Loewy lengths of centers of blocks II

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Abstract

Let ZB be the center of a p-block B of a finite group with defect group D. We show that the Loewy length $LL(ZB)$ of ZB is bounded by $\frac{|D|}{p} + p - 1$ provided D is not cyclic. If D is non-abelian, we prove the stronger bound $LL(ZB) < \min\{p^{d-1}, 4p^{d-2}\}$ where $|D| = p^d$. Conversely, we classify the blocks B with $LL(ZB) \geq \min\{p^{d-1}, 4p^{d-2}\}$. This extends some results previously obtained by the present authors. Moreover, we characterize blocks with uniserial center.

Keywords: center of blocks, Loewy length
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1 Introduction

The aim of this paper is to extend some results on Loewy lengths of centers of blocks obtained in [8, 11]. In the following we will reuse some of the notation introduced in [8]. In particular, B is a block of a finite group G with respect to an algebraically closed field F of characteristic $p > 0$. Moreover, let D be a defect group of B. The second author has shown in [11, Corollary 3.3] that the Loewy length of the center of B is bounded by

$$LL(ZB) \leq |D| - \frac{|D|}{\exp(D)} + 1$$

where $\exp(D)$ is the exponent of D. It was already known to Okuyama [9] that this bound is best possible if D is cyclic. The first and the third author have given in [8, Theorem 1] the optimal bound $LL(ZB) \leq LL(FD)$ for blocks with abelian defect groups. Our main result of the present paper establishes the following bound for blocks with non-abelian defect groups:

$$LL(ZB) < \min\{p^{d-1}, 4p^{d-2}\}$$

where $|D| = p^d$. As a consequence we obtain

$$LL(ZB) \leq p^{d-1} + p - 1$$

for all blocks with non-cyclic defect groups. It can be seen that this bound is optimal whenever B is nilpotent and $D \cong C_{p^{d-1}} \times C_p$.

In the second part of the paper we show that $LL(ZB)$ depends more on $\exp(D)$ than on $|D|$. We prove for instance that $LL(ZB) \leq d^2 \exp(D)$ unless $d = 0$. Finally, we use the opportunity to improve a result of Willems [14] about blocks with uniserial center.

In addition to the notation used in the papers cited above, we introduce the following objects. Let $\mathrm{Cl}(G)$ be the set of conjugacy classes of G. A p-subgroup $P \leq G$ is called a defect group of $K \in \mathrm{Cl}(G)$ if P is a Sylow

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p-subgroup of $\mathrm{C}_G(x)$ for some $x \in K$. Let $\mathrm{Cl}_P(G)$ be the set of conjugacy classes with defect group P. Let $K^+ := \sum_{x\in K} x \in FG$ and

$$\begin{aligned} I_P(G) &:= \langle K^+ : K \in \mathrm{Cl}_P(G)\rangle \subseteq ZFG, \\ I_{\le P}(G) &:= \sum_{Q\le P} I_Q(G) \trianglelefteq ZFG, \\ I_{<P}(G) &:= \sum_{Q<P} I_Q(G) \trianglelefteq ZFG. \end{aligned}$$

2 Results

We begin by restating a lemma of Passman [12, Lemma 2]. For the convenience of the reader we provide a (slightly easier) proof.

Lemma 1 (Passman)**.** *Let P be a central p-subgroup of G. Then $I_{\le P}(G) \cdot JZFG = I_{\le P}(G) \cdot JFP$.*

Proof. Let K be a conjugacy class of G with defect group P, and let $x \in K$. Then P is the only Sylow p-subgroup of $\mathrm{C}_G(x)$, and the p-factor u of x centralizes x. Thus $u \in P$. Hence u is the p-factor of every element in K, and $K = uK'$ where K' is a p-regular conjugacy class of G with defect group P. This shows that $I := I_{\le P}(G)$ is a free FP-module with the p-regular class sums with defect group P as an FP-basis. The canonical epimorphism $\nu : FG \to F[G/P]$ maps I into $I_1(G/P) \subseteq SF[G/P]$. Thus $\nu(I \cdot JZFG) \subseteq SF[G/P] \cdot JZF[G/P] = 0$. Hence $I \cdot JZFG \subseteq I \cdot JFP$. The other inclusion is trivial. □

Lemma 2. *Let $P \le G$ be a p-subgroup of order p^n. Then*

(i) $I_{\le P}(G) \cdot JZFG^{LL(F\,\mathrm{Z}(P))} \subseteq I_{<P}(G)$.

