On blocks with abelian defect groups of small rank

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Abstract

Let B be a p-block of a finite group with abelian defect group D. Suppose that D has no elementary abelian direct summand of order p^4. Then we show that B satisfies Brauer's $k(B)$-Conjecture (i.e. $k(B) \le |D|$). Together with former results, it follows that Brauer's $k(B)$-Conjecture holds for all blocks of defect at most 3. We also obtain some related results.

Keywords: blocks, abelian defect groups, defect 3, Brauer's $k(B)$-Conjecture
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1 Introduction

Let B be a p-block of a finite group G with abelian defect group D. Using the classification of the finite simple groups, it has been shown by Kessar and Malle [12] that all irreducible characters of B have height 0. This confirms one direction of Brauer's Height Zero Conjecture. As a consequence of this strong result, we concluded in [23] (see also [22, Chapter 14]) that $k(B) \le |D|^{\frac{3}{2}}$ where $k(B)$ denotes the number of irreducible characters in B. This improves the famous Brauer-Feit [2] bound which states that $k(B) \le |D|^2$ without any hypotheses on D. Both inequalities are motivated by Brauer's $k(B)$-Conjecture asserting $k(B) \le |D|$.

For p-solvable groups G, Brauer's $k(B)$-Conjecture has been reduced to the so-called $k(GV)$-problem by Nagao [15]. Eventually, the $k(GV)$-problem has been solved again by relying on the classification (see [28]). Nevertheless, Brauer's Conjecture remains open for arbitrary groups G. The aim of the present paper is to carry over some arguments from the $k(GV)$-problem to the general case.

In a recent paper [26, Theorem 20] by the present author, it became clear that the bounds on $k(B)$ depend more on the rank of the (abelian) defect group D than on just the order $|D|$. In particular, we verified $k(B) \le |D|$ under the condition that D has no elementary abelian direct summand of order p^3. The proof relies on the existence of certain regular orbits of coprime linear groups. Since such regular orbits do not always exist, the methods used in that paper were limited. In particular, we could not deal with the case $|D| = 7^3$. Now in the present paper we settle this special case and show that (surprisingly) the larger primes do not cause such trouble. This is related to the fact that linear groups over "large" fields usually have regular orbits. This observation has already been used in [20, Theorem 2.1(iii)]. Eventually, we show the following.

Theorem A. *Let B be a p-block with abelian defect group D. Suppose that D has no elementary abelian direct summand of order p^4. Then $k(B) \le |D|$.*

This applies of course whenever D is abelian of rank at most 3. In this case, Brauer [1, (7D)] has obtained that $k(B) \le |D|^{\frac{5}{3}}$ and this was subsequently improved to $k(B) \le |D|^{\frac{4}{3}}$ in [23, Proposition 2]. Note that Brauer [1, (7D)] has already shown his conjecture provided D is abelian of rank 2 (see also [6, Theorem VII.10.13]).

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For the proof of Theorem A, we make use of a result by Köhler-Pahlings [14] about large orbits of linear quasisimple groups. As another ingredient, we use computer calculations with GAP [7] to identify linear groups of small dimensions over finite fields and to enumerate certain configurations of generalized decomposition matrices. These computations are easy to implement, but cannot be replaced by reasonable hand calculations. Since we are using the result by Kessar-Malle mentioned in the beginning, our proofs implicitly rely on the classification of the finite simple groups. Many of our arguments also depend on the fact that the rank 3 is a prime number. One should point out that the general situation seems to be much harder than the $k(GV)$-problem. For instance, there is not even a reduction to the case where the inertial quotient of B acts irreducibly on D (see e. g. [26, Remark before Proposition 22], cf. [28, Proposition 3.1a]).

Since Brauer's $k(B)$-Conjecture for the non-abelian defect groups of order p^3 has been settled in [26, Theorem 5], we obtain the following corollary.

Theorem B. *Every p-block B of defect at most 3 satisfies $k(B) \le p^3$.*

Previously, this has been known only for blocks of defect at most 2 (see [6, Theorem VII.10.14]). Finally, we also improve the general bound $k(B) \le |D|^{\frac{3}{2}}$ obtained in [23] (see Theorem 6).

