WEIGHTS AND NILPOTENT SUBGROUPS

GABRIEL NAVARRO AND BENJAMIN SAMBALE

Abstract. In a finite group G, we consider nilpotent weights, and prove a π-version of the Alperin Weight Conjecture for certain π-separable groups. This widely generalizes an earlier result by I. M. Isaacs and the first author.

1. Introduction

Let G be a finite group and let p be a prime. The celebrated Alperin Weight Conjecture asserts that the number of conjugacy classes of G consisting of elements of order not divisible by p is exactly the number of G-conjugacy classes of p-weights. Recall that a **p-weight** is a pair (Q, γ), where Q is a p-subgroup of G and $\gamma \in \mathrm{Irr}(\mathbf{N}_G(Q)/Q)$ is an irreducible complex character with p-defect zero (that is, such that the p-part $\gamma(1)_p = |\mathbf{N}_G(Q)/Q|_p$).

In the main result of this paper, we replace p by a set of primes π as follows:

Theorem A. *Let G be a π-separable group with a solvable Hall π-subgroup. Then the number of conjugacy classes of π'-elements of G is the number of G-conjugacy classes of pairs (Q, γ), where Q is a nilpotent π-subgroup of G and $\gamma \in \mathrm{Irr}(\mathbf{N}_G(Q)/Q)$ has p-defect zero for every $p \in \pi$.*

Recall that a finite group is called **π-separable** if all its composition factors are π-groups or π'-groups. Let us restate Theorem A in the (presumably trivial) case where G itself is a (solvable) π-group. In this case, there is only one conjugacy class of π'-elements of G. On the other hand, if Q is a nilpotent subgroup of G, then $\gamma \in \mathrm{Irr}(\mathbf{N}_G(Q)/Q)$ has p-defect zero for every $p \in \pi$ if and only if $\mathbf{N}_G(Q) = Q$. Amazingly enough, there is only one conjugacy class of self-normalizing nilpotent subgroups: the Carter subgroups of G (see p. 281 in [R]).

Of course, if $\pi = \{p\}$, then Theorem A is the p-solvable case of the Alperin Weight Conjecture (AWC). As a matter of fact, AWC was proven for π-separable groups with a nilpotent Hall π-subgroup by Isaacs and the first author [IN]. Now we realize that

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the nilpotency hypothesis can be dropped if one counts nilpotent weights instead. The solvability hypothesis is still needed, as shown by $G = \mathsf{A}_5$ and $\pi = \{2, 3, 5\}$.

There is a price to pay, however. The proof in [IN] relied on the so called Okuyama–Wajima argument, a definitely non-trivial but accessible tool involving extensions of Glauberman correspondents. In order to prove Theorem A, however, we shall need to appeal to a deeper theorem of Dade and Puig (which uses Dade's classification of the endo-permutation modules).

As it is often the case in a "solvable" framework, the equality of cardinalities in Theorem A has a hidden structure which we are going to explain now. For sake of convenience we interchange from now on the roles of π and π' (of course, π-separable is equivalent to π'-separable). Recall that in a π-separable group G, the set $\mathbf{I}_\pi(G)$ of irreducible π-partial characters of G is the exact π-version (when π is the complement of a prime p) of the irreducible Brauer characters $\mathrm{IBr}(G)$ of a p-solvable group (see next section for precise definitions). Each $\varphi \in \mathbf{I}_\pi(G)$ has canonically associated a G-conjugacy class of π'-subgroups Q, which are called the **vertices** of φ. If $\mathbf{I}_\pi(G|Q)$ is the set of irreducible π-partial characters with vertex Q, unless $\pi = p'$, it is not in general true that $|\mathbf{I}_\pi(G|Q)| = |\mathbf{I}_\pi(\mathbf{N}_G(Q)|Q)|$. Instead we will prove the following theorem.

Theorem B. *Suppose that G is π-separable with a solvable Hall π-complement. Let R be a nilpotent π'-subgroup of G and let $\mathcal{Q}$ be the set of π'-subgroups Q of G such that R is a Carter subgroup of Q. Then*

$$\Big| \bigcup_{Q \in \mathcal{Q}} \mathbf{I}_\pi(G|Q) \Big| = |\mathbf{I}_\pi(\mathbf{N}_G(R)|R)| \,.$$

Since $|\mathbf{I}_\pi(\mathbf{N}_G(R)|R)|$ is just the number of π'-weights with first component R (see Lemma 6.28 of [I2]), Theorem B implies Theorem A.

As happens in the classical case where $\pi = p'$, and following the ideas of Dade, Knörr and Robinson, one can define chains of π'-subgroups and relate them with π-defect of characters. This shall be explored elsewhere. Similarly, one can attach every weight to a π'-block B of G by using Slattery's theory [S]. In this setting we expect that the number of π-partial characters belonging to B equals the number of nilpotent weights attached to B.