(ii) $I_{\le P}(G) \cdot JZFG^{(p^{n+1}-1)/(p-1)} = 0$.

Proof.

(i) Let $\mathrm{Br}_P : ZFG \to ZF\,\mathrm{C}_G(P)$ be the Brauer homomorphism. Since $\mathrm{Ker}(\mathrm{Br}_P) \cap I_{\le P}(G) = I_{<P}(G)$, we need to show that $\mathrm{Br}_P(I_{\le P}(G) \cdot JZFG^{LL(F\,\mathrm{Z}(P))}) = 0$. By Lemma 1 we have

$$\begin{aligned} \mathrm{Br}_P(I_{\le P}(G) \cdot JZFG^{LL(F\,\mathrm{Z}(P))}) &\subseteq I_{\le \mathrm{Z}(P)}(\mathrm{C}_G(P)) \cdot JZF\,\mathrm{C}_G(P)^{LL(F\,\mathrm{Z}(P))} \\ &= I_{\le \mathrm{Z}(P)}(\mathrm{C}_G(P)) \cdot JF\,\mathrm{Z}(P)^{LL(F\,\mathrm{Z}(P))} = 0. \end{aligned}$$

(ii) We argue by induction on n. The case $n = 1$ follows from $I_1(G) \subseteq SFG$. Now suppose that the claim holds for $n-1$. Since $LL(F\,\mathrm{Z}(P)) \le |P| = p^n$, (i) implies

$$\begin{aligned} I_{\le P}(G) \cdot JZFG^{(p^{n+1}-1)/(p-1)} &= I_{\le P}(G) \cdot JZFG^{p^n} JZFG^{(p^n-1)/(p-1)} \\ &\subseteq I_{<P}(G) \cdot JZFG^{(p^n-1)/(p-1)} \\ &= \sum_{Q<P} I_{\le Q}(G) \cdot JZFG^{(p^n-1)/(p-1)} = 0. \end{aligned}$$ □

Recall from [8, Lemma 9] the following group

$$W_{p^d} := \langle x, y, z \mid x^{p^{d-2}} = y^p = z^p = [x,y] = [x,z] = 1,\ [y,z] = x^{p^{d-3}}\rangle.$$

Note that W_{p^d} is a central product of $C_{p^{d-2}}$ and an extraspecial group of order p^3. Now we prove our main theorem which improves [8, Theorem 12].

Theorem 3. *Let B be a block of FG with non-abelian defect group D of order p^d. Then one of the following holds*

(i) $LL(ZB) < 3p^{d-2}$.

(ii) $p \geq 5$, $D \cong W_{p^d}$ and $LL(ZB) < 4p^{d-2}$.

In any case we have

$$LL(ZB) < \min\{p^{d-1}, 4p^{d-2}\}.$$

Proof. By [8, Proposition 15], we may assume that $p > 2$. Since D is non-abelian, $|D : \mathrm{Z}(D)| \geq p^2$ and $LL(F\,\mathrm{Z}(D)) \leq p^{d-2}$. Let Q be a maximal subgroup of D. If Q is cyclic, then $D \cong M_{p^n}$ and the claim follows from [8, Proposition 10]. Hence, we may assume that Q is not cyclic. Then $LL(F\,\mathrm{Z}(Q)) \leq p^{d-2} + p - 1$. Now setting $\lambda := \frac{p^{d-1}-1}{p-1}$ it follows from Lemma 2 that

$$\begin{aligned} JZB^{2p^{d-2}+p-1+\lambda} &\subseteq 1_B JZFG^{2p^{d-2}+p-1+\lambda} \subseteq I_{\leq D}(G) \cdot JZFG^{2p^{d-2}+p-1+\lambda} \\ &\subseteq I_{<D}(G) \cdot JZFG^{p^{d-2}+p-1+\lambda} = \sum_{Q<D} I_{\leq Q}(G) \cdot JZFG^{p^{d-2}+p-1+\lambda} \\ &\subseteq \sum_{Q<D} I_{<Q}(G) \cdot JZFG^{\lambda} = 0. \end{aligned}$$

