# A counterexample to a conjecture of Kiyota, Murai and Wada

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**Abstract**

Kiyota, Murai and Wada conjectured in 2002 that the largest eigenvalue of the Cartan matrix $C$ of a block of a finite group is rational if and only if all eigenvalues of $C$ are rational. We provide a counterexample to this conjecture and discuss related questions.

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Let $B$ be a block of a finite group $G$ with respect to an algebraically closed field of characteristic $p > 0$. It is well-known that the Cartan matrix $C \in \mathbb{Z}^{l\times l}$ of $B$ is symmetric, positive definite, non-negative and indecomposable (here $l = l(B)$ is the number of simple modules of $B$). Let $ED$ (respectively $EV$) be the multiset of elementary divisors (respectively eigenvalues) of $C$. Note that these multisets do not depend on the order of the simple modules of $B$. Let $D$ be a defect group of $B$. Then the elementary divisors of $C$ divide $|D|$ and $|D|$ occurs just once in $ED$. On the other hand, the eigenvalues of $C$ are real, positive algebraic integers. By Perron-Frobenius theory, the largest eigenvalue $\rho(C)$ (i. e. the *spectral radius*) of $C$ occurs with multiplicity 1 in $EV$. Moreover, $\prod_{\lambda\in ED} \lambda = \det(C) = \prod_{\lambda\in EV} \lambda$. Apart from these facts, there seems little correlation between $EV$ and $ED$.

According to (the weak) Donovan's Conjecture, there should be an upper bound on $\rho(C)$ in terms of $|D|$. However, it can happen that $\rho(C) > l(B)|D|$. For example, if $B$ is the principal 2-block of $G = \mathrm{PSp}(4, 4).4$, a computation with GAP [1] shows that $\rho(C) > 7201 > 5 \cdot 2^{10} = l(B)|D|$. This is even more striking than the observation $\mathrm{tr}(C) > l(B)|D|$ made in [7] for the same block. Conversely, $|D|$ cannot be bounded in terms of $\rho(C)$: for $p \geq 5$ the principal $p$-block of $\mathrm{SL}(2, p)$ satisfies $\rho(C) < 4 < p = |D|$ (see [4, Example on p. 3843]).

If $\lambda \in EV \cap \mathbb{Z}$, then $|D|/\lambda$ is an eigenvalue of $|D|C^{-1} \in \mathbb{Z}^{l\times l}$ and therefore it is an algebraic integer. This shows that $\lambda$ divides $|D|$. By a similar argument, $\lambda$ is divisible by the smallest elementary divisor of $C$. In [3, Questions 1 and 2], Kiyota, Murai and Wada proposed the following conjecture on the rationality of eigenvalues (see also [10, Conjecture]).

**Conjecture 1** (Kiyota-Murai-Wada). *The following assertions are equivalent:*

*(1)* $EV = ED$.

*(2)* $\rho(C) = |D|$.

*(3)* $\rho(C) \in \mathbb{Z}$.

*(4)* $EV \subseteq \mathbb{Z}$.

Clearly, (1) ⇒ (2) ⇒ (3) ⇐ (4) ⇐ (1) holds and it remains to prove (3) ⇒ (1). This has been done for blocks of finite or tame representation type (see [3, Propositions 3 and 4]). For $p$-solvable $G$ we have (1) ⇔ (2) ⇔ (4) and $\rho(C) \leq |D|$ (see [3, Theorem 1], [8, Corollary 3.6] and [4, Corollary 3.6]). Other special cases were considered in [5, 6, 9, 12]. If $D \trianglelefteq G$, then (1)–(4) are satisfied (see [3, Proposition 2]). This holds in particular for the Brauer correspondent $b$ of $B$ in the normalizer $\mathrm{N}_G(D)$. In view of Broué's Abelian Defect Group Conjecture, Kiyota, Murai and Wada [3, Question 3] raised the following question.

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**Question 2** (Kiyota-Murai-Wada)**.** *If $D$ is abelian and $\rho(C) = |D|$, are $B$ and $b$ Morita equivalent?*

It was proved in [6, 5] that the answer to Question 2 is yes for principal $p$-blocks whenever $p \in \{2, 3\}$. However, the following counterexample shows not only that Conjecture 1 is false, but also that Question 2 has a negative answer (for principal blocks) in general:

Let $B$ be the principal 5-block of $G = \mathrm{PSU}(3, 4)$. The Atlas of Brauer characters [2] (or [11]) gives

$$C = \begin{pmatrix} 10 & 10 & 5 \\ 10 & 13 & 6 \\ 5 & 6 & 7 \end{pmatrix}.$$

It follows that $EV = \left\{\frac{1}{2}(5 + \sqrt{5}), \frac{1}{2}(5 - \sqrt{5}), 25\right\}$ and $ED = \{1, 5, 25\}$. Therefore, $\rho(C) = 25 = |D|$, but $EV \neq ED$. Moreover, $D$ is abelian since $|D| = 25$, but $B$ cannot be Morita equivalent to $b$, since the eigenvalues of the Cartan matrix of $b$ are rational integers as explained above.

We do not know whether the implications (3) $\Rightarrow$ (2), (4) $\Rightarrow$ (1) or (4) $\Rightarrow$ (2) in Conjecture 1 might hold in general. Wada [9, Decomposition Conjecture] strengthened all three implications as follows.

**Conjecture 3** (Wada)**.** *There exist partitions $EV = E_1 \sqcup \ldots \sqcup E_n$ and $ED = F_1 \sqcup \ldots \sqcup F_n$ of multisets such that*

- $|E_i| = |F_i|$ *for* $i = 1, \ldots, n$.
- $\prod_{\lambda \in E_i} \lambda = \prod_{\lambda \in F_i} \lambda$ *for* $i = 1, \ldots, n$.
- $\prod_{\lambda \in E_i} (X - \lambda) \in \mathbb{Z}[X]$ *is irreducible for* $i = 1, \ldots, n$.
- $\rho(C) \in E_1$, $|D| \in F_1$.

Again we found a counterexample: The group $\mathrm{PSU}(3, 3)$ has a faithful 7-dimensional representation over $\mathbb{F}_3$. Let $G = \mathbb{F}_3^7 \rtimes \mathrm{PSU}(3, 3)$ be the corresponding semidirect product, and let $B$ be the principal 3-block of $G$. This group and its character table can be accessed as `PrimitiveGroup`$(3^7, 35)$ and `CharacterTable("P49/G1/L1/V1/ext3")` in GAP. In this way we obtain $9 \in EV$, but $9 \notin ED$. Obviously, this contradicts Conjecture 3.

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