The groups described in Theorem A are sometimes called π-solvable. We did not find a counterexample in the wider class of so-called π-selected groups. Here, π-**selected** means that the order of every composition factor is divisible by at most one prime in π. P. Hall [H] has shown that these groups still have solvable Hall π-subgroups. Since every finite group is p-selected for every prime p, this version of the conjecture includes the full AWC.

Unfortunately, Theorem A does not hold for arbitrary groups even if they possess nilpotent Hall π-subgroups. It is not so easy to find a counterexample, though.

The fourth Janko group $G = J_4$ has a cyclic Hall π-subgroup of order 35 (that is, $\pi = \{5, 7\}$). The normalizers of the non-trivial π-weights are contained in a maximal subgroup M of type $2^{3+12}.(S_5 \times L_3(2))$. However, $l(G) - k_0(G) = 25 \neq 30 = l(M) - k_0(M)$, where $l(G)$ denotes the number of π'-conjugacy classes and $k_0(G)$ is the number of π-defect zero characters of G. (The fact that J_4 was a counterexample for the π-version of the McKay conjecture for groups with a nilpotent Hall π-subgroup was noticed by Pham H. Tiep and the first author.)

We take the opportunity to thank the developers of [GAP]. Without their tremendous work the present paper probably would not exist.

The paper is organized as follows: In the next section we review π-partial characters which were introduced by Isaacs. In Section 3 we present two general lemmas on characters in π-separable groups. Afterwards we prove Theorem B. In the final section we construct a natural bijection explaining Theorem B in the presence of a normal Hall π-subgroup.

2. Review of π-theory

Isaacs' π-theory is the π-version in π-separable groups of the p-modular representation theory for p-solvable groups. When $\pi = p'$, the complement of a prime, then $\mathbf{I}_\pi(G) = \mathrm{IBr}(G)$ and we recover most of the well-known classical results. In what follows G is a finite π-separable group, where π is a set of primes. All the references for π-theory can now be found together in Isaacs' recent book [I2]. For the reader's convenience, we review some of the main features. If n is a natural number and p is a prime, recall that n_p is the largest power of p dividing n. If π is a set of primes, then $n_\pi = \prod_{p\in\pi} n_p$. The number n is a π-number if $n = n_\pi$.

If G is a π-separable group, then G^0 is the set of elements of G whose order is a π-number. A π-partial character of G is the restriction of a complex character of G to G^0. A π-partial character is **irreducible** if it is not the sum of two π-partial characters. We write $\mathbf{I}_\pi(G)$ for the set of irreducible π-partial characters of G. Notice that if $\mu \in \mathbf{I}_\pi(G)$ by definition there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi^0 = \mu$, where χ^0 denotes the restriction of χ to the π-elements of G. Also, it is clear by the definition, that every π-partial character is a sum of irreducible π-partial characters. Notice that if G is a π-group, then $\mathbf{I}_\pi(G) = \mathrm{Irr}(G)$.

Theorem 2.1 (Isaacs)**.** *Let G be a finite π-separable group. Then $\mathbf{I}_\pi(G)$ is a basis of the space of class functions defined on G^0. In particular, $|\mathbf{I}_\pi(G)|$ is the number of conjugacy classes of π-elements of G.*

Proof. This is Theorem 3.3 of [I2]. □

We can induce and restrict π-partial characters in a natural way. If H is a subgroup of G and $\varphi \in \mathbf{I}_\pi(G)$, then $\varphi_H = \sum_{\mu \in \mathbf{I}_\pi(H)} a_\mu \mu$ for some uniquely defined nonnegative integers a_μ. We write $\mathbf{I}_\pi(G|\mu)$ to denote the set of $\varphi \in \mathbf{I}_\pi(G)$ such that $a_\mu \neq 0$.

A non-trivial result is that Clifford's theory holds for π-partial characters. If $N \triangleleft G$, it is then clear that G naturally acts on $\mathbf{I}_\pi(N)$ by conjugation.

Theorem 2.2 (Isaacs). *Suppose that G is π-separable and $N \triangleleft G$.*

(a) If $\varphi \in \mathbf{I}_\pi(G)$, then $\varphi_N = e(\theta_1 + \cdots + \theta_t)$, where $\theta_1, \ldots, \theta_t$ are all the G-conjugates of some $\theta \in \mathbf{I}_\pi(N)$.

(b) If $\theta \in \mathbf{I}_\pi(N)$ and $T = G_\theta$ is the stabilizer of θ in G, then induction defines a bijection $\mathbf{I}_\pi(T|\theta) \to \mathbf{I}_\pi(G|\theta)$.

Proof. See Corollary 5.7 and Theorem 5.11 of [I2]. □

In part (b) of Theorem 2.2, if $\mu^G = \varphi$, where $\mu \in \mathbf{I}_\pi(T|\theta)$, then μ is called the **Clifford correspondent** of φ over θ, and sometimes it is written $\mu = \varphi_\theta$.

It is not a triviality to define vertices for π-partial characters (a concept that in classical modular representation theory has little to do with character theory). This was first accomplished in [IN] (generalizing a result of Huppert on Brauer characters of p-solvable groups).