In addition to the notation already introduced, we use the following. The number of irreducible Brauer characters of B is denoted by $l(B)$. Let b_D be a Brauer correspondent of B in $\mathrm{C}_G(D)$. Then $I(B) := \mathrm{N}_G(D, b_D)/\mathrm{C}_G(D)$ is the *inertial quotient* (recall that D is abelian). It is known that $I(B)$ is a p'-group and we will often regard it as a subgroup of $\mathrm{Aut}(D)$. A *subsection* for B is a pair (u, b_u) where $u \in D$ and b_u is a Brauer correspondent of B in $\mathrm{C}_G(u)$. A cyclic group of order n is denoted by C_n. For convenience: $C_n^m = C_n \times \ldots \times C_n$ (m copies).

2 Results

We first deal with a special case.

Lemma 1. *Let B be a block of a finite group with defect group $D \cong C_7^2$ and $I(B) \cong \mathrm{SL}(2,3)$ such that $D \rtimes I(B) \cong$ `SmallGroup`(1176,214). Then there exists a basic set for B such that the Cartan matrix C of B satisfies $\mathrm{tr}(C) \le 49$.*

Proof. We will construct C with the algorithm described in [22, Section 4.2]. There is a set of representatives for the $I(B)$-conjugacy classes of D of the form $1, x, x^2, x^4, y$. The corresponding orbit lengths are $1, 8, 8, 8, 24$. It follows that y is $I(B)$-conjugate to all its non-trivial powers, since otherwise there would be a second regular orbit. If (y, b_y) is a corresponding B-subsection, then $I(b_y) = \mathrm{C}_{I(B)}(y) = 1 = l(b_y)$ and the generalized decomposition numbers $d^y_{\chi\psi}$ contribute one integral column in the generalized decomposition matrix Q.

Now we investigate the generalized decomposition numbers with respect to the subsection (x, b_x). Since b_x dominates a block $\overline{b_x}$ of $\mathrm{C}_G(x)/\langle x\rangle$ with defect 1, it is easy to see that $l(b_x) = 3$ and the Cartan matrix C_x of b_x has the form

$$C_x = 7\begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix} \tag{2.1}$$

up to basic sets (see [4, Theorem 8.6]). Let $Q_x \in \mathbb{C}^{k(B)\times 3}$ be the part of the generalized decomposition matrix such that $C_x = Q_x^{\mathrm{T}}\overline{Q_x}$. Since x and x^{-1} are $I(B)$-conjugate, it may happen that two columns of Q_x are complex conjugates of each other. In this case, $\mathrm{N}_G(\langle x\rangle, b_x)$ interchanges two irreducible Brauer characters of b_x. The same holds for $\overline{b_x}$. Since $\overline{b_x}$ has defect 1, we have $k(\overline{b_x}) = 5$. It follows from Dade's theory of cyclic defect groups that there are one or two pairs of non-stable irreducible characters in $\mathrm{Irr}(\overline{b_x})$ under the action of $\overline{N} := \mathrm{N}_G(\langle x\rangle, b_x)/\langle x\rangle$. Since $|\overline{N} : \mathrm{C}_G(x)/\langle x\rangle| = |\mathrm{N}_G(\langle x\rangle, b_x) : \mathrm{C}_G(x)| = 2$, there are seven or four characters in $\mathrm{Irr}(\overline{N})$ lying over $\overline{b_x}$ (in the sense of [16, Lemma 5.5.7]). Every block $B_{\overline{N}}$ of $\overline{N}$ which covers $\overline{b_x}$ has defect 1 and therefore $k(B_{\overline{N}}) \in \{5, 7\}$. Consequently, $B_{\overline{N}} = \overline{b_x}^{\overline{N}}$ is the only block covering $\overline{b_x}$ and $k(B_{\overline{N}}) = 7$. Then however $l(B_{\overline{N}}) \ne 3$, but we should have $l(B_{\overline{N}}) = 3$. This contradiction shows that Q_x is a real matrix.

