On a fixed point formula of Navarro–Rizo

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Abstract

Let G be a π-separable group with a Hall π-subgroup H or order n. For $x \in H$ let $\lambda(x)$ be the number of Hall π-subgroups of G containing x. We show that $\prod_{d|n} \prod_{x\in H} \lambda(x^d)^{\frac{n}{d}\mu(d)} = 1$, where μ is the Möbius function. This generalizes fixed point formulas for coprime actions by Brauer, Wielandt and Navarro–Rizo. We further investigate an additive version of this formula.

Keywords: fixed points, coprime action, Sylow subgroups, p-solvable groups, π-separable groups
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1 Introduction

Navarro and Rizo [5] proved the following fixed point equation related to formulas of Brauer and Wielandt.

Theorem 1 (Navarro–Rizo)**.** *Let P be a finite p-group acting on a p'-group N. Then*

$$|\mathrm{C}_N(P)| = \Big(\prod_{x\in P} \frac{|\mathrm{C}_N(x)|}{|\mathrm{C}_N(x^p)|^{1/p}}\Big)^{\frac{p}{|P|(p-1)}}. \tag{1.1}$$

For a finite group G and $x \in P \in \mathrm{Syl}_p(G)$ let $\lambda_G(x)$ be the number of Sylow p-subgroups of G containing x. In the situation of Theorem 1, the Sylow p-subgroups of $G := N \rtimes P$ have the form nPn^{-1} for $n \in N$. Note that $\mathrm{N}_G(P) = \mathrm{C}_N(P)P$. Suppose that $x \in P \cap nPn^{-1}$. Then there exists $y \in P$ such that $x = nyn^{-1}$. Since $[n, y] = xy^{-1} \in P \cap N = 1$, it follows that $x = y$ and $n \in \mathrm{C}_N(x)$. Hence, $\lambda_G(x) = |\mathrm{C}_N(x) : \mathrm{C}_N(P)|$ for all $x \in P$. Now (1.1) turns into the more elegant formula:

$$\prod_{x\in P} \lambda_G(x^p) = \prod_{x\in P} \lambda_G(x)^p. \tag{1.2}$$

We show that this holds more generally for all p-solvable groups. In fact, our main theorem applies to π-separable groups, where π is any set of primes. Recall that a π-separable group G has a unique conjugacy class of Hall π-subgroups. For a π-element $x \in G$ let $\lambda_G(x)$ be the number of Hall π-subgroups of G containing x.

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Theorem 2. *Let G be a π-separable group with a Hall π-subgroup H of order n. Then*

$$\prod_{d\,|\,n}\Bigl(\prod_{x\in H}\lambda_G\bigl(x^d\bigr)^{\frac{n}{d}}\Bigr)^{\mu(d)}=1, \tag{1.3}$$

where μ is the Möbius function.

For $\pi=\{p\}$, (1.3) becomes (1.2). We will show in Proposition 4 that (1.3) holds for arbitrary groups whenever they have a cyclic Hall π-subgroup (by a result of Wielandt, the Hall π-subgroups are conjugate in this situation too, see [2, Satz III.5.8]). In general, the left hand side of (1.2) can be larger or smaller than the right hand side (consider $G=A_5$ and $G=\mathrm{GL}(3,2)$ for $p=2$). We did not find a non-solvable group fulfilling (1.2) for $p=2$.

In the last section we obtain the following additive version of (1.3).

Theorem 3. *Let G be a finite group with a Hall subgroup H of order n. Then*

$$\frac{1}{n^2}\sum_{d\,|\,n}\mu(d)\sum_{h\in H}\lambda_G(h^d)^{\frac{n}{d}}$$

is a non-negative integer, which is zero if and only if $1\ne H\trianglelefteq G$.

