Cartan matrices and Brauer's $k(B)$-conjecture II

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Abstract

This paper continues [27]. We show that the methods developed there also work for odd primes. In particular we prove Brauer's $k(B)$-conjecture for defect groups which contain a central, cyclic subgroup of index at most 9. As a consequence, the $k(B)$-conjecture holds for 3-blocks of defect at most 3. In the second part of the paper we illustrate the limits of our methods by considering an example. Then we use the work of Kessar, Koshitani and Linckelmann [13] (and thus the classification) to show that the $k(B)$-conjecture is satisfied for 2-blocks of defect 5 except for the extraspecial defect group $D_8 * D_8$. As a byproduct we also obtain the block invariants of 2-blocks with minimal nonmetacyclic defect groups. Some proofs rely on computer computations with GAP [10].

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1 Introduction

Let G be a finite group and let B be a p-block of G for a prime number p. We denote the number of ordinary irreducible characters by $k(B)$, and the number of irreducible Brauer characters by $l(B)$. In [27] we showed that for a 2-block B the number $k(B)$ can be bounded by the Cartan invariants of major subsections (see Lemma 3 in [27]). Our first aim here is to generalize this for all primes p.

Lemma 1. *Let (u,b) be a major subsection associated with the block B. Let $C_b = (c_{ij})$ be the Cartan matrix of b up to equivalence. Then for every positive definite, integral quadratic form $q(x_1,\ldots,x_{l(b)}) = \sum_{1\le i\le j\le l(b)} q_{ij}x_ix_j$ we have*

$$k(B) \le \sum_{1\le i\le j\le l(b)} q_{ij}c_{ij}.$$

In particular

$$k(B) \le \sum_{i=1}^{l(b)} c_{ii} - \sum_{i=1}^{l(b)-1} c_{i,i+1}. \tag{1}$$

Proof. Let us consider the generalized decomposition numbers d^u_{ij} associated with the subsection (u,b). We write $d_i := (d^u_{i1}, d^u_{i2}, \ldots, d^u_{i,l(b)})$ for $i = 1,\ldots,k(B)$. Since (u,b) is major, none of the rows d_i vanishes (see (4C) in [4]). Let $Q = (\widetilde{q}_{ij})^{l(b)}_{i,j=1}$ with

$$\widetilde{q}_{ij} := \begin{cases} q_{ij} & \text{if } i = j, \\ q_{ij}/2 & \text{if } i \ne j \end{cases}.$$

Then we have

$$\sum_{1\le i\le j\le l(b)} q_{ij}c_{ij} = \sum_{1\le i\le j\le l(b)} \sum_{r=1}^{k(B)} q_{ij}d_{ri}^u\overline{d_{rj}^u} = \sum_{r=1}^{k(B)} d_r Q\overline{d_r}^{\mathrm{T}},$$

and it suffices to show

$$\sum_{r=1}^{k(B)} d_r Q\overline{d_r}^{\mathrm{T}} \ge k(B). \tag{2}$$

For this, let p^n be the order of u. Then d_{ij}^u lies in the ring of integers $\mathbb{Z}[\zeta]$ of the p^n-th cyclotomic field $\mathbb{Q}(\zeta)$ for $\zeta := e^{2\pi i/p^n}$. Since Q is positive definite, $\alpha_r := d_r Q\overline{d_r}^{\mathrm{T}}$ is positive algebraic integer for $r = 1, \ldots, k(B)$. Let $\mathcal{G}$ be the Galois group of $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$. Then it is known that $\mathcal{G}$ permutes the set $\{\alpha_r : 1 \le r \le k(B)\}$. Hence, $\prod_{r=1}^{k(B)} \alpha_r \in \mathbb{Z}[\zeta]$ is rational and thus integral. Since all α_r are positive, we get $\prod_{r=1}^{k(B)} \alpha_r \ge 1$. Now (2) follows from the inequality of the arithmetic and geometric means. For the second claim we take the quadratic form corresponding to the Dynkin diagram of type $A_{l(b)}$ for q. □

2 3-Blocks of defect 3

Let D be a defect group of B, and let b_D be a Brauer correspondent of B in $D\,\mathrm{C}_G(D)$. Then $\mathrm{N}_G(D, b_D)$ is the inertial group of b_D in $\mathrm{N}_G(D)$, and the number $e(B) := |\mathrm{N}_G(D, b_D)/D\,\mathrm{C}_G(D)|$ is called inertial index of B. It is well known that $e(B)$ is a p'-divisor of the order of the automorphism group of D. As an application of Lemma 1 we show the following generalization of Theorem 3 in [27].