Let $\zeta \in \mathbb{C}$ be a primitive 7-th root of unity, and let $\tau_i := \zeta^i + \zeta^{-i}$ for $i = 2, 3$. The columns of Q_x have the form $q_i = a_i + b_i\tau_2 + c_i\tau_3$ with $a_i, b_i, c_i \in \mathbb{Z}^{k(B)}$ for $i = 1, 2, 3$. We are interested in the pairwise scalar products of the a_i, b_i and c_i. Let γ be the Galois automorphism of $\mathbb{Q}(\zeta)$ such that $\gamma(\zeta) = \zeta^2$. Then $\gamma(Q_x) = Q_{x^2}$. By the orthogonality relation, we have $Q_x^{\mathrm{T}} Q_{x^2} = 0$. Let A_1 be the matrix with columns a_1, b_1 and c_1. Then

$$(1, \tau_2, \tau_3) A_1^{\mathrm{T}} A_1 \begin{pmatrix} 1 \\ \tau_2 \\ \tau_3 \end{pmatrix} = 21.$$

This can be rephrased as a linear system of the form

$$((1, \tau_2, \tau_3) \otimes (1, \tau_2, \tau_3)) X = 21$$

where X is the vectorization of $A_1^{\mathrm{T}} A_1$ and $\otimes$ denotes the Kronecker product. Together with the other orthogonality relations we obtain

$$A_1^{\mathrm{T}} A_1 = 3 \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Moreover, if A is the matrix with columns $a_1, b_1, \ldots, c_3$, then it follows that

$$A^{\mathrm{T}} A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix} \otimes \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 3 & 3 & 6 & 2 & 2 & 6 & 2 & 2 \\ 3 & 6 & 3 & 2 & 4 & 2 & 2 & 4 & 2 \\ 3 & 3 & 6 & 2 & 2 & 4 & 2 & 2 & 4 \\ 6 & 2 & 2 & 9 & 3 & 3 & 6 & 2 & 2 \\ 2 & 4 & 2 & 3 & 6 & 3 & 2 & 4 & 2 \\ 2 & 2 & 4 & 3 & 3 & 6 & 2 & 2 & 4 \\ 6 & 2 & 2 & 6 & 2 & 2 & 9 & 3 & 3 \\ 2 & 4 & 2 & 2 & 4 & 2 & 3 & 6 & 3 \\ 2 & 2 & 4 & 2 & 2 & 4 & 3 & 3 & 6 \end{pmatrix}.$$

Since it is too difficult to compute A from this information alone, we consider the situation more closely. By Brauer's permutation lemma ([6, Lemma IV.6.10]), there are three triples of algebraic conjugate characters in $\mathrm{Irr}(B)$. The remaining characters are all p-rational ($p = 7$). It follows that the columns (b_i, c_i) contain blocks of the form

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha - \beta \\ \beta - \alpha & -\alpha \end{pmatrix}.$$

If one of these entries is ± 2, then (b_i, c_i) contains only one block of this type. But then there is no possibility such that $(b_i, b_j) = 4$. Hence, $\alpha, \beta \in \{0, \pm 1\}$ and there are exactly three such blocks in (b_i, c_i). These blocks account for all nine non-p-rational characters. We may assume that the three blocks in (b_1, c_1) fulfill $\alpha = \beta = 1$. We may also assume that the same holds for the first block of (b_2, c_2). Similar arguments eventually show that the nine blocks have the following form

$$\left(\begin{array}{cc|cc|cc} 1 & 1 & 1 & 1 & 1 & . \\ -1 & . & -1 & . & . & 1 \\ . & -1 & . & -1 & -1 & -1 \\ \hline 1 & 1 & 1 & . & 1 & 1 \\ -1 & . & . & 1 & -1 & . \\ . & -1 & -1 & -1 & . & -1 \\ \hline 1 & 1 & . & 1 & . & 1 \\ -1 & . & -1 & -1 & -1 & -1 \\ . & -1 & 1 & . & 1 & . \end{array}\right).$$

In particular, the columns b_i and c_i are essentially unique. These information suffice in order to enumerate the possible matrices A by using Plesken's algorithm [17] in GAP [7]. It turns out that $11 \le k(B) \le 18$ (Alperin's Conjecture would imply $k(B) = 17$). For each choice of A we have to add one integral column for (y, b_y) in such a way that the orthogonality relations are satisfied. In order to reduce the possibilities for this column, we use the Broué-Puig $*$-construction [3]. Observe that there is an $I(B)$-stable generalized character λ of D such that $\lambda(1) = 0$, $\lambda(x) = \lambda(x^2) = \lambda(x^4) = 21$ and $\lambda(y) = 28$. Let $M_x := Q_x 49 C_x^{-1} Q_x^{\mathrm{T}}$ be the contribution matrix with respect to (x, b_x). Similarly, we define M_u for every $u \in D$. Then $21(M_x + M_{x^2} + M_{x^4}) + 28M_y \equiv 0 \pmod{49}$ and

