An answer to a question of Bonnafé

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Abstract

We give a negative answer to a question of Bonnafé on the Loewy length of a character ring of a finite group.

A question frequently addressed in group theory asks which properties of a finite group G can be determined by its ordinary complex characters. The Grothendieck group of these characters becomes a ring $\mathbb{Z}\operatorname{Irr}(G)$, called the *character ring*, with respect to tensor products. To make things more interesting consider a prime divisor p of $|G|$ and a complete discrete valuation ring $\mathcal{O}$ such that $F = \mathcal{O}/J(\mathcal{O})$ is an algebraically closed field of characteristic p. We may take scalar extensions $\mathcal{O}\operatorname{Irr}(G) = \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}\operatorname{Irr}(G)$ and reduced modulo $J(\mathcal{O})$ to obtain $F\operatorname{Irr}(G)$. Bonnafé [1] has shown that $F\operatorname{Irr}(G)$ decomposes into local algebras, called blocks, and those are parametrized by the p-regular conjugacy classes of G. Recall that the *p'-section* of a p'-element $g \in G$ consists of all $h \in G$ such that the p'-factor of h is conjugate to g in G. Let e_S be the characteristic function of the p'-section S of g (i. e. $e_S(h) = 1$ if $h \in S$ and 0 otherwise). It can be shown that $e_S \in \mathcal{O}\operatorname{Irr}(G)$ and we may consider e_S as an idempotent of $F\operatorname{Irr}(G)$. In fact, e_S is a primitive idempotent, so every block of $F\operatorname{Irr}(G)$ has the form $B_S = F\operatorname{Irr}(G)e_S$ for some p'-section S. The block $B_p = B_p(G)$ corresponding to the p'-section $S = G_p$ of all p-elements is called the *principal* block by Bonnafé. It should be mentioned that the more familiar block decomposition of FG has little to do with the decomposition of $F\operatorname{Irr}(G)$ (not even the number of blocks is the same).

Bonnafé discovered a remarkable similarity between $B_p(G)$ and $B_p(\mathrm{N}_G(P))$ where P is a Sylow p-subgroup of G. He showed that both algebras have the same Loewy length whenever P is abelian. This can perhaps be seen as a variation of McKay's, Alperin's or Broué's conjecture. After checking many more cases, Bonnafé asked at the end of his paper whether $B_p(G)$ and $B_p(\mathrm{N}_G(P))$ always have the same Loewy length.

The aim of this short note is to report on the following counterexample to Bonnafé's question: Let $p = 2$ and let G be the solvable group

$$\texttt{SmallGroup}(768, 1085354) \cong C_8^2 \rtimes C_3 \rtimes C_4$$

of order $768 = 2^8 \cdot 3$. Here $B_p(G)$ has Loewy length 5 while $B_p(\mathrm{N}_G(P))$ has Loewy length 6. The computation was performed with GAP [2] using the following strategy: The character ring $\mathbb{Z}\operatorname{Irr}(G)$ can be realized as a structure constant algebra with respect to the canonical basis $\operatorname{Irr}(G)$. The structure constants are the scalar products $[\chi\psi, \varphi]$ where $\chi, \psi, \varphi \in \operatorname{Irr}(G)$. We only need to reduce these integers modulo p to obtain $\mathbb{F}_p\operatorname{Irr}(G)$. Similarly, the coefficient of e_{G_p} with respect to $\chi \in \operatorname{Irr}(G)$ is $\frac{1}{|G|}\sum_{g \in G_p} \chi(g)$; a rational number by elementary Galois theory. Since $e_{G_p} \in \mathcal{O}\operatorname{Irr}(G)$, these coefficients can again be reduced modulo p. In this way we construct the subalgebra $\mathbb{F}_p\operatorname{Irr}(G)e_{G_p}$. Now the Loewy length does not change under the scalar extension $B_p(G) = F \otimes_{\mathbb{F}_p} \mathbb{F}_p\operatorname{Irr}(G)e_{G_p}$ since $J(B_p(G)) = F \otimes_{\mathbb{F}_p} J(\mathbb{F}_p\operatorname{Irr}(G)e_{G_p})$. Finally, the computation of the Loewy length was done with the command `ProductSpace` in GAP. According to the `SmallGroupsInformation` command, the first 1085323 groups of order 768 in the small groups library have a normal Sylow 2-subgroup or a normal Sylow 3-subgroup. In those cases Bonnafé's question has an affirmative answer by [1, Proposition 4.7] (in part (d) of that proposition H should be replaced by G/N).

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References

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