Since $2p^{d-2} + p - 1 + \lambda \leq 4p^{d-2}$, we are done in case $p \geq 5$ and $D \cong W_{p^d}$. If $p = 3$ and $D \cong W_{p^d}$, then the claim follows from [8, Lemma 11]. Now suppose that $D \not\cong W_{p^d}$. If $\mathrm{Z}(D)$ is cyclic of order p^{d-2}, then the claim follows from [8, Lemma 9 and Proposition 10]. Hence, suppose that $\mathrm{Z}(D)$ is non-cyclic or $|\mathrm{Z}(D)| < p^{d-2}$. Then $d \geq 4$ and $LL(F\,\mathrm{Z}(D)) \leq p^{d-3} + p - 1$. The arguments above give $LL(ZB) \leq p^{d-2} + p^{d-3} + 2p - 2 + \lambda$, hence we are done whenever $p > 3$.

In the following we assume that $p = 3$. Here we have $LL(ZB) \leq 3^{d-2} + 3^{d-3} + 4 + \frac{1}{2}(3^{d-1} - 1)$ and it suffices to handle the case $d = 4$. By [11, Theorem 3.2], there exists a non-trivial B-subsection (u, b) such that

$$LL(ZB) \leq (|\langle u \rangle| - 1)LL(Z\overline{b}) + 1$$

where $\overline{b}$ is the unique block of $F\,\mathrm{C}_G(u)/\langle u \rangle$ dominated by b. We may assume that $\overline{b}$ has defect group $\mathrm{C}_D(u)/\langle u \rangle$ (see [13, Lemma 1.34]). If $u \notin \mathrm{Z}(D)$, we obtain $LL(ZB) < |\mathrm{C}_D(u)| \leq 27$ as desired. Hence, let $u \in \mathrm{Z}(D)$. Then $D/\langle u \rangle$ is not cyclic. Moreover, by our assumption on $\mathrm{Z}(D)$, we have $|\langle u \rangle| = 3$. Now it follows from [8, Theorem 1, Proposition 10 and Lemma 11] applied to $\overline{b}$ that

$$LL(ZB) \leq 2LL(Z\overline{b}) + 1 \leq 23 < 27.$$

□

We do not expect that the bounds in Theorem 3 are sharp. In fact, we do not know if there are p-blocks B with non-abelian defect groups of order p^d such that $p > 2$ and $LL(ZB) > p^{d-2}$. See also Proposition 7 below.

Corollary 4. *Let B be a block of FG with non-cyclic defect group of order p^d. Then*

$$LL(ZB) \leq p^{d-1} + p - 1.$$

Proof. By Theorem 3, we may assume that B has abelian defect group D. Then [8, Theorem 1] implies $LL(ZB) \leq LL(FD) \leq p^{d-1} + p - 1$. □

We are now in a position to generalize [8, Corollary 16].

Corollary 5. *Let B be a block of FG with defect group D of order p^d such that $LL(ZB) \geq \min\{p^{d-1}, 4p^{d-2}\}$. Then one of the following holds*

(i) *D is cyclic.*

(ii) $D \cong C_{p^{d-1}} \times C_p$.

(iii) *$D \cong C_2 \times C_2 \times C_2$ and B is nilpotent.*

Proof. Again by Theorem 3 we may assume that D is abelian. By [8, Corollary 16], we may assume that $p > 2$. Suppose that D is of type $(p^{a_1}, \ldots, p^{a_s})$ such that $s \geq 3$. Then

$$\begin{aligned}\min\{p^{d-1}, 4p^{d-2}\} &\leq LL(ZB) = p^{a_1} + \ldots + p^{a_s} - s + 1 \\ &\leq p^{a_1} + p^{a_2} + p^{a_3 + \ldots + a_s} - 2 \leq p^{d-2} + 2(p-1).\end{aligned}$$

This clearly leads to a contradiction. Therefore, $s \leq 2$ and the claim follows. □

In case (i) of Corollary 5 it is known conversely that $LL(ZB) = \frac{p^d - 1}{l(B)} + 1 > p^{d-1}$ (see [6, Corollary 2.8]).