$$M_y \equiv M_x + M_{x^2} + M_{x^4} \pmod 7$$

(cf. [24, Section 2.5]). Moreover, there are restrictions on the p-adic valuation of the entries of M_x (see [22, Proposition 1.36]). This reduces the choices for A down to 23 possibilities. In particular, $k(B) \in \{13, 14, 16, 17, 18\}$. Now one can obtain the Cartan matrix C of B up to basic sets by computing the integral orthogonal space of the columns we have found (cf. [22, Section 4.2]). After one applies the LLL reduction algorithm the claim follows easily. □

In the proof above, it is interesting to see that $A^{\mathrm{T}}A$ has elementary divisors $1, 1, 1, 1, 7, 7, 7, 7, 49$. In contrast to that the matrices C_x, C_{x^2} and C_{x^4} give (in total) elementary divisors $7, 7, 7, 7, 7, 7, 49, 49, 49$. Moreover, the elementary divisors of C in the local case are $1, 1, 1, 1, 7, 7, 49$.

Now we prove our main theorem which generalizes [26, Proposition 22].

Theorem 2. *Let B be a p-block with abelian defect group D. Suppose that D has no elementary abelian direct summand of order p^4. Then $k(B) \le |D|$.*

Proof. We decompose $D = D_1 \oplus \ldots \oplus D_k$ into $I(B)$-invariant indecomposable summands. Then each D_i is a homocyclic group, i. e. a direct product of isomorphic cyclic groups (see [8, Theorem 5.2.2]). If D_i is not elementary abelian, then by [26, Proposition 19] we find an element $x_i \in D_i$ such that $\mathrm{C}_{I(B)}(x_i) = \mathrm{C}_{I(B)}(D_i)$. The same is true if $|D_i| = p$. If D_i is elementary abelian of order p^2, then there exists an element $x \in D_i$ such that $[D_i, \mathrm{C}_{I(B)}(x_i)]$ has order at most p. In this case also $|[D, \mathrm{C}_{I(B)}(x)]| \le p$ for $x := x_1 \ldots x_k$. Then the claim follows from [26, Proposition 11]. Thus, we may assume that D_1 is elementary abelian of order p^3. In the first part of the proof we will assume that $p \ne 7$. By [22, Lemma 14.5] it suffices to find $x \in D_1$ such that $\mathrm{C}_{I(B)}(x)/\mathrm{C}_{I(B)}(D_1) \le S_3$. In order to do so, we may assume that $D = D_1$.

As we have seen, $I(B)$ acts irreducibly on D. Assume first that $I(B)$ is non-solvable. Then $I(B)$ is in fact absolutely irreducible, since otherwise the representation $I(B) \to \mathrm{GL}(3, \overline{\mathbb{F}_p})$ where $\overline{\mathbb{F}_p}$ is the algebraic closure of $\mathbb{F}_p$ would decompose completely into representations of degree 1 and $I(B)$ would be abelian (see [11, Corollary 9.23]). Let $\mathrm{E}(I(B))$ be the layer of $I(B)$. Then $\mathrm{E}(I(B))$ is a central product of components $\mathrm{E}(I(B)) = K_1 * \ldots * K_r$. Each K_i is a quasisimple group. By Clifford's Theorem (see [11, Theorem 6.5]), $\mathrm{E}(I(B))$ and K_1 are also (absolutely) irreducible. Thus, Schur's Lemma (see [11, Theorem 9.2]) implies $\mathrm{C}_{\mathrm{GL}(3,p)}(K_1) = \mathrm{Z}(\mathrm{GL}(3, p))$. Hence $r = 1$, i. e. $\mathrm{E}(I(B))$ is quasisimple. Now, by [14, Theorems 2.1 and 2.2], there is always an element $x \in D$ such that $|\mathrm{C}_{I(B)}(x)| \le 2$.