2 The proof of Theorem 2

In the first step we reduce Theorem 2 to π-nilpotent groups. Let $\alpha(G)$ be the left hand side of (1.3). Since

$$\alpha(G)=\Bigl(\prod_{d\,|\,n}\prod_{x\in H}\lambda_G(x^d)^{\mu(d)/d}\Bigr)^n,$$

we may replace n by the product of the prime divisors of $|H|$. Suppose that $p\in\pi$ does not divide n. Then $\langle x^d\rangle=\langle x^{dp}\rangle$ for all $x\in H$ and $d\mid n$. Since $\lambda_G(x^d)$ only depends on $\langle x^d\rangle$, it follows that

$$\prod_{d\,|\,np}\prod_{x\in H}\lambda_G(x^d)^{\frac{np}{d}\mu(d)}=\prod_{d\,|\,n}\prod_{x\in H}\lambda_G(x^d)^{\frac{np}{d}\mu(d)}\lambda_G(x^d)^{-\frac{n}{d}\mu(d)}=\alpha(G)^{p-1}.$$

Hence, we can assume that $n=\prod_{p\in\pi}p$.

Suppose that $N:=\mathrm{O}_\pi(G)\ne 1$. Since N lies in every Hall π-subgroup of G, we have $\lambda_G(xy)=\lambda_G(x)$ for all $x\in H$ and $y\in N$. Moreover, $\lambda_G(x)=\lambda_{G/N}(xN)$. By induction on $|G|$, we obtain

$$\alpha(G)=\prod_{x\in H}\prod_{d\,|\,n}\lambda_G(x^d)^{\frac{n}{d}\mu(d)}=\prod_{xN\in H/N}\Bigl(\prod_{d\,|\,n}\lambda_{G/N}(x^dN)^{\frac{n}{d}\mu(d)}\Bigr)^{|N|}=\alpha(G/N)^{|N|}=1.$$

Thus, we may assume that $\mathrm{O}_\pi(G)=1$. Then $N:=\mathrm{O}_{\pi'}(G)\ne 1$. By the argument from the introduction, we have $\lambda_{HN}(x)=|\mathrm{C}_N(x):\mathrm{C}_N(H)|$ for $x\in H$. If x lies in another Hall π-subgroup K, then

$$\lambda_{HN}(x)=|\mathrm{C}_N(x):\mathrm{C}_N(H)|=|\mathrm{C}_N(x):\mathrm{C}_N(K)|=\lambda_{KN}(x),$$

because H and K are conjugate in G. It follows that $\lambda_G(x)=\lambda_{HN}(x)\lambda_{G/N}(xN)$. Hence, by the same argument as before, $\alpha(G)=\alpha(HN)\alpha(G/N)$. By induction on $|G|$, we may assume that $G=HN$, i.e. G is a π-nilpotent group.

Next, we reduce to the case where H is cyclic. Under this assumption, the result holds for arbitrary groups.

Proposition 4. *Let G be a group with a cyclic Hall subgroup H. Then* (1.3) *holds.*

Proof. For every generator h of H we have $\lambda_G(h) = 1$. Now let $h \in H$ be of order $\frac{n}{e} < n$. Then for every $d \mid e$ there exist exactly d elements $x \in H$ such that $x^d = h$. Hence, the exponent of $\lambda_G(h)$ in (1.3) is

$$\sum_{d \mid e} d\frac{n}{d}\mu(d) = n\sum_{d \mid e}\mu(d) = 0. \qquad \square$$

For the general case, let $\mathcal{Z}$ be the set of cyclic subgroups of H. Instead of running over all elements of H, we run over $Z \in \mathcal{Z}$ and then over $z \in Z$. In order to track multiplicity, we use the Möbius function μ of the lattice $\mathcal{Z}$. Since the subgroups of a cyclic group of order d are in bijection to the divisors of d, we have $\mu(Z, W) = \mu(|W/Z|)$ whenever $Z \le W$ and $\mu(Z, W) = 0$ otherwise. We define $f(Z) := \sum_{W \in \mathcal{Z}} \mu(Z, W)$. Recall the inversion formula for Euler's totient function:

$$\varphi(n) = \sum_{d \mid n} \frac{n}{d}\mu(d). \tag{2.1}$$

For $W \in \mathcal{Z}$, let $[W]$ be the set of generators of W. For any function $\gamma : G \to \mathbb{Z}$, we have

$$\prod_{w \in W} \gamma(w) = \prod_{Z \le W} \prod_{z \in [Z]} \gamma(z).$$

By Möbius inversion, it follows that $\prod_{w \in [W]} \gamma(w) = \prod_{Z \le W} \prod_{z \in Z} \gamma(z)^{\mu(Z,W)}$ and

$$\prod_{x \in H} \gamma(x) = \prod_{W \in \mathcal{Z}} \prod_{w \in [W]} \gamma(w) = \prod_{W \in \mathcal{Z}} \prod_{Z \le W} \prod_{z \in Z} \gamma(z)^{\mu(Z,W)} = \prod_{Z \in \mathcal{Z}} \Big(\prod_{z \in Z} \gamma(z)\Big)^{f(Z)}. \tag{2.2}$$

Counting the number of factors on both sides also reveals that

$$|H| = \sum_{Z \in \mathcal{Z}} |Z| f(Z). \tag{2.3}$$

Assuming $G = NH$, we have

$$\lambda_G(x) = |\mathrm{C}_N(x) : \mathrm{C}_N(H)| = |\mathrm{C}_N(x) : \mathrm{C}_N(Z)||\mathrm{C}_N(Z) : \mathrm{C}_N(H)| = \lambda_{ZN}(x)|\mathrm{C}_N(Z) : \mathrm{C}_N(H)| \tag{2.4}$$

for $x \in Z \in \mathcal{Z}$. Now we can put everything together and apply Proposition 4:

$$\begin{aligned}\alpha(G) &\overset{(2.2)}{=} \prod_{Z \in \mathcal{Z}} \Big(\prod_{z \in Z} \prod_{d \mid n} \lambda_G(z^d)^{\frac{n}{d}\mu(d)}\Big)^{f(Z)} \overset{(2.4)}{=} \prod_{Z \in \mathcal{Z}} \Big(\alpha(NZ) \prod_{d \mid n} |\mathrm{C}_N(Z) : \mathrm{C}_N(H)|^{|Z|\frac{n}{d}\mu(d)}\Big)^{f(Z)} \\ &\overset{(2.1)}{=} \prod_{Z \in \mathcal{Z}} |\mathrm{C}_N(Z) : \mathrm{C}_N(H)|^{\varphi(n)|Z|f(Z)} \overset{(2.3)}{=} \Big(|\mathrm{C}_N(H)|^{-|H|} \prod_{Z \in \mathcal{Z}} |\mathrm{C}_N(Z)|^{|Z|f(Z)}\Big)^{\varphi(n)}.\end{aligned}$$

At this point, the claim follows from Wielandt's formula [8, Satz 2.3], which we prove for sake of self-containment.

Theorem 5 (Wielandt)**.** *Let H be a group acting coprimely on a group N. Then*

$$|\mathrm{C}_N(H)|^{|H|} = \sum_{Z \in \mathcal{Z}} |\mathrm{C}_N(Z)|^{|Z|f(Z)}.$$

Proof. Since Wielandt's paper is hard to follow (even for a German native speaker), we use some modern ingredients. We consider N as a H-set via conjugation. By a theorem of Hartley–Turull [1, Lemma 2.6.2] (see also [7, Satz 9.20]), there exist a direct product A of elementary abelian groups and an isomorphism of H-sets $\varphi : N \to A$, i. e. $\varphi(n^h) = \varphi(n)^h$ for $n \in N$ and $h \in H$. It follows that $\varphi(\mathrm{C}_N(Z)) = \mathrm{C}_A(Z)$ for $Z \in \mathcal{Z}$. Thus, we may replace N by A. Then N decomposes into its (characteristic) Sylow subgroups $N = N_1 \times \ldots \times N_k$. Since $\mathrm{C}_N(Z) = \mathrm{C}_{N_1}(Z) \times \ldots \times \mathrm{C}_{N_k}(Z)$, we may assume further that $N = N_1$ is elementary abelian. Thus, N is an $\mathbb{F}_pH$-module for some prime p not dividing $|H|$. The corresponding Brauer character $\chi : H \to \mathbb{C}$ can be regarded as an ordinary character since $|H|$ is coprime to p. We further extend χ to the complex group algebra $\mathbb{C}H$. For $S \subseteq H$ let $S^+ := \sum_{s\in S} s \in \mathbb{C}H$. The additive version of (2.2) reads