Our next result gives a more precise bound by invoking the exponent of a defect group.

Theorem 6. *Let B be a block of FG with defect group D of order $p^d > 1$ and exponent p^e. Then*

$$LL(ZB) \leq \Big(\frac{d}{e} + 1\Big)\Big(\frac{d}{2} + \frac{1}{p-1}\Big)(p^e - 1).$$

In particular, $LL(ZB) \leq d^2 p^e$.

Proof. Let $\alpha := \lfloor d/e \rfloor$. Let $P \leq D$ be abelian of order p^{ie+j} with $0 \leq i \leq \alpha$ and $0 \leq j < e$. If P has type $(p^{a_1}, \ldots, p^{a_r})$, then $a_i \leq e$ for $i = 1, \ldots, r$ and

$$LL(FP) = (p^{a_1} - 1) + \ldots + (p^{a_r} - 1) + 1 \leq i(p^e - 1) + p^j.$$

Arguing as in Theorem 3, we obtain

$$\begin{aligned}LL(ZB) &\leq \sum_{i=0}^{\alpha} \sum_{j=0}^{e-1} i(p^e - 1) + p^j = e(p^e - 1)\Big(\sum_{i=0}^{\alpha} i\Big) + (\alpha + 1)\frac{p^e - 1}{p - 1} \\ &= e(p^e - 1)\frac{\alpha(\alpha + 1)}{2} + (\alpha + 1)\frac{p^e - 1}{p - 1} \\ &\leq \Big(\frac{d}{e} + 1\Big)\Big(\frac{d}{2} + \frac{1}{p-1}\Big)(p^e - 1).\end{aligned}$$

This proves the first claim. For the second claim we note that

$$\Big(\frac{d}{e} + 1\Big)\Big(\frac{d}{2} + \frac{1}{p-1}\Big) \leq (d+1)\Big(\frac{d}{2} + 1\Big) \leq d^2$$

unless $d \leq 3$. In these small cases the claim follows from Theorem 3 and Corollary 4. □

If $2e > d$ and p is large, then the bound in Theorem 6 is approximately dp^e. The groups of the form $G = D = C_{p^e} \times \ldots \times C_{p^e}$ show that there is no bound of the form $LL(ZB) \leq Cp^e$ where C is an absolute constant. A more careful argumentation in the proof above gives the stronger (but opaque) bound

$$LL(ZB) \leq \alpha(p^e - 1)\Big(\frac{e(\alpha - 1)}{2} + \frac{1}{p-1} + d - \alpha e\Big) + \beta(p^e - 1) + \frac{p^{d - \alpha e} - 1}{p - 1} + p^{d - 2 - \beta e}$$

for non-abelian defect groups where $\alpha := \lfloor \frac{d-1}{e} \rfloor$ and $\beta := \lfloor \frac{d-2}{e} \rfloor$. We omit the details.

In the next result we compute the Loewy length for $d = e + 1$.

Proposition 7. *Let B be a block of FG with non-abelian defect group of order p^d and exponent p^{d-1}. Then*

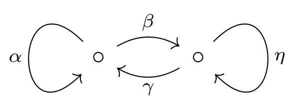
$$LL(ZB) \leq \begin{cases} 2^{d-2} + 1 & \text{if } p = 2, \\ p^{d-2} & \text{if } p > 2 \end{cases}$$

and both bounds are optimal for every $d \geq 3$.

Proof. Let D be a defect group of B. If $p > 2$, then $D \cong M_{p^d}$ and we have shown $LL(ZB) \le p^{d-2}$ in [8, Proposition 10]. Equality holds if and only if B is nilpotent.