Now assume that $I(B)$ is solvable. In order to find an element x as above, we may assume that $I(B)$ is a maximal solvable subgroup of $\mathrm{GL}(3, p)$. Then there are three cases for $I(B)$ given in [30, Theorem 21.6]. In the first case $I(B)$ is imprimitive on D. In particular, $I(B)$ contains an abelian normal subgroup A such that $I(B)/A \cong S_3$. It is well known that A has a regular orbit on D. Thus, we may choose $x \in D$ such that $\mathrm{C}_A(x) = 1$. Then $\mathrm{C}_{I(B)}(x) \le S_3$ and we are done. In the second case, $I(B)$ is the normalizer of a Singer cycle of order $p^3 - 1$. Then $I(B) \cong C_{p^3-1} \rtimes C_3$ and we find $x \in D$ such that $\mathrm{C}_{I(B)}(x) \le C_3$. Finally, in the third case $|I(B)| = 6^3(p-1)$ and $p \equiv 1 \pmod 3$. More precisely, the structure of $I(B)$ is given as follows (see [5, Lemma 2.1(7)]):

$$A := \mathrm{Z}(I(B)) \cong C_{p-1}, \qquad N := \mathrm{F}(I(B)) \cong E * A, \qquad I(B)/N \cong \mathrm{SL}(2, 3).$$

Here A acts freely on D, E is extraspecial of order 27 and exponent 3, and $I(B)/N$ acts naturally on $N/A \cong C_3^2$. Let $1 \ne x \in D$, and let $\alpha \in \mathrm{C}_{I(B)}(x)$. Then we may assume that α has prime order q. Since $\alpha \notin A$, we have $q \in \{2, 3\}$. Now the idea is to count the elements α and consider $D \setminus \bigcup_\alpha \mathrm{C}_D(\alpha)$. Suppose first that $q = 2$. Then $\langle \alpha N \rangle = \mathrm{Z}(I(B)/N)$. We may multiply each α by the unique involution z in A. Since α acts by inversion on N/A, there are $2 \cdot 9 = 18$ choices for α. For each α we have $|\mathrm{C}_D(\alpha)| \le p^2$. Suppose that $|\mathrm{C}_D(\alpha)| = |\mathrm{C}_D(\alpha z)| = p^2$. Then we get the contradiction $1 \ne \mathrm{C}_D(\alpha) \cap \mathrm{C}_D(\alpha z) \le \mathrm{C}_D(z) = 1$. Hence, $|\mathrm{C}_D(\alpha) \cup \mathrm{C}_D(\alpha z)| \le p^2 + p - 1$.

Next suppose that $q = 3$. Obviously, $I(B)$ has four Sylow 3-subgroups. Suppose that $\alpha \in I(B) \setminus N$ has order 3. Then we may choose generators a and b of N/A such that $\alpha = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ with respect to a and b. Then $(\alpha b)^3 = \alpha b \alpha^{-1} \cdot \alpha^2 b \alpha^{-2} \cdot b = aba^2b^2 \ne 1$, since E is non-abelian. Hence, the group $E\langle\alpha\rangle$ contains at most $26 + 2 \cdot 9$ elements of order 3. Excluding the elements in A, $24 + 2 \cdot 9$ are remaining. Taking the three other Sylow 3-subgroups into account, we get $24 + 8 \cdot 9 = 96$ choices for α (of order 3). However, $\mathrm{C}_D(\alpha) = \mathrm{C}_D(\alpha^{-1})$.

Therefore, it suffices to take only 48 of those. By [5, Lemma 2.4], the elements in $\alpha \in N$ satisfy $|\mathrm{C}_D(\alpha)| \le p$. If $\alpha \in I(B) \setminus N$ and $1 \ne z \in \mathrm{Z}(E)$, then $|\mathrm{C}_D(\alpha) \cup \mathrm{C}_D(\alpha z) \cup \mathrm{C}_D(\alpha z^{-1})| \le p^2 + 2p - 2$ as above. Altogether we obtain

$$\Bigl|\bigcup_{g \in I(B)} \mathrm{C}_D(g)\Bigr| \le p^2 + 8(p^2 - p) + 9(p-1) + 12(p^2 - p) + 36(p-1) = 21p^2 + 25p - 45.$$

Hence if $p \ge 23$ we find a regular orbit for $I(B)$. Since $p \equiv 1 \pmod 3$, it remains to handle the cases $p \in \{7, 13, 19\}$. If $p \in \{13, 19\}$, one can show with GAP that there is always an element $x \in D$ such that $\mathrm{C}_{I(B)}(x) \le S_3$.