Theorem 2.3. *Suppose that G is π-separable, and let $\varphi \in \mathbf{I}_\pi(G)$. Then there exist a subgroup U of G and $\alpha \in \mathbf{I}_\pi(U)$ of π-degree such that $\alpha^G = \varphi$. Furthermore, if Q is a Hall π-complement of U, then the G-conjugacy class of Q is uniquely determined by φ.*

Proof. This is Theorem 5.17 of [I2]. □

The uniquely defined G-class of π'-subgroups Q associated to φ by Theorem 2.3 is called the set of **vertices** of φ. If Q is a π'-subgroup of G, then we write $\mathbf{I}_\pi(G|Q)$ to denote the set of $\varphi \in \mathbf{I}_\pi(G)$ which have Q as a vertex. By definition, notice in this case that

$$\varphi(1)_{\pi'} = |G : Q|_{\pi'} .$$

Our last important ingredient is the Glauberman correspondence.

Theorem 2.4 (Glauberman). *Let S be a finite solvable group acting via automorphisms on a finite group G such that $(|S|, |G|) = 1$. Then there exists a canonical bijection, called the S-**Glauberman correspondence**,*

$$\mathrm{Irr}_S(G) \to \mathrm{Irr}(C), \qquad \chi \mapsto \chi^*,$$

where $\mathrm{Irr}_S(G)$ is the set of S-invariant irreducible characters of G and $C = \mathbf{C}_G(S)$. Here, χ^* is a constituent of the restriction χ_C. Also, if $T \triangleleft S$, then the T-Glauberman correspondence is an isomorphism of S-sets. Moreover, if X acts by automorphism on $G \rtimes S$ fixing S setwise, then the S-Glauberman correspondence commutes with the action of X.

Proof. See Theorem 13.1 of [I1]. The last claim is proven in Lemma 2.10 of [N] under the assumption that S is a p-group. The general case can be obtain by induction on $|S|$. □

3. Preliminaries

If G is a finite group, π is a set of primes, and $\chi \in \mathrm{Irr}(G)$, then we say that χ has **π-defect zero** if $\chi(1)_\pi = |G|_\pi$.

Lemma 3.1. *If $\chi \in \mathrm{Irr}(G)$ has π-defect zero, then $\mathbf{O}_\pi(G) = 1$.*

Proof. Let $N \triangleleft G$, and let $\theta \in \mathrm{Irr}(N)$ be under χ. Then we have that $\theta(1)$ divides $|N|$ and $\chi(1)/\theta(1)$ divides $|G : N|$ by Corollary 11.29 of [I1]. Thus $\chi(1)_\pi = |G|_\pi$ if and only if $\theta(1)_\pi = |N|_\pi$ and $(\chi(1)/\theta(1))_\pi = |G : N|_\pi$. The result is now clear applying this to $N = \mathbf{O}_\pi(G)$. □

We shall use the following notation. Suppose G is π-separable, $N \triangleleft G$, $\tau \in \mathbf{I}_\pi(N)$ and Q is a π'-subgroup of G. Then

$$\mathbf{I}_\pi(G|Q, \tau) = \mathbf{I}_\pi(G|Q) \cap \mathbf{I}_\pi(G|\tau)\,.$$

Lemma 3.2. *Suppose G is π-separable and that $N \triangleleft G$. Let Q be a π'-subgroup of G.*

(a) Suppose that $\mu \in \mathbf{I}_\pi(G|Q)$. Then there is a unique $\mathbf{N}_G(Q)$-orbit of $\tau \in \mathbf{I}_\pi(N)$ such that $\mu_\tau \in \mathbf{I}_\pi(G_\tau|Q)$, where μ_τ is the Clifford correspondent of μ over τ. Every such τ is Q-invariant.

(b) Suppose that $\tau \in \mathbf{I}_\pi(N)$ is Q-invariant. Let $\mathcal{U}$ be a complete set of representatives of the G_τ-orbits on the set $\{Q^g \mid g \in G, Q^g \subseteq G_\tau\}$. Then

$$|\mathbf{I}_\pi(G|Q, \tau)| = \sum_{U \in \mathcal{U}} |\mathbf{I}_\pi(G_\tau|U, \tau)|\,.$$

Thus, if $G_\tau \mathbf{N}_G(Q) = G$, then

$$|\mathbf{I}_\pi(G|Q, \tau)| = |\mathbf{I}_\pi(G_\tau|Q, \tau)|\,.$$

Proof. (a) Let $\nu \in \mathbf{I}_\pi(N)$ be under μ, and let $\mu_\nu \in \mathbf{I}_\pi(G_\nu|\nu)$ be the Clifford correspondent of μ over ν. If R is a vertex of μ_ν, then R is a vertex of μ, by Theorem 2.3. Therefore $R = Q^g$ for some $g \in G$. If $\tau = \nu^{g^{-1}}$, then we have that μ_τ has vertex Q. Suppose now that $\rho \in \mathbf{I}_\pi(N)$ is under μ such that μ_ρ has vertex Q. By Theorem 2.2(a), there exists $g \in G$ such that $\tau^g = \rho$. Thus Q^g is a vertex of μ_ρ. Then there is $x \in G_\rho$ such that $Q^{gx} = Q$. Since $\tau^{gx} = \rho$, the proof of part (a) is complete.