Therefore, we may assume $p = 2$ in the following. The modular groups M_{2^d} are still handled by [8, Proposition 10]. Hence, it remains to consider the defect groups of maximal nilpotency class, i. e. $D \in \{D_{2^d}, Q_{2^d}, SD_{2^d}\}$. By [8, Proposition 10], we may assume that $d \ge 4$. The isomorphism type of ZB is uniquely determined by D and the fusion system of B (see [2]). The possible cases are listed in [13, Theorem 8.1]. If B is nilpotent, [8, Proposition 8] gives $LL(ZB) = LL(ZFD) \le LL(FD') = 2^{d-2}$. Moreover, in the case $D \cong D_{2^d}$ and $l(B) = 3$ we have $LL(ZB) \le k(B) - l(B) + 1 = 2^{d-2} + 1$ by [11, Proposition 2.2]. In the remaining cases we present B by quivers with relations which were constructed originally by Erdmann [3]. We refer to [4, Appendix B].

(i) $D \cong D_{2^d}$, $l(B) = 2$:



$$\beta\eta = \eta\gamma = \gamma\beta = \alpha^2 = 0,\quad \alpha\beta\gamma = \beta\gamma\alpha,\quad \eta^{2^{d-2}} = \gamma\alpha\beta.$$

By [4, Lemma 2.3.3], we have

$$ZB = \operatorname{span}\{1, \beta\gamma, \alpha\beta\gamma, \eta^i : i = 1, \ldots, 2^{d-2}\}.$$

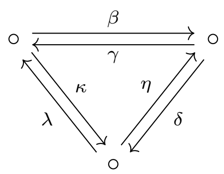
It follows that $JZB^2 = \langle \eta^2 \rangle$ and $LL(ZB) = 2^{d-2} + 1$.

(ii) $D \cong Q_{2^d}$, $l(B) = 2$: Here [15, Lemma 6] gives the isomorphism type of ZB directly as a quotient of a polynomial ring

$$ZB \cong F[U, Y, S, T]/(Y^{2^{d-2}+1}, U^2 - Y^{2^{d-2}}, S^2, T^2, SY, SU, ST, UY, UT, YT).$$

It follows that $JZB^2 = (Y^2)$ and again $LL(ZB) = 2^{d-2} + 1$.

(iii) $D \cong Q_{2^d}$, $l(B) = 3$:



$$\beta\delta = (\kappa\lambda)^{2^{d-2}-1}\kappa,\ \eta\gamma = (\lambda\kappa)^{2^{d-2}-1}\lambda,\quad \delta\lambda = \gamma\beta\gamma,\ \kappa\eta = \beta\gamma\beta,\ \lambda\beta = \eta\delta\eta,\quad \gamma\kappa = \delta\eta\delta,\ \gamma\beta\delta = \delta\eta\gamma = \lambda\kappa\eta = 0.$$

By [4, Lemma 2.5.15],

$$ZB = \operatorname{span}\{1, \beta\gamma + \gamma\beta, (\kappa\lambda)^i + (\lambda\kappa)^i, \delta\eta + \eta\delta, (\beta\gamma)^2, (\lambda\kappa)^{2^{d-2}}, (\delta\eta)^2 : i = 1, \ldots, 2^{d-2} - 1\}.$$

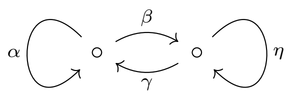
We compute

$$\begin{aligned}
(\beta\gamma + \gamma\beta)^2 &= (\beta\gamma)^2 + (\gamma\beta)^2 = (\beta\gamma)^2 + \delta\lambda\beta = (\beta\gamma)^2 + (\delta\eta)^2,\\
(\beta\gamma + \gamma\beta)(\kappa\lambda + \lambda\kappa) &= \beta\gamma\kappa\lambda = \beta\delta\eta\delta\lambda = \beta\delta\eta\gamma\beta\gamma = 0,\\
(\beta\gamma + \gamma\beta)(\delta\eta + \eta\delta) &= \gamma\beta\delta\eta = 0,\\
(\beta\gamma + \gamma\beta)(\beta\gamma)^2 &= (\beta\gamma)^3 = \beta\gamma\beta\delta\lambda = 0,\\
(\beta\gamma + \gamma\beta)(\lambda\kappa)^{2^{d-2}} &= 0,\\
(\beta\gamma + \gamma\beta)(\delta\eta)^2 &= \gamma\beta\delta\eta\delta\eta = 0,\\
((\kappa\lambda)^{2^{d-2}-1} + (\lambda\kappa)^{2^{d-2}-1})(\kappa\lambda + \lambda\kappa) &= \kappa\eta\gamma + (\lambda\kappa)^{2^{d-2}} = (\beta\gamma)^2 + (\lambda\kappa)^{2^{d-2}},\\
(\kappa\lambda + \lambda\kappa)(\delta\eta + \eta\delta) &= \lambda\kappa\eta\delta = 0,\\
(\kappa\lambda + \lambda\kappa)(\beta\gamma)^2 &= \kappa\lambda\beta\gamma\beta\gamma = \kappa\eta\delta\eta\gamma\beta\gamma = 0,\\
(\kappa\lambda + \lambda\kappa)(\lambda\kappa)^{2^{d-2}} &= \lambda\kappa\eta\gamma\kappa = 0,
\end{aligned}$$