Finally for the remainder of the proof we deal with the case $p = 7$. Here a GAP computation shows that only the maximal solvable group $I(B)/\mathrm{C}_{I(B)}(D_1)$ of order 6^4 considered above causes problems. As in the beginning of the proof, we choose $x \in D$ such that $[D, \mathrm{C}_{I(B)}(x)] \cong C_p^2$. More precisely, we find an element $x_1 \in D_1$ such that $\mathrm{C}_{I(B)}(x) \cong \mathrm{SL}(2,3)$. Let (x, b_x) be a B-subsection. Then $I(b_x) \cong \mathrm{C}_{I(B)}(x)$. Let $Z := \mathrm{C}_D(I(b_x))$, and let β_Z be a Brauer correspondent of b_x in $\mathrm{C}_G(Z)$ $(\subseteq \mathrm{C}_G(x))$. By [32, Theorem 1], $k(\beta_Z) = k(b_x)$ and $l(\beta_Z) = l(b_x)$. Moreover, β_Z dominates a block $\overline{\beta_Z}$ of $\mathrm{C}_G(Z)/Z$ with defect group $D/Z \cong C_p^2$ and $I(\overline{\beta_Z}) \cong I(\beta_Z) \cong I(b_x) \cong \mathrm{SL}(2,3)$. It turns out that $\overline{\beta_Z}$ is the block considered in Lemma 1. Let $\overline{C_Z}$ be the Cartan matrix of $\overline{\beta_Z}$. Then the Cartan matrix of β_Z is given by $C_Z = |Z|\overline{C_Z}$. In order to get from C_Z to the Cartan matrix C_x of b_x we use an argument from [26, proof of Proposition 16]. Let Q_x be the decomposition matrix of b_x. Then by the Broué-Puig $*$-construction, every row of Q_x appears $|Z|$ times (see [26, Lemma 10]). Hence, the algorithm from [22, Section 4.2] applied to b_x is essentially same as the application to $\overline{\beta_Z}$. In particular, we get the same matrices for C_x as we have computed in Lemma 1 (multiplied by $|Z|$). Thus, $\mathrm{tr}(C_x) \le |Z| \,\mathrm{tr}(\overline{C_Z}) \le 49|Z| = |D|$ by Lemma 1. Now the claim follows from [22, Theorem 4.2]. □

Corollary 3. *Let B be a block of a finite group with abelian defect group D of rank at most 3. Then $k(B) \le |D|$.*

We mention that there is a stronger result for $p = 2$ (see [26, Proposition 21]). On the other hand, we have already pointed out in [23, p. 794] that the elementary abelian defect group of order 3^4 is more difficult to handle (and still open).

The next result generalizes [26, Theorem 5].

Theorem 4. *Let B be a block of a finite group with defect group D such that $D/\langle z\rangle$ is abelian of rank at most 2 for some $z \in \mathrm{Z}(D)$. Then $k(B) \le |D|$.*

Proof. By Corollary 3, we may assume that D is non-abelian. Then the rank of $D/\langle z\rangle$ must be 2 and the claim follows from [26, Theorems 5]. □

Corollary 5. *Brauer's $k(B)$-Conjecture holds for the blocks of defect at most 3.*

Now we give a minor improvement on the bound obtained in [23, Theorem 1].

Theorem 6. *Let B be a p-block with abelian defect group D of order p^d. Suppose that the largest elementary abelian direct summand of D has rank $r \ge 1$. Then $k(B) \le p^{d + \frac{r-1}{2}}$.*

