (ii) $\Rightarrow$ (i) in Theorem 9.7.4] that there is an indecomposable direct summand $\mathcal{M}$ of the (B_1,B_2)-bimodule $kG_1\,j_1\otimes_{kP} j_2\,kG_2$ such that the pair $(\mathcal{M},\mathcal{M}^\vee)$ induces a Morita equivalence between B_1 and B_2 Set $S_2 := k_{G_1}\otimes_{B_1}\mathcal{M}$. Then, though we do not know that $\mathcal{M}=\mathrm{Sc}(G\times H,\Delta P)$ yet, Corollary A.1 yields that

$(S_2\, j_2)_\varphi \cong k_{G_1}\, j_1$ as right $j_1\, B_1\, j_1$-modules. Hence by sending these via the canonical epimorphism $kG_i \twoheadrightarrow k\overline{G_i}$,

$$(\overline{S_2}\,\overline{j_2})_\Phi \cong (\overline{S_2}\,\overline{j_2})_{\overline{\varphi}} \cong k_{\overline{G_1}}\,\overline{j_1} \quad \text{as right } \overline{j_1}\,\overline{B_1}\,\overline{j_1}\text{-modules.} \tag{8}$$

Now, we claim that $\mathcal{M} \cong M$. Since the Scott module $\mathfrak{M}$ induces a Morita equivalence between $\overline{B_1}$ and $\overline{B_2}$ and $\mathfrak{M}$ is related to $\overline{j_i}$ for $i = 1, 2$ (see (7)) and Φ, it follows from Corollary A.1 and [KL20a, Lemma 3.4(a)] that

$$k_{\overline{G_1}}\,\overline{j_1} \;\cong\; (k_{\overline{G_1}} \otimes_{\overline{B_1}} \mathfrak{M}\,\overline{j_2})_\Phi \;\cong\; (k_{\overline{G_2}}\,\overline{j_2})_\Phi \quad \text{as right } \overline{j_1}\overline{B_1}\,\overline{j_1}\text{-modules.} \tag{9}$$

Hence (8) and (9) yield that $(\overline{S_2}\,\overline{j_2})_\Phi \cong (k_{\overline{G_2}}\,\overline{j_2})_\Phi$ as right $\overline{j_1}\overline{B_1}\,\overline{j_1}$-modules, and hence $\overline{S_2}\,\overline{j_2} \cong k_{\overline{G_2}}\,\overline{j_2}$ as right $\overline{j_2}\overline{B_2}\,\overline{j_2}$-modules. Since the pair $(\overline{B_2}\,\overline{j_2}, \overline{j_2}\overline{B_2})$ induces the canonical Morita equivalence between $\overline{B_2}$ and its source algebra $\overline{j_2}\,\overline{B_2}\,\overline{j_2}$, we get that $\overline{S_2} \cong k_{\overline{G_2}}$ as right $\overline{B_2}$-modules, say as right $k\overline{G_2}$-modules. Since S_2 and k_{G_2} are both simple right kG_2-modules, Z is in their kernels. Thus, $\mathrm{Def}^{G_2}_{\overline{G_2}} S_2 \cong \mathrm{Def}^{G_2}_{\overline{G_2}} k_{G_2}$ as right $k\overline{G_2}$-modules where Def is the deflation. Hence $S_2 \cong k_{G_2}$ as right kG_2-modules. This means by the definition of S_2 that the bimodule $\mathcal{M}$ transposes k_{G_1} to k_{G_2}. Thus, the adjunction in [Rou01, Line 9 on p.105] implies that $\mathrm{Hom}_{(k(G_1\times G_2))^{\mathrm{op}}}(\mathcal{M}, k_{(G_1\times G_2)}) \neq \{0\}$. Therefore $\mathcal{M} \cong \mathrm{Sc}(G_1 \times G_2, \Delta P) = M$ by [Thé95, Exercise (27.5)]. We are done. □

Remark A.3. On the right-hand side of line 2 of [KL20b, Lemma 3.1(b)], $\mathrm{Sc}(\overline{G}\times\overline{H}, \Delta\overline{P})$ must be replaced by $\mathrm{Sc}(\overline{G} \times \overline{H}, \Delta\overline{P}) \oplus \mathcal{N}$ for a *possibly non-zero* $k(\overline{G} \times \overline{H})$-module $\mathcal{N}$ as in (6). As a result, the proof of [KL20b, Proposition 3.3(b)] as given in [KL20b] holds only in the case in which $\mathcal{N} = \{0\}$. However, Theorem A.2 now proves that [KL20b, Proposition 3.3(b)] is correct, also in the case in which $\mathcal{N} \neq \{0\}$. As a consequence, [KL20b] and [KLS22, proof of Proposition 5.2], where [KL20b, Proposition 3.3(b)] are used, are not affected and remain true with the given proofs. Moreover, we would like to mention that the results of [KS23], together with further explicit calculations, give an alternative way to establish the validity of [KL20b, Proposition 3.3(b)] in special cases, e.g. when the defect groups are generalised quaternion or semi-dihedral 2-groups.

Acknowledgements. The authors are thankful to the referees for helpful comments which led to a substantial clarification of this manuscript. They would like to thank Gerhard Hiss and Gunter Malle for useful information on the modular representation theory of the finite groups of Lie type involved in this article. They would like to thank Naoko Kunugi for pointing out a gap in the proof of [KL20b, Proposition 3.3(b)] (see Appendix A), as well as Markus Linckelmann and Yuanyang Zhou for answering several of their questions on Puig's theory.

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S. Koshitani, Department of Mathematics and Informatics, Graduate School of Science, Chiba University, 1-33 Yayoi-cho, Inage-ku,Chiba 263-8522, Japan
Email address: koshitan@math.s.chiba-u.ac.jp

C. Lassueur, Institut für Algebra, Zahlentheorie und Diskrete Mathematik, Leibniz Universität Hannover, Welfengarten 1, D-30167 Hannover Germany
Email address: lassueur@mathematik.uni-kl.de

B. Sambale, Institut für Algebra, Zahlentheorie und Diskrete Mathematik, Leibniz Universität Hannover, Welfengarten 1, D-30167 Hannover, Germany
Email address: sambale@math.uni-hannover.de