$$\begin{aligned}
(\kappa\lambda + \lambda\kappa)(\delta\eta)^2 &= 0,\\
(\delta\eta + \eta\delta)^2 = (\delta\eta)^2 + (\eta\delta)^2 &= (\delta\eta)^2 + \lambda\beta\delta = (\delta\eta)^2 + (\lambda\kappa)^{2^{d-2}},\\
(\delta\eta + \eta\delta)(\beta\gamma)^2 &= 0,\\
(\delta\eta + \eta\delta)(\lambda\kappa)^{2^{d-2}} &= \eta\delta(\lambda\kappa)^{2^{d-2}} = \eta\delta\eta\gamma\kappa = 0,\\
(\delta\eta + \eta\delta)(\delta\eta)^2 &= \delta\lambda\beta\delta\eta = \gamma\beta\gamma\beta\delta\eta = 0,\\
(\beta\gamma)^2(\beta\gamma)^2 &= (\beta\gamma)^2(\lambda\kappa)^{2^{d-2}} = (\beta\gamma)^2(\delta\eta)^2 = 0,\\
(\lambda\kappa)^{2^{d-2}}(\lambda\kappa)^{2^{d-2}} &= (\lambda\kappa)^{2^{d-2}}(\delta\eta)^2 = 0,\\
(\delta\eta)^2(\delta\eta)^2 &= \gamma\kappa\eta(\delta\eta)^2 = \gamma\beta\gamma\beta(\delta\eta)^2 = 0.
\end{aligned}$$

Hence, $JZB^2 = \langle(\lambda\kappa)^2 + (\kappa\lambda)^2, (\beta\gamma)^2 + (\delta\eta)^2\rangle$ and $JZB^3 = \langle(\lambda\kappa)^3 + (\kappa\lambda)^3\rangle$. This implies $LL(ZB) = 2^{d-2} + 1$.

(iv) $D \cong SD_{2^d}$, $k(B) = 2^{d-2} + 3$ and $l(B) = 2$:



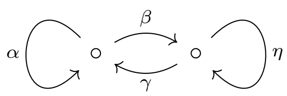
$$\begin{gathered}
\gamma\beta = \eta\gamma = \beta\eta = 0,\\
\alpha^2 = \beta\gamma,\ \alpha\beta\gamma = \beta\gamma\alpha,\\
\eta^{2^{d-2}} = \gamma\alpha\beta.
\end{gathered}$$

By [5, Section 5.1], we have

$$ZB = \operatorname{span}\{1, \beta\gamma, \alpha\beta\gamma, \eta^i : i = 1, \ldots, 2^{d-2}\}.$$

As in (i) we obtain $JZB^2 = \langle\eta^2\rangle$ and $LL(ZB) = 2^{d-2} + 1$.