Proof. We may decompose $D = D_1 \oplus D_2$ with $I(B)$-invariant summands such that D_1 is elementary abelian of rank r. By [26, Proposition 19] there is an element $x_2 \in D_2$ such that $\mathrm{C}_{I(B)}(x_2) = \mathrm{C}_{I(B)}(D_2)$. By [9, Corollary 1.2] there are elements $x_1, y \in D_1$ such that $\mathrm{C}_{I(B)}(x_1) \cap \mathrm{C}_{I(B)}(y) = \mathrm{C}_{I(B)}(D_1)$. Let $x := x_1 x_2$. Then $Q := [D, \mathrm{C}_{I(B)}(x)] \le D_1$ has rank at most $r - 1$. Let (x, b_x) be a subsection for B. By [32, Theorem 1], there exists a block $\overline{b_x}$ with defect group Q and $l(\overline{b_x}) = l(b_x)$. We may assume that $y \in Q$ (otherwise $Q = 1$ and $k(B) \le p^d$). Let (y, β_y) be a subsection for $\overline{b_x}$. Then β_y has inertial index 1. Hence, β_y is nilpotent and $l(\beta_y) = 1$. By a result of Robinson (see [22, Proposition 4.7]), it follows that $k(\overline{b_x}) \le |Q| = p^{r-1}$. Therefore, $l(b_x) \le p^{r-1}$. Finally, [22, Proposition 4.12] yields $k(B) \le |D|\sqrt{l(b_x)} \le p^{d + \frac{r-1}{2}}$. □

In the worst case of Theorem 6, D is elementary abelian. Then the bound coincides with [23, Theorem 1]. Using [30] we are able to deal with a special case for $p = 2$.

Proposition 7. *Let B be a 2-block of a finite group with elementary abelian defect group D of rank r. Suppose that r is a prime and $I(B)$ acts irreducibly on D. Then $k(B) \le |D|$.*

Proof. By the Feit-Thompson Theorem, $I(B)$ is solvable. By [26, Proposition 11], it suffices to find an element $x \in D$ such that $\mathrm{C}_{I(B)}(x)$ has prime order (or is trivial). In order to do so, we may assume that $I(B)$ is a maximal solvable irreducible subgroup of $\mathrm{GL}(r,2)$. By [30, Theorem 21.6 and the following remarks], $I(B) \cong C_{2^r-1} \rtimes C_r$. Hence, there exists $x \in D$ such that $\mathrm{C}_{I(B)}(x) \le C_r$, and we are done. $\square$

Our last result concerns the sharpness of Brauer's $k(B)$-Conjecture. This is motivated by corresponding results in the local case (see [21, 27, 29]).

Proposition 8. *Let B be a p-block of a finite group with defect group D of order p^2 such that $k(B) = |D|$. Then one of the following holds*

(i) B is nilpotent, i. e. $I(B) = 1$.

(ii) $|\mathrm{C}_D(I(B))| = p$ and $I(B) \cong C_{p-1}$.

(iii) $D \rtimes I(B) \cong (C_p \rtimes C_{p-1})^2$.

(iv) $I(B)$ acts regularly on $D \setminus \{1\}$. Moreover, $I(B) \le \Gamma\mathrm{L}(1,p^2) \cong C_{p^2-1} \rtimes C_2$ provided $p > 59$.

(v) $p = 3$ and $I(B) \cong D_8$ or $I(B) \cong SD_{16}$.

(vi) $p = 5$ and $I(B) \cong \mathrm{SL}(2,3) \rtimes C_4$.

Proof. We may assume that B is non-nilpotent, i. e. $I(B) \ne 1$. Let (x, b_x) be a non-trivial B-subsection. Then $I(b_x) \cong \mathrm{C}_{I(B)}(x)$ acts faithfully on $D/\langle x\rangle$. In particular, $\mathrm{C}_{I(B)}(x)$ is cyclic of order $d_x \mid p-1$. Moreover, the Cartan matrix C_x of b_x is given by $p(m + \delta_{ij})$ with $m := \frac{p-1}{d_x}$ (see [22, Theorem 8.6]). Now [22, Theorem 4.2] implies $p^2 = k(B) \le p\big(m + d_x\big) \le p^2$. This forces $d_x \in \{1, p-1\}$ for all $x \in D \setminus \{1\}$. If $x \in \mathrm{C}_D(I(B)) \setminus \{1\}$, then clearly $I(B) = \mathrm{C}_{I(B)}(x) \cong C_{p-1}$. Hence, we may assume that $\mathrm{C}_D(I(B)) = 1$ in the following. Similarly, we may assume that $I(B)$ acts irreducibly on D, because otherwise (iii) holds.