$$H^+ = \sum_{Z\in\mathcal{Z}} f(Z)Z^+.$$

By the first orthogonality relation, $\chi(Z^+) = |Z|[\chi_Z, 1_Z]$, where $[\chi_Z, 1_Z]$ is the multiplicity of the trivial character 1_Z as a constituent of the restriction χ_Z. On the other hand, we have $|\mathrm{C}_N(Z)| = p^{[\chi_Z,1_Z]}$. It follows that

$$|\mathrm{C}_N(H)|^{|H|} = p^{|H|[\chi,1_H]} = p^{\chi(H^+)} = p^{\sum_{Z\in\mathcal{Z}} |Z|[\chi_Z,1_Z]f(Z)} = \prod_{Z\in\mathcal{Z}} |\mathrm{C}_N(Z)|^{|Z|f(Z)}. \qquad \square$$

The proof of Theorem 5 relies on the Feit–Thompson theorem to guarantee that H or N is solvable. In comparison, the proof of Theorem 1 does not require representation theory, but uses the fact that Sylow subgroups are nilpotent.

3 The proof of Theorem 3

A π-element $x \in G$ lies in a Hall π-subgroup H if and only if $x \in \mathrm{N}_G(H)$. Hence, the map $\lambda_G : H \to \mathbb{Z}$ is the permutation character of the conjugation action of H on the set $\mathrm{Hall}_\pi(G)$ of all Hall π-subgroups of G (we do not assume that these subgroups are conjugate in G). We use the following recipe to construct a related character.

Theorem 6. *Let χ be a character of a finite group H and let α be a character of a permutation group $A \le S_n$. For $a \in A$ let $c_i(a)$ be the number of cycles of a of length i. Then the map $\chi_\alpha : H \to \mathbb{C}$ with*

$$\chi_\alpha(h) = \frac{1}{|A|}\sum_{a\in A}\alpha(a)\prod_{i=1}^{n}\chi(h^i)^{c_i(a)}$$

for $h \in H$ is a character or the zero map.

Proof. See [3, Theorem 7.7.7 and Eq. (7.7.9)]. $\square$

We assume the notation of Theorem 3 and choose a cyclic subgroup $A \le S_n$ generated by a cycle of length n. Let $\alpha \in \mathrm{Irr}(A)$ be a faithful character. For $B \le A$ of order d, $\sum_{b\in[B]} \alpha(b)$ is the sum of the primitive roots of unity of order d. A simple Möbius inversion shows that this sum equals $\mu(d)$. Moreover, every $b \in [B]$ is a product of n/d disjoint cycles of length d. Thus, $c_d(b) = n/d$ and $c_i(b) = 0$ for $i \neq d$.

We apply Theorem 6 with $\chi=\lambda_G$. For $h\in H$ we compute

$$\chi_\alpha(h)=\frac{1}{n}\sum_{d\,|\,n}\sum_{\substack{a\in A\\|\langle a\rangle|=d}}\alpha(a)\lambda_G(h^d)^{\frac{n}{d}}=\frac{1}{n}\sum_{d\,|\,n}\mu(d)\lambda_G(h^d)^{\frac{n}{d}}.$$

Taking the scalar product of χ_α and the trivial character of H, shows that

$$\beta_G(H):=\frac{1}{n^2}\sum_{h\in H}\sum_{d\,|\,n}\mu(d)\lambda_G(h^d)^{\frac{n}{d}}$$

is a non-negative integer. This confirms the first part of Theorem 3.