(b) We have that induction defines a bijection $\mathbf{I}_\pi(G_\tau|\tau) \to \mathbf{I}_\pi(G|\tau)$. Notice that

$$\bigcup_{U \in \mathcal{U}} \mathbf{I}_\pi(G_\tau|U)$$

is a disjoint union. It suffices to observe, again, that if $\xi \in \mathbf{I}_\pi(G_\tau|\tau)$ has vertex U, then ξ^G has vertex U. □

4. Proofs

The deep part in our proofs comes from the following result.

Theorem 4.1. *Suppose that L is a normal π-subgroup of G, Q is a solvable π'-subgroup of G such that $LQ \lhd G$. Suppose that $M \subseteq \mathbf{Z}(G)$ is contained in L and that $\varphi \in \mathrm{Irr}(M)$. Then $|\mathbf{I}_\pi(G|Q,\varphi)| = |\mathbf{I}_\pi(\mathbf{N}_G(Q)|Q,\varphi)|$.*

Proof. Let $\mathcal{A}$ be a complete set of representatives of $\mathbf{N}_G(Q)$-orbits on $\mathrm{Irr}_Q(L|\varphi)$, the Q-invariant members of $\mathrm{Irr}(L|\varphi)$. Using Lemma 3.2, we have that

$$\mathbf{I}_\pi(G|Q,\varphi) = \bigcup_{\tau\in\mathcal{A}} \mathbf{I}_\pi(G|Q,\tau)$$

is a disjoint union. Let $\mathcal{A}^*$ be the set of the Q-Glauberman correspondents of the elements of $\mathcal{A}$. Notice that $\mathcal{A}^*$ is a complete set of representatives of $\mathbf{N}_G(Q)$-orbits on $\mathrm{Irr}(\mathbf{C}_L(Q)|\varphi)$. Moreover, $\mathbf{N}_{G_\tau}(Q) = \mathbf{N}_{G_{\tau^*}}(Q)$. Then, as before,

$$\mathbf{I}_\pi(\mathbf{N}_G(Q)|Q,\varphi) = \bigcup_{\tau\in\mathcal{A}} \mathbf{I}_\pi(\mathbf{N}_G(Q)|Q,\tau^*)$$

is a disjoint union. Thus

$$|\mathbf{I}_\pi(\mathbf{N}_G(Q)|Q,\varphi)| = \sum_{\tau\in\mathcal{A}} |\mathbf{I}_\pi(\mathbf{N}_{G_\tau}(Q)|Q,\tau^*)|\,.$$

Thus we need to prove that

$$|\mathbf{I}_\pi(G_\tau|Q,\tau)| = |\mathbf{I}_\pi(\mathbf{N}_{G_\tau}(Q)|Q,\tau^*)|\,.$$

We may assume that τ is G-invariant.

Now, since $LQ \lhd G$ and τ is G-invariant, by Lemma 6.30 of [I2], we have that Q is contained as a normal subgroup in some vertex of θ, whenever $\theta \in \mathbf{I}_\pi(G)$ lies over τ. Therefore $\theta \in \mathbf{I}_\pi(G|\tau)$ has vertex Q if and only if $\theta(1)_{\pi'} = |G:Q|_{\pi'}$. Similarly, $\theta \in \mathbf{I}_\pi(\mathbf{N}_G(Q)|\tau^*)$ has vertex Q if and only if $\theta(1)_{\pi'} = |\mathbf{N}_G(Q):Q|_{\pi'} = |G:Q|_{\pi'}$ since $G = L\mathbf{N}_G(Q)$ by the Frattini argument and the Schur–Zassenhaus theorem.

Now we use the Dade–Puig theory on the character theory above Glauberman correspondents, which is thoroughly explained in [T]. By Theorem 6.5 of [T], in the language of Chapter 11 of [I1] (see Definition 11.23 of [I1]), we have that the character triples (G, L, τ) and $(\mathbf{N}_G(Q), \mathbf{C}_L(Q), \tau^*)$ are isomorphic. Write $^* : \mathrm{Irr}(G|\tau) \to \mathrm{Irr}(\mathbf{N}_G(Q)|\tau^*)$ for the associated bijection of characters. By Lemma 6.21 of [I2], there exists a unique bijection

$$^* : \mathbf{I}_\pi(G|\tau) \to \mathbf{I}_\pi(\mathbf{N}_G(Q)|\tau^*)$$

such that if $\chi^0 = \phi \in \mathbf{I}_\pi(G|\tau)$ and $\chi \in \mathrm{Irr}(G)$ (which necessarily lies over τ), then $(\chi^*)^0 = \phi^*$. Since $\chi(1)/\tau(1) = \chi^*(1)/\tau^*(1)$ (by Lemma 11.24 of [I1]), it follows that $\chi(1)_{\pi'} = \chi^*(1)_{\pi'}$. We deduce that

$$|\mathbf{I}_\pi(G|Q,\tau)| = |\mathbf{I}_\pi(\mathbf{N}_G(Q)|Q,\tau^*)|\,,$$

as desired. □

In order to prove Theorem B, we argue by induction on the index of a normal π-subgroup M of G. Theorem B follows from the special case $M = 1$.