Theorem 1. *Brauer's $k(B)$-conjecture holds for defect groups which contain a central, cyclic subgroup of index at most 9.*

Proof. If $p \notin \{2, 3\}$, then the defect groups in the hypothesis are abelian of rank at most 2. In this case it is known that the $k(B)$-conjecture holds. The case $p = 2$ was done in [27]. Thus, it suffices to consider blocks B with elementary abelian defect groups D of order 9. For this, we use the work [14] by Kiyota. We have $e(B) \in \{1, 2, 4, 8, 16\}$. As usual, we may assume $e(B) > 1$. We denote the Cartan matrix of B by C.

Case 1: $e(B) = 2$.
By [29] we may assume that $G = D \rtimes C_2$ (observe that there are two essentially different actions of C_2 on D). It is easy to show that C is given by

$$\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} \text{ or } \begin{pmatrix} 6 & 3 \\ 3 & 6 \end{pmatrix}.$$

Hence, the claim follows from Inequality (1).

Case 2: $e(B) = 4$.
If the inertial group $I(B)$ is cyclic, we obtain C up to equivalence as follows

$$\begin{pmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{pmatrix}$$

from [24]. If $I(B)$ is noncyclic, we have to deal with twisted group algebras of $D \rtimes C_2^2$ as in [23]. Let γ be the corresponding 2-cocycle. Then there are just two possibilities for γ. In particular there are at most two equivalence classes for C. If γ is trivial, the C is equivalent to

$$\begin{pmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{pmatrix}.$$

Here we can use Lemma 1 with the quadratic form q corresponding to the positive definite matrix

$$\frac{1}{2}\begin{pmatrix} 2 & -1 & 1 & -1 \\ -1 & 2 & -1 & . \\ 1 & -1 & 2 & -1 \\ -1 & . & -1 & 2 \end{pmatrix}.$$

In the other case Kiyota gives the following example: Let Q_8 act on D with kernel $\mathrm{Z}(Q_8)$ (this action is essentially unique). Then we can take the nonprincipal block of $D \rtimes Q_8$ for B. In this case $l(B) = 1$, so the claim follows.

Case 3: $I(B) \cong C_8$.
Then $I(B)$ acts regularly on $D \setminus \{1\}$. Thus, there are just two B-subsections $(1, B)$ and (u, b) with $l(b) = 1$ up to conjugation. Kiyota did not obtain the block invariants in this case. Hence, we have to consider some possibilities. By Lemma (1D) in [14] we have $k(B) \in \{3, 6, 9\}$. Since u is conjugate to u^{-1} in $I(B)$, the generalized decomposition numbers d^u_{ij} are integers. Suppose $k(B) = 3$. Then the column corresponding to (u, b) in the generalized decomposition matrix has the form $(\pm2, \pm2, \pm1)^{\mathrm{T}}$. Hence, C is equivalent to

$$\begin{pmatrix} 5 & 1 \\ 1 & 2 \end{pmatrix}.$$

In the case $k(B) = 6$ the column corresponding to (u, b) is given by $(\pm2, \pm1, \pm1, \pm1, \pm1, \pm1)^{\mathrm{T}}$, and C is equivalent to

$$\begin{pmatrix} 2 & 1 & 1 & 1 & . \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ . & 1 & 1 & 1 & 3 \end{pmatrix}.$$

Finally in the case $k(B) = 9$ we get the following Cartan matrix:

$$\begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \end{pmatrix}.$$

As before, the claim follows from Inequality 1 in all cases.

Case 4: $I(B) \cong D_8$.
By Proposition (2F) in [14] there are two possibilities: $(k(B), l(B)) \in \{(9, 5), (6, 2)\}$. In both cases there are three subsections $(1, B)$, (u_1, b_1) and (u_2, b_2) with $l(b_1) = l(b_2) = 2$ up to conjugation. The Cartan matrix of b_1 and b_2 is given by $\left(\begin{smallmatrix} 6 & 3 \\ 3 & 6 \end{smallmatrix}\right)$. In the case $k(B) = 9$ and $l(B) = 5$ the numbers $d^{u_1}_{ij}$ and $d^{u_2}_{ij}$ are integers (see Subcase (a) on page 39 in [14]). Thus, we may assume that the numbers $d^{u_1}_{ij}$ form the two columns

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & . & . & . \\ . & . & . & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^{\mathrm{T}}.$$

Now we use a GAP program to enumerate the possibilities for the columns $(d^{u_2}_{1j}, d^{u_2}_{2j}, \ldots, d^{u_2}_{9j})$ $(j = 1, 2)$. It turns out that C is equivalent to

$$\begin{pmatrix} 3 & . & 1 & . & 1 \\ . & 3 & 1 & . & 1 \\ 1 & 1 & 3 & 1 & . \\ . & . & 1 & 3 & 1 \\ 1 & 1 & . & 1 & 3 \end{pmatrix}$$

in all cases. Here we can take the positive definite quadratic form q corresponding to the matrix

$$\frac{1}{2}\begin{pmatrix} 2 & . & -1 & . & -1 \\ . & 2 & -1 & 1 & -1 \\ -1 & -1 & 2 & -1 & 1 \\ . & 1 & -1 & 2 & -1 \\ -1 & -1 & 1 & -1 & 2 \end{pmatrix}$$

in Lemma 1.

In the case $k(B) = 6$ and $l(B) = 2$ the columns $d_1 := (d_{11}^{u_1}, d_{21}^{u_1}, \ldots, d_{61}^{u_1})$ and $d_2 := (d_{12}^{u_1}, d_{22}^{u_1}, \ldots, d_{62}^{u_1})$ do not consist of integers only. We write $d_1 = a + b\zeta$ with $a, b \in \mathbb{Z}^6$ and $\zeta := e^{2\pi i/3}$. Then $d_2 = a + b\overline{\zeta}$. The orthogonality relations show that

$$\begin{aligned} 6 &= (d_1 \mid d_1) = (a \mid a) + (b \mid b) - (a \mid b), \\ 3 &= (d_1 \mid d_2) = (a \mid a) + 2(a \mid b)\zeta + (b \mid b)\overline{\zeta} = (a \mid a) - (b \mid b) + (2(a \mid b) - (b \mid b))\zeta. \end{aligned}$$

This shows $(a \mid a) = 5$, $(b \mid b) = 2$ and $(a \mid b) = 1$. Hence, we can arrange d_1 in the following way:

$$(1, 1, 1, 1, 1 + \zeta, 1 + \overline{\zeta} = -\zeta)^{\mathrm{T}}.$$

It is easy to see that there are essentially two possibilities for the column $(d_{11}^{u_2}, d_{21}^{u_2}, \ldots, d_{61}^{u_2})^{\mathrm{T}}$:

$$(1 + \zeta, -\zeta, -1, -1, 1, 1)^{\mathrm{T}} \text{ or } (1 + \zeta, -\zeta, -1, 1, -1, -1)^{\mathrm{T}}.$$

The second possibility is impossible, since then C would have determinant 81. Thus, the first possibility occurs, and C is

$$\begin{pmatrix} 5 & 1 \\ 1 & 2 \end{pmatrix}$$

up to equivalence.

Case 5: $I(B) \cong Q_8$.
Then $I(B)$ acts regularly on $D \setminus \{1\}$. Hence, the result follows as in the case $I(B) \cong C_8$.

Case 6: $e(B) = 16$.
Then there are two B-subsections $(1, B)$ and (u, b) up to conjugation. This time we have $l(b) = 2$. By [31] we have $k(B) = 9$ and $l(B) = 7$. The Cartan matrix of b is given by $\left(\begin{smallmatrix} 6 & 3 \\ 3 & 6 \end{smallmatrix}\right)$. By way of contradiction, we assume that the columns $d_1 := (d_{11}^u, d_{21}^u, \ldots, d_{91}^u)$ and $d_2 := (d_{12}^u, d_{22}^u, \ldots, d_{92}^u)$ are 3-conjugate. Then an argument as in Case 4 shows the contradiction $k(B) \le 6$. Hence, the columns d_1 and d_2 have the form

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & . & . & . \\ . & . & . & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^{\mathrm{T}}.$$

Thus, we obtain C as follows:

$$\begin{pmatrix} 2 & 1 & . & . & . & . & 1 \\ 1 & 2 & . & . & . & . & 1 \\ . & . & 2 & 1 & . & . & 1 \\ . & . & 1 & 2 & . & . & 1 \\ . & . & . & . & 2 & 1 & 1 \\ . & . & . & . & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 3 \end{pmatrix}.$$

In this case we can take the positive definite quadratic form q corresponding to the matrix

$$\frac{1}{2}\begin{pmatrix} 2 & -1 & . & . & . & . & -1 \\ -1 & 2 & . & . & . & . & . \\ . & . & 2 & -1 & . & . & -1 \\ . & . & -1 & 2 & . & 1 & . \\ . & . & . & . & 2 & -1 & -1 \\ . & . & . & 1 & -1 & 2 & . \\ -1 & . & -1 & . & -1 & . & 2 \end{pmatrix}$$

in Lemma 1. $\square$

We deduce an important consequence.

Corollary 1. *Brauer's $k(B)$-conjecture holds for 3-blocks of defect at most 3.*

Hendren obtained some results about blocks with nonabelian defect groups of order p^3 (see [12, 11]). In particular he showed that Brauer's $k(B)$-conjecture is satisfied in the exponent p^2 case. However, he was not able to prove this in the exponent p case, even for $p = 3$ (see Section 6.1 in [12]).

We add a similar result in the same spirit for $p = 2$ which will be needed later.

Theorem 2. *Brauer's $k(B)$-conjecture holds for all 2-blocks with minimal nonabelian defect groups. Moreover, let Q be a minimal nonabelian 2-group, but not of type $\langle x, y \mid x^{2^r} = y^{2^r} = [x, y]^2 = [x, x, y] = [y, x, y] = 1\rangle$ with $r \geq 3$, $[x, y] := xyx^{-1}y^{-1}$ and $[x, x, y] := [x, [x, y]]$ (these groups have order $2^{2r+1} \geq 128$). Then Brauer's $k(B)$-conjecture holds for defect groups which are central extensions of Q by a cyclic group.*

Proof. This follows from a part of the author's PhD thesis (see [26]). $\square$

3 A counterexample

Külshammer and Wada [16] have shown that there is not always a positive definite quadratic form q such that we have equality in Lemma 1 (for $u = 1$). However, it is not clear if there is always a quadratic form q such that

$$\sum_{1 \leq i \leq j \leq l(B)} q_{ij} c_{ij} \leq p^d, \tag{3}$$

where d is the defect of the block B. (Of course, this would imply the $k(B)$-conjecture in general.)

We consider an example. Let $D \cong C_2^4$, $A \in \mathrm{Syl}_3(\mathrm{Aut}(D))$, $G = D \rtimes A$ and $B = B_0(G)$. Then $k(B) = 16$, $l(B) = 9$, and the decomposition matrix Q and the Cartan matrix C of B are

$$Q = \begin{pmatrix}
1 & . & . & . & . & . & . & . & . \\
. & 1 & . & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & . \\
. & . & . & . & 1 & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & 1 & . & . \\
. & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & 1 \\
1 & 1 & 1 & . & . & . & . & . & . \\
1 & . & . & . & . & 1 & 1 & . & . \\
. & . & . & 1 & . & 1 & . & 1 & . \\
. & . & . & . & 1 & . & 1 & . & 1 \\
. & 1 & . & . & 1 & . & . & 1 & . \\
. & . & 1 & 1 & . & . & . & . & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}, \quad C = \begin{pmatrix}
4 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 \\
2 & 4 & 2 & 1 & 2 & 1 & 1 & 2 & 1 \\
2 & 2 & 4 & 2 & 1 & 1 & 1 & 1 & 2 \\
1 & 1 & 2 & 4 & 1 & 2 & 1 & 2 & 2 \\
1 & 2 & 1 & 1 & 4 & 1 & 2 & 2 & 2 \\
2 & 1 & 1 & 2 & 1 & 4 & 2 & 2 & 1 \\
2 & 1 & 1 & 1 & 2 & 2 & 4 & 1 & 2 \\
1 & 2 & 1 & 2 & 2 & 2 & 1 & 4 & 1 \\
1 & 1 & 2 & 2 & 2 & 1 & 2 & 1 & 4
\end{pmatrix}.$$

We will see that in this case there is no positive definite quadratic form q such that Inequality (3) is satisfied. In order to do so, we assume that q is given by the matrix $\frac{1}{2}A$ with $A = (a_{ij}) \in \mathbb{Z}^{9 \times 9}$. Since A is symmetric, we only consider the upper triangular half of A. Then the rows of Q are 1-roots of q, i. e. $rAr^{\mathrm{T}} = 2$ for every row r of Q (see Corollary B in [16]). If we take the first nine rows of Q, it follows that $a_{ii} = 2$ for $i = 1, \ldots, 9$. Now assume $|a_{12}| \geq 2$. Then

$$(1, -\operatorname{sgn} a_{12}, 0, \ldots, 0)A(1, -\operatorname{sgn} a_{12}, 0, \ldots, 0)^{\mathrm{T}} \leq 0,$$

and