If $H\trianglelefteq G$, then $\lambda_G(h)=1$ for all $h\in H$ and it follows that $\beta_G(H)=0$ unless $H=1$ (where $\beta_G(H)=1$). Now assume that H is not normal in G. In particular, $n>1$. Let $t:=|\operatorname{Hall}_\pi(G)|$. If $t=2$, then $|G:\mathrm{N}_G(H)|=2$ and $\mathrm{N}_G(H)\trianglelefteq G$. But this would imply that $H^g=\mathrm{O}_\pi(\mathrm{N}_G(H))=H$ for every $g\in G$. Hence, $t\geq 3$.

Next we investigate the contribution of $d=1$ to $\beta_G(H)$. Note that $\lambda_G(h)^n$ is the number of fixed points of h on $\operatorname{Hall}_\pi(G)^n$, acting diagonally. The number of orbits of H on $\operatorname{Hall}_\pi(G)$ is at most t^n/n. Using Burnside's lemma and the trivial estimate $\lambda_G(h^d)\leq t$ for $d\neq 1$ and $h\in H$, we obtain

$$n\beta_G(H)\geq\frac{1}{n}\sum_{h\in H}\lambda_G(h)^n-\frac{1}{n}\sum_{h\in H}\sum_{1\neq d\,|\,n}t^{\frac{n}{d}}\geq\frac{t^n}{n}-\sum_{1\neq d\,|\,n}t^{\frac{n}{d}}.$$

It suffices to show that

$$n\sum_{1\neq d\,|\,n}t^{\frac{n}{d}}<t^n.$$

If n is a prime, this reduces to $nt<t^n$ and we are done as $t\geq 3$. If $n=4$ and $t=3$, the claim can be checked directly. In all other cases, one can verify that $n\leq t^{\frac{n}{2}-1}$ and

$$n\sum_{1\neq d\,|\,n}t^{\frac{n}{d}}\leq\sum_{k=0}^{n-1}t^k=\frac{t^n-1}{t-1}<t^n.$$

This finishes the proof.

We remark that the degree $\chi_\alpha(1)=\frac{1}{n}\sum_{d|n}\mu(d)t^{n/d}$ has several interesting interpretations. For instance, if t is a prime power, then $\chi_\alpha(1)$ is the number of irreducible polynomials of degree n over the finite field $\mathbb{F}_t$ (see [6, Corollary 10.2.3]). In general, $\chi_\alpha(1)$ is the rank of the n-th quotient of the lower central series of a free group of rank t (see [4, Theorem 5.11 and Corollary 5.12]).

We now give an interpretation of $\beta_G(H)$ in a special case. Suppose that H is nilpotent with regular Sylow subgroups (for all primes). Then the d-powers form a subgroup H^d, and for every $h\in H^d$ there exist exactly $|H:H^d|$ elements $x\in H$ with $x^d=h$ (see [2, Hauptsatz III.10.5 and Satz III.10.6]). Burnside's lemma applied to H^d acting on $\operatorname{Hall}_\pi(G)^{n/d}$ yields

$$\beta_G(H)=\frac{1}{n}\sum_{d\,|\,n}\mu(d)f_{n/d}(H^d),$$

where $f_{n/d}(H^d)$ is the number of orbits of H^d on $\operatorname{Hall}_\pi(G)^{n/d}$.

Finally, we comment on a curiosity. In order to produce similar quantities as $\beta_G(H)$, we may replace λ_G by any character of any finite group G. For instance, let τ be the conjugation character of $G = A_5$ on $\mathrm{Syl}_3(G)$. We compute

$$\frac{1}{60^2}\sum_{d\,|\,60}\mu(d)\sum_{g\in G}\tau(g^d)^{\frac{n}{d}} = 27777777777777777777777777777333333333275480383275809093 3.$$

Similar curious numbers arise from $G \in \{S_5, \mathrm{PSL}(2,9), \mathrm{PGL}(2,9)\}$. This can be explained by the presents of large powers of $\tau(1) = 10$.

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