Theorem 4.2. *Suppose that G is π-separable with a solvable Hall π-complement. Let R be a nilpotent π'-subgroup of G. Let $M \triangleleft G$ be a normal π-subgroup, and let $\varphi \in \mathrm{Irr}(M)$ be G-invariant. Let $\mathcal{Q}$ be the set of π'-subgroups Q of G such that R is a Carter subgroup of Q. Then*

$$\Big|\bigcup_{Q\in\mathcal{Q}} \mathbf{I}_\pi(G|Q,\varphi)\Big| = |\mathbf{I}_\pi(M\mathbf{N}_G(R)|R,\varphi)|\,.$$

Proof. We argue by induction on $|G : M|$.

By Lemma 3.11 of [I2], let (G^*, M^*, φ^*) be a character triple isomorphic to (G, M, φ), where M^* is a central π-subgroup of G^*. If Q is a π'-subgroup of G, notice that we can write $(QM)^* = M^* \times Q^*$, for a unique π'-subgroup Q^* of G^*. If R is contained in a π'-subgroup Q, then R is a Carter subgroup of Q if and only if RM/M is a Carter subgroup of QM/M, using that Q is naturally isomorphic to QM/M. This happens if and only if $(RM)^*$ is a Carter subgroup of $(QM)^*$, which again happens if and only if R^* is a Carter subgroup of Q^*. Notice further that if R is a Carter subgroup of Q, then R is a Carter subgroup of every Hall π-complement Q_1 of QM that happens to contain R (again using the isomorphism between QM/M and Q). We easily check now that the set of π'-subgroups of G^* that contain R^* as a Carter subgroup is exactly $\mathcal{Q}^* = \{Q^*|Q \in \mathcal{Q}\}$.

By the Frattini argument and the Schur–Zassenhaus theorem, notice that $\mathbf{N}_G(MR) = M\mathbf{N}_G(R)$. By Lemma 6.21 and the proof of Lemma 6.32 of [I2], there is a bijection $^* : \mathbf{I}_\pi(G|\varphi) \to \mathbf{I}_\pi(G^*|\varphi^*)$ such that η has vertex Q if and only if η^* has vertex Q^*. From all these arguments, it easily follows that we may assume that M is central. In particular, $M \le \mathbf{N}_G(R)$.

Let $K = \mathbf{O}_{\pi'}(G)$. Suppose that there exists some $\mu \in \mathbf{I}_\pi(G|Q,\varphi)$ for some $Q \in \mathcal{Q}$. By Lemma 6.30 of [I2] (in the notation of that lemma, K is 1 and Q is K), we have that K is contained in Q. Hence, it is no loss if we only consider $Q \in \mathcal{Q}$ such that $K \subseteq Q$.

Suppose that $\mathbf{N}_K(R)$ is not contained in R. Then there cannot be weights (R,γ), where $\gamma \in \mathrm{Irr}(\mathbf{N}_G(R)/R)$ has π-defect zero by Lemma 3.1. So the right hand side is zero. Suppose that there exists some $\mu \in \mathbf{I}_\pi(G|Q,\varphi)$ for some $Q \in \mathcal{Q}$ (with $K \subseteq Q$). Since R is a Carter subgroup of Q, then R is a Carter subgroup of KR, and therefore

$\mathbf{N}_K(R)$ is contained in R. Therefore may assume that $\mathbf{N}_K(R)$ is contained in R. We claim that R is a Carter subgroup of Q if and only if RK/K is a Carter subgroup of Q/K. One implication is known (see 9.5.3 in [R]). Suppose that RK/K is a Carter subgroup of Q/K. Since $\mathbf{N}_Q(R)$ normalizes RK, it is contained in RK. Hence $\mathbf{N}_Q(R) = \mathbf{N}_{KR}(R) = R$, and R is a Carter subgroup of Q. In this situation the Frattini argument yields $\mathbf{N}_G(R)K = \mathbf{N}_G(RK)$.