(v) $D \cong SD_{2^d}$, $k(B) = 2^{d-2} + 4$ and $l(B) = 2$:



$$\begin{gathered}
\beta\eta = \alpha\beta\gamma\alpha\beta,\ \gamma\beta = \eta^{2^{d-2}-1},\\
\eta\gamma = \gamma\alpha\beta\gamma\alpha,\\
\beta\eta^2 = \eta^2\gamma = \alpha^2 = 0.
\end{gathered}$$

By [5, Section 5.2.2], we have

$$ZB = \operatorname{span}\{1, \alpha\beta\gamma + \beta\gamma\alpha + \gamma\alpha\beta, \beta\gamma\alpha\beta\gamma, (\alpha\beta\gamma)^2, \eta^i, \eta + \alpha\beta\gamma\alpha : i = 2, \ldots, 2^{d-2}\}.$$

Since $(\alpha\beta\gamma)^2 = \beta\eta\gamma = (\beta\gamma\alpha)^2$ and $(\gamma\alpha\beta)^2 = \eta\gamma\beta = \eta^{2^{d-2}}$, it follows that

$$(\alpha\beta\gamma + \beta\gamma\alpha + \gamma\alpha\beta)^2 = (\alpha\beta\gamma)^2 + (\beta\gamma\alpha)^2 + (\gamma\alpha\beta)^2 = \eta^{2^{d-2}}.$$

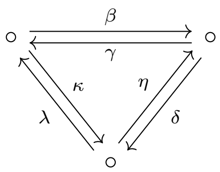
Similarly,

$$\begin{aligned}
(\alpha\beta\gamma + \beta\gamma\alpha + \gamma\alpha\beta)\beta\gamma\alpha\beta\gamma &= 0,\\
(\alpha\beta\gamma + \beta\gamma\alpha + \gamma\alpha\beta)(\alpha\beta\gamma)^2 &= 0,\\
(\alpha\beta\gamma + \beta\gamma\alpha + \gamma\alpha\beta)\eta^2 &= 0,\\
(\alpha\beta\gamma + \beta\gamma\alpha + \gamma\alpha\beta)(\eta + \alpha\beta\gamma\alpha) &= 0,\\
(\beta\gamma\alpha\beta\gamma)^2 &= 0,\\
\beta\gamma\alpha\beta\gamma(\alpha\beta\gamma)^2 &= 0,\\
\beta\gamma\alpha\beta\gamma\eta^2 = \beta\gamma\alpha\beta\eta^2\gamma &= 0,\\
\beta\gamma\alpha\beta\gamma(\eta + \alpha\beta\gamma\alpha) = \beta\gamma(\alpha\beta\gamma)^2\alpha &= 0,\\
(\alpha\beta\gamma)^2(\alpha\beta\gamma)^2 &= 0,
\end{aligned}$$

$$
\begin{aligned}
(\alpha\beta\gamma)^2\eta^2 &= 0,\\
(\alpha\beta\gamma)^2(\eta+\alpha\beta\gamma\alpha) &= 0,\\
\eta^2(\eta+\alpha\beta\gamma\alpha) &= \eta^3,\\
(\eta+\alpha\beta\gamma\alpha)^2 &= \eta^2.
\end{aligned}
$$

Consequently, $JZB^2 = \langle \eta^2 \rangle$ and $LL(ZB) = 2^{d-2}+1$.

(vi) $D \cong SD_{2^d}$, $l(B) = 3$:



$$
\kappa\eta = \eta\gamma = \gamma\kappa = 0,\ \delta\lambda = (\gamma\beta)^{2^{d-2}-1}\gamma,\\
\beta\delta = \kappa\lambda\kappa,\ \lambda\beta = \eta.
$$

From [4, Lemma 2.4.16] we get

$$
ZB = \operatorname{span}\{1, (\beta\gamma)^i + (\gamma\beta)^i, \kappa\lambda + \lambda\kappa, (\beta\gamma)^{2^{d-2}}, (\lambda\kappa)^2, \delta\eta : i = 1, \ldots, 2^{d-2}-1\}.
$$

