A COUNTEREXAMPLE TO FEIT'S PROBLEM VIII ON DECOMPOSITION NUMBERS

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Abstract. We find a counterexample to Feit's Problem VIII on the bound of decomposition numbers. This also answers a question raised by T. Holm and W. Willems.

Richard Brauer asked if the Cartan invariants $c_{\varphi\psi}$ of a p-block B of a finite group G are at most p^d where d is the defect of B. It is well-known that Peter Landrock showed that the Suzuki group Sz(8) with $p = 2$ is a counterexample to Brauer's question. Since

$$c_{\varphi\varphi} = \sum_{\chi \in \mathrm{Irr}(B)} (d_{\chi\varphi})^2,$$

Walter Feit, in his list of open problems in Representation Theory, proposed the following weaker question on decomposition numbers in his book [1, Problem (VIII), p. 169]:

(VIII) *Is* $(d_{\chi\varphi})^2 \leq p^d$ *whenever* χ, φ *lie in a block of defect* d*?*

To the present authors' surprise apparently no one has noticed that the Atlas of Brauer characters [3] contains two counterexamples to Feit's problem: $\mathrm{PSp}_4(4).4$ and $\mathrm{Sz}(32).5$ both for $p = 2$, both in the principal block. In the first case, 44 occurs as a decomposition number and in the second case, 47 occurs (see [5] for instance). For both groups we have $|G|_2 = 2^{10}$.

It is interesting to speculate on whether or not (VIII) (or even Brauer's original question) has a positive answer for odd primes. This would prove, using Brauer's $k(B)$-conjecture, that $c_{\varphi\psi} \leq \max\{c_{\varphi\varphi}, c_{\psi\psi}\} \leq p^{2d}$. In fact, we are not aware of any counterexample to this, even for $p = 2$. Also, we are not aware of any example where the inequality $d_{\chi\varphi} \leq p^d$ does not hold.

Another way of relaxing Brauer's question was proposed by Holm and Willems. They ask in [2, Question 2.6] if

$$\mathrm{tr}\, C \leq l(B)p^d$$

holds for every p-block B with defect d and Cartan matrix C. An easy computation with GAP [4] shows that our examples above provide counterexamples also for this question.

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