Next, we will replace G by G/K. By Lemma 6.31 of [I2] (the roles of K and M are interchanged in that lemma),

$$|\mathbf{I}_\pi(G|Q,\varphi)| = |\mathbf{I}_\pi(G/K|Q/K,\hat{\varphi})|,$$

where $\hat{\varphi} \in \mathrm{Irr}(MK/K)$ corresponds to φ via the natural isomorphism. Similarly,

$$|\mathbf{I}_\pi(\mathbf{N}_G(R)|R,\varphi)| = |\mathbf{I}_\pi(\mathbf{N}_G(R)K/K|RK/K,\hat{\varphi})| = |\mathbf{I}_\pi(\mathbf{N}_G(RK)/K|RK/K,\hat{\varphi})|.$$

Hence, for the remainder of the proof we may assume that $\mathbf{O}_{\pi'}(G) = K = 1$.

Suppose now that $L = \mathbf{O}_\pi(G)$. Let $\mathcal{A}$ be a complete set of $\mathbf{N}_G(R)$-representatives of the R-invariant characters in $\mathrm{Irr}(L|\varphi)$. If $L = M$, then $L = G$ by the Hall-Higman Lemma 1.2.3, and G is a π-group. In this case, $R = 1 = Q$, and there is nothing to prove. Thus, we may assume that $|G:L| < |G:M|$.

For each $\tau \in \mathcal{A}$, let $\mathcal{Q}_\tau$ be the set of π'-subgroups Q of G_τ such that R is a Carter subgroup of Q. By induction,

$$\Big|\bigcup_{Q\in\mathcal{Q}_\tau} \mathbf{I}_\pi(G_\tau|Q,\tau)\Big| = |\mathbf{I}_\pi(L\mathbf{N}_{G_\tau}(R)|R,\tau)|\,.$$

Since L is a π-group and R is a π'-subgroup, we have that $L\mathbf{N}_G(R) = \mathbf{N}_G(LR)$. Also, $\mathbf{N}_G(LR)_\tau = L\mathbf{N}_G(R)_\tau$. By Lemma 3.2, we have that

$$|\mathbf{I}_\pi(L\mathbf{N}_{G_\tau}(R)|R,\tau)| = |\mathbf{I}_\pi(L\mathbf{N}_G(R)|R,\tau)|\,.$$

Also,

$$|\mathbf{I}_\pi(L\mathbf{N}_G(R)|R,\varphi)| = \sum_{\tau\in\mathcal{A}} |\mathbf{I}_\pi(L\mathbf{N}_G(R)|R,\tau)|\,,$$

by the first paragraph of the proof of Theorem 4.1. By Theorem 4.1,

$$|\mathbf{I}_\pi(L\mathbf{N}_G(R)|R,\varphi)| = |\mathbf{I}_\pi(\mathbf{N}_G(R)|R,\varphi)|\,.$$

Therefore,

$$\sum_{\tau\in\mathcal{A}}\Big|\bigcup_{Q\in\mathcal{Q}_\tau} \mathbf{I}_\pi(G_\tau|Q,\tau)\Big| = |\mathbf{I}_\pi(\mathbf{N}_G(R)|R,\varphi)|\,.$$

We are left to show that

$$\Big|\bigcup_{Q\in\mathcal{Q}} \mathbf{I}_\pi(G|Q,\varphi)\Big| = \sum_{\tau\in\mathcal{A}}\Big|\bigcup_{Q\in\mathcal{Q}_\tau} \mathbf{I}_\pi(G_\tau|Q,\tau)\Big|\,.$$

Let $\mathcal{R}$ be a complete set of representatives of $\mathbf{N}_G(R)$-orbits in $\mathcal{Q}$, and notice that

$$\bigcup_{Q\in\mathcal{Q}} \mathbf{I}_\pi(G|Q,\varphi) = \bigcup_{Q\in\mathcal{R}} \mathbf{I}_\pi(G|Q,\varphi)$$

is a disjoint union. Indeed, if $\mu \in \mathbf{I}_\pi(G|Q_1,\varphi)\cap \mathbf{I}_\pi(G|Q_2,\varphi)$ for $Q_i \in \mathcal{Q}$, then we have that $Q_1 = Q_2^g$ for some $g\in G$ by the uniqueness of vertices. Hence R^g and R are Carter subgroups of Q_1, and therefore $R^{gx} = R$ for some $x \in Q_1$. It follows that $Q_1 = Q_2^{gx}$ are $\mathbf{N}_G(R)$-conjugate.

Now fix $Q \in \mathcal{R}$. For each $\mu \in \mathbf{I}_\pi(G|Q,\varphi)$, we claim that there is a unique $\tau\in\mathcal{A}$ such that $\mu_\tau \in \mathbf{I}_\pi(G_\tau|Q^x,\tau)$, for some $x\in \mathbf{N}_G(R)$. We know that there is $\nu \in \mathrm{Irr}(L|\varphi)$ such that $\mu_\nu \in \mathbf{I}_\pi(G_\nu|Q,\nu)$ by Lemma 3.2(a). Now, $\nu^x = \tau$ for some $x \in \mathbf{N}_G(R)$ and $\tau \in \mathcal{A}$, and it follows that $\mu_\tau \in \mathbf{I}_\pi(G_\tau|Q^x,\tau)$. Suppose that $\mu_\epsilon \in \mathbf{I}_\pi(G_\epsilon|Q^y,\epsilon)$, for some $y \in \mathbf{N}_G(R)$ and $\epsilon\in\mathcal{A}$. Now, $\epsilon = \tau^g$ for some $g \in G$, by Clifford's theorem. Thus $Q^{xgt} = Q^y$ for some $t \in G_\epsilon$, by the uniqueness of vertices. Thus $xgty^{-1} \in \mathbf{N}_G(Q)$. Since R is a Carter subgroup of Q, by the Frattini argument we have that $xgty^{-1} = qv$, where $q\in Q$ and $v \in G$ normalizes Q and R. Since Q^x fixes τ, then Q fixes $\tau^{x^{-1}}$. Now