We compute

$$
\begin{aligned}
(\beta\gamma+\gamma\beta)((\beta\gamma)^{2^{d-2}-1} + (\gamma\beta)^{2^{d-2}-1}) &= (\beta\gamma)^{2^{d-2}} + \delta\lambda\beta = (\beta\gamma)^{2^{d-2}} + \delta\eta,\\
(\beta\gamma+\gamma\beta)(\kappa\lambda+\lambda\kappa) &= \beta\gamma\kappa\lambda = 0,\\
(\beta\gamma+\gamma\beta)(\beta\gamma)^{2^{d-2}} &= \beta\delta\lambda\beta\gamma = \kappa\lambda\kappa\eta\gamma = 0,\\
(\beta\gamma+\gamma\beta)(\lambda\kappa)^2 &= 0,\\
(\beta\gamma+\gamma\beta)\delta\eta &= \gamma\beta\delta\eta = \gamma\kappa\lambda\kappa\eta = 0,\\
(\kappa\lambda+\lambda\kappa)^2 &= \beta\delta\lambda + (\lambda\kappa)^2 = (\beta\gamma)^{2^{d-2}} + (\lambda\kappa)^2,\\
(\kappa\lambda+\lambda\kappa)(\beta\gamma)^{2^{d-2}} &= \kappa\lambda\beta\gamma(\beta\gamma)^{2^{d-2}-1} = \kappa\eta\gamma(\beta\gamma)^{2^{d-2}-1} = 0,\\
(\kappa\lambda+\lambda\kappa)(\lambda\kappa)^2 &= \lambda(\beta\gamma)^{2^{d-2}}\kappa = \eta\gamma(\beta\gamma)^{2^{d-2}-1}\kappa = 0,\\
(\kappa\lambda+\lambda\kappa)\delta\eta &= 0,\\
(\beta\gamma)^{2^{d-2}}(\beta\gamma)^{2^{d-2}} &= (\beta\gamma)^{2^{d-2}}(\lambda\kappa)^2 = (\beta\gamma)^{2^{d-2}}\delta\eta = 0,\\
(\lambda\kappa)^2(\lambda\kappa)^2 &= (\lambda\kappa)^2\delta\eta = 0,\\
(\delta\eta)^2 &= \delta\lambda\beta\delta\eta = \delta\lambda\kappa\lambda\kappa\eta = 0.
\end{aligned}
$$

Hence, $JZB^2 = \langle (\beta\gamma)^2 + (\gamma\beta)^2, (\kappa\lambda)^2 + \delta\eta \rangle$ and $JZB^3 = \langle (\beta\gamma)^3 + (\gamma\beta)^3 \rangle$. This implies $LL(ZB) = 2^{d-2}+1$. □

It is interesting to note the difference between even and odd primes in Proposition 7. For $p = 2$, non-nilpotent blocks gives larger Loewy lengths while for $p > 2$ the maximal Loewy length is only assumed for nilpotent blocks.

Recall that a *lower defect group* of a block B of FG is a p-subgroup $Q \le G$ such that

$$
I_{<Q}(G)1_B \ne I_{\le Q}(G)1_B.
$$

In this case Q is conjugate to a subgroup of a defect group D of B and conversely D is also a lower defect group since $1_B \in I_{\le D}(G) \setminus I_{<D}(G)$. It is clear that in the proofs of Theorem 3 and Theorem 6 it suffices to sum over the lower defect groups of B. In particular there exists a chain of lower defect groups $Q_1 < \ldots < Q_n = D$ such that $LL(ZB) \le \sum_{i=1}^n LL(F\,\mathrm{Z}(Q_i))$. Unfortunately, it is hard to compute the lower defect groups of a given block.

The following proposition generalizes [14, Theorem 1.5].

Proposition 8. *Let B be a block of FG. Then ZB is uniserial if and only if B is nilpotent with cyclic defect groups.*

Proof. Suppose first that ZB is uniserial. Then $ZB \cong F[X]/(X^n)$ for some $n \in \mathbb{N}$; in particular, ZB is a symmetric F-algebra. Then [10, Theorems 3 and 5] implies that B is nilpotent with abelian defect group D. Thus, by a result of Broué and Puig [1] (see also [7]), B is Morita equivalent to FD; in particular, FD is also uniserial. Thus D is cyclic.

Conversely, suppose that B is nilpotent with cyclic defect group D. Then the Broué-Puig result mentioned above implies that B is Morita equivalent of FD. Thus $ZB \cong ZFD = FD$. Since FD is uniserial, the result follows. □

A similar proof shows that ZB is isomorphic to the group algebra of the Klein four group over an algebraically closed field of characteristic 2 if and only if B is nilpotent with Klein four defect groups.

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