Suppose that $I(B)$ acts freely on $D \setminus \{1\}$, i. e. $d_x = 1$ for all x. Then the Cartan matrix C of B has determinant p^2 (see [22, Proposition 1.46]). Hence, [25, Theorem 5] implies $p^2 = k(B) \le \frac{p^2-1}{l(B)} + l(B) \le p^2$. It follows that $l(B) \in \{1, p^2-1\}$. The case $l(B) = 1$ contradicts [22, Theorem 1.35] (cf. [19]). Hence, $l(B) = p^2 - 1$ and $I(B)$ must act regularly on $D \setminus \{1\}$. Suppose that $p > 59$. Then, by work of Hering (see e. g. [22, Theorem 15.1]), $I(B)$ lies in the semilinear group $\Gamma\mathrm{L}(1,p^2) \cong C_{p^2-1} \rtimes C_2$.

Thus, in the following we may assume that there is at least one $x \in D \setminus \{1\}$ such that $d_x = p - 1$. Suppose first that $I(B)$ is non-solvable. Similarly as in the proof of Theorem 2, the layer $\mathrm{E}(I(B))$ is quasisimple. Let $Z := \mathrm{Z}(\mathrm{GL}(2,p))$. By a result of Dickson (see [10, Hauptsatz II.8.27]), we have $5 \mid p^2-1$ and $\mathrm{E}(I(B))Z/Z \cong A_5$. Moreover by Schur's Lemma, $\mathrm{C}_{I(B)}(\mathrm{E}(I(B))) \le Z$. This implies $I(B)Z/Z \le S_5$. Since $\mathrm{C}_Z(x) = 1$, we have $\mathrm{C}_{I(B)}(x) \cong \mathrm{C}_{I(B)}(x)Z/Z \le S_5$. Hence, we conclude that $p \le 7$. But then $5 \nmid p^2 - 1$.

Finally, suppose that $I(B)$ is solvable. Then $I(B)$ lies in a maximal irreducible solvable subgroup $S \le \mathrm{GL}(2,p)$ given in [30, Theorem 21.6]. Assume first that S acts imprimitively, interchanging the subgroups $\langle x\rangle$ and $\langle y\rangle$ of D. Let $A \le I(B)$ be the subgroup which normalizes $\langle x\rangle$ and $\langle y\rangle$. If A acts freely on D, then there must be an element $z \in D$ such that $d_z = |I(B)/A| = 2$. Hence, $p = 3$, but it is easy to see that this is impossible. Thus, A does not act freely and we may assume that $|\mathrm{C}_A(x)| = p - 1$. Since $I(B)$ acts irreducibly, there is an element in $I(B)$ which interchanges $\langle x\rangle$ and $\langle y\rangle$. This yields $|\mathrm{C}_A(y)| = p - 1$ and $I(B) = S$. It follows that $d_{xy} = 2$, $p = 3$ and $I(B) \cong D_8$ (dihedral group of order 8).

In the next case S is the semilinear group $S \cong C_{p^2-1} \rtimes C_2$. Since the Singer cycle acts regularly on $D \setminus \{1\}$, there is an element $x \in D$ such that $d_x = 2$. This leads to $p = 3$ and $I(B) \cong SD_{16}$ (semidihedral group of order 16). Finally, assume that S has a normal subgroup $Z \cong C_{p-1}$ such that $S/Z \cong S_4$. By [5, Lemma 2.4], Z acts freely on $D \setminus \{1\}$. Hence, $\mathrm{C}_{I(B)}(x) \cong \mathrm{C}_{I(B)}(x)Z/Z \le S_4$. Therefore, $p \in \{3, 5\}$. A computation with GAP shows that $p = 5$ and $I(B) \cong \mathrm{SL}(2,3) \rtimes C_4$ where C_4 acts faithfully on $\mathrm{SL}(2,3)$ (the action is essentially unique). $\square$

Case (i) of Proposition 8 is well understood by a theorem of Puig [18]. Also case (ii) is well understood by results of Watanabe [32, 31]. In both cases the converse is also true. It is clear that there are examples for case (iii). Now assume that case (iv) occurs. Then the Sylow subgroups of $I(B)$ are cyclic or quaternion groups. In particular, the Schur multiplier of $I(B)$ is trivial. Hence, in view of Alperin's Conjecture, $I(B)$ should by abelian and thus cyclic. Therefore, the hypothesis $p > 59$ seems to be superfluous. However, this is not even clear if $p = 3$ and $I(B) \cong Q_8$ (see [13]). On the other hand, case (v) actually occurs (see [13, 33]). Finally, case (vi) would contradict Alperin's Conjecture.

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