$$\epsilon^{y^{-1}} = (\tau^{gt})^{y^{-1}} = \tau^{x^{-1}xgty^{-1}} = \tau^{x^{-1}qv} = \tau^{x^{-1}v}\,.$$

So ϵ and τ are $\mathbf{N}_G(R)$-conjugate, and thus they are equal.

Now we define a map

$$f : \bigcup_{Q\in\mathcal{R}} \mathbf{I}_\pi(G|Q,\varphi) \to \bigcup_{\tau\in\mathcal{A}} \left(\left(\bigcup_{Q\in\mathcal{Q}_\tau} \mathbf{I}_\pi(G_\tau|Q,\tau)\right)\times\{\tau\}\right)$$

given by $f(\mu) = (\mu_\tau,\tau)$, where $\tau\in\mathcal{A}$ is the unique element in $\mathcal{A}$ such that $\mu_\tau \in \mathbf{I}_\pi(G_\tau|Q^x,\tau)$, for some $x \in \mathbf{N}_G(R)$. Since $\mu_\tau^G = \mu$, we have that f is injective. If we have that $\gamma \in \bigcup_{Q\in\mathcal{Q}_\tau}\mathbf{I}_\pi(G_\tau|Q,\tau)$ then $\gamma^G \in \bigcup_{Q\in\mathcal{Q}}\mathbf{I}_\pi(G|Q,\varphi)$, so f is surjective. $\square$

Some of the difficulties in Theorem 4.2 are caused by the fact that Clifford correspondence does not necessarily respect vertices, even in quite restricted situations. Suppose that N is a normal p'-subgroup of G, $\tau\in\mathrm{Irr}(N)$, Q is a p-subgroup of G and τ is Q-invariant. Then it is not necessarily true that induction defines a bijection $\mathrm{IBr}(G_\tau|Q,\tau)\to\mathrm{IBr}(G|Q,\tau)$. For instance, take $p=2$ and $G = \texttt{SmallGroup}(216,158)$. This group has a unique normal subgroup N of order 3. The Fitting subgroup F of G is $F = N\times M$, where M is a normal subgroup of type $C_3\times C_3$, and $G/F = D_8$. Let $1\neq\tau\in\mathrm{Irr}(N)$. Then $G_\tau \lhd G$ has index 2, and $G_\tau/N = S_3\times S_3$. Now τ has a unique extension $\hat{\tau}\in\mathrm{IBr}(G_\tau)$. The group G_τ has three conjugacy classes of subgroups Q of order 2. Take Q_1 that corresponds to $C_2\times 1$ and Q_2 that corresponds to $1\times C_2$. Then Q_1 and Q_2 are not G_τ-conjugate but G-conjugate. So $|\mathrm{IBr}(G_\tau|Q_1,\tau)| = 1$ and $|\mathrm{IBr}(G|Q_1,\tau)| = 2$.

5. A canonical bijection

If G has a normal Hall π-subgroup, then we have a canonical bijection in Theorem 4.2. This seems worth exploring.

Lemma 5.1. *Suppose that $G = NH$ where N is a normal π-subgroup and H is a π'-subgroup. Then $\mathbf{N}_G(Q) = \mathbf{C}_N(Q)\mathbf{N}_H(Q)$ for every $Q \leq H$.*

Proof. First note that

$$Q = Q(N \cap H) = QN \cap H \lhd N\mathbf{N}_G(Q) \cap H \leq \mathbf{N}_H(Q).$$

Let $xh \in \mathbf{N}_G(Q)$ where $x \in N$ and $h \in H$. Then $h = x^{-1}(xh) \in N\mathbf{N}_G(Q) \cap H \leq \mathbf{N}_H(Q)$. This shows $\mathbf{N}_G(Q) = \mathbf{N}_N(Q)\mathbf{N}_H(Q) = \mathbf{C}_N(Q)\mathbf{N}_H(Q)$. □

Lemma 5.2. *Suppose that $G = NH$ where N is a normal π-subgroup and H is a solvable π'-subgroup. Let $R \leq H$, and let $\tau \in \mathrm{Irr}(\mathbf{C}_N(R))$ be such that $\mathbf{N}_G(R)_\tau = \mathbf{C}_N(R) \times R$. Let $\gamma \in \mathrm{Irr}_R(N)$ be the Glauberman correspondent of τ. Then $R = \mathbf{N}_{H_\gamma}(R)$.*

Proof. Suppose that $R < S \leq H_\gamma$, where $R \lhd S$. Then S acts on the R-Glauberman correspondence. Since S fixes γ, therefore it fixes $\gamma^* = \tau$. But this gives the contradiction $S \subseteq \mathbf{N}_G(R)_\tau = \mathbf{C}_N(R) \times R$. □

Theorem 5.3. *Suppose that $G = NH$ where N is a normal π-subgroup and H is a solvable π'-subgroup. Let R be a nilpotent subgroup of H. Let $\mathcal{Q}$ be the set of subgroups $Q \subseteq H$ such that R is a Carter subgroup of Q. Then there is a natural bijection*

$$\bigcup_{Q \in \mathcal{Q}} \mathbf{I}_\pi(G|Q) \to \mathbf{I}_\pi(\mathbf{N}_G(R)|R)\,.$$

Proof. Let $Q \in \mathcal{Q}$. By the Frattini argument, notice that $\mathbf{N}_G(Q) = Q(\mathbf{N}_G(Q) \cap \mathbf{N}_G(R))$, and that $Q \cap (\mathbf{N}_G(Q) \cap \mathbf{N}_G(R)) = R$.

Let $\phi \in \mathbf{I}_\pi(G|Q)$. By Lemma 3.2, there exists a Q-invariant $\theta \in \mathrm{Irr}(N)$ under ϕ. Then $T = G_\theta = QN$ using Corollary 8.16 in [I1] for instance. If θ_1 is another such choice, then $\theta_1 = \theta^g$ for some $g \in \mathbf{N}_G(Q)$. Thus, we may assume that $g \in \mathbf{N}_G(Q) \cap \mathbf{N}_G(R)$. Let $\theta^* \in \mathbf{C}_N(R)$ be the R-Glauberman correspondent of θ. Now, by Lemma 5.1 applied in T, we have that $\mathbf{N}_T(R) = \mathbf{C}_N(R)\mathbf{N}_Q(R) = \mathbf{C}_N(R) \times R$. We claim that $\mathbf{N}_T(R)$ is the stabilizer of θ^* in $\mathbf{N}_G(R)$. If $x \in \mathbf{N}_G(R)$ fixes θ^*, then x fixes θ, and thus $x \in \mathbf{N}_T(R)$, as claimed. Now $\phi^* := (\theta^* \times 1_R)^{\mathbf{N}_G(R)}$ is irreducible, and belongs to $\mathbf{I}_\pi(\mathbf{N}_G(R)|R)$. Since θ_1 is $\mathbf{N}_G(Q) \cap \mathbf{N}_G(R)$-conjugate to θ, ϕ^* is independent of the choice of θ.

Suppose that $\phi^* = \mu^*$, where $\phi \in \mathbf{I}_\pi(G|Q_1)$ and $\mu \in \mathbf{I}_\pi(G|Q_2)$, where R is a Carter subgroup of Q_i and $Q_i \subseteq H$. Suppose that we picked θ for ϕ and ϵ for μ, so that $\phi^* = (\theta^* \times 1_R)^{\mathbf{N}_G(R)}$ and $\mu^* = (\epsilon^* \times 1_R)^{\mathbf{N}_G(R)}$. Then $\mathbf{N}_G(R) = \mathbf{C}_N(R)\mathbf{N}_H(R)$, and θ^* and ϵ^* are $\mathbf{N}_H(R)$-conjugate, say $(\theta^*)^x = \epsilon^*$. Then $\theta^x = \epsilon$. By replacing (Q_1, θ) by (Q_1^x, θ^x), we may assume that $\theta = \epsilon$. But then $Q_1 = H_\theta = H_\epsilon = Q_2$. Since π-partial character are determined on the π-elements, we must have $\phi = \mu$ now.

Suppose conversely that $\tau \in \mathbf{I}_\pi(\mathbf{N}_G(R)|R)$. Then τ is induced from $\mathbf{C}_N(R) \times R$. Let $\mu \in \mathrm{Irr}(\mathbf{C}_N(R))$ such that $\mu \times 1_R$ induces τ. Then the stabilizer of μ in $\mathbf{N}_G(R)$ is $\mathbf{C}_N(R) \times R$. If $\rho \in \mathrm{Irr}_R(N)$ is the R-Glauberman correspondent of μ, then by Lemma 5.2 we know that R is a Carter subgroup of $Q = H_\rho$, where Q is the stabilizer in H of ρ. Thus with the notation of the first part of the proof we obtain $\tau = \mu^*$ where μ is induced from $G_\rho = QN$. $\square$

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Department of Mathematics, Universitat de València, 46100 Burjassot, València, Spain

Email address: gabriel.navarro@uv.es

Institut für Mathematik, Friedrich-Schiller-Universität Jena, 07737 Jena, Germany

Email address: benjamin.sambale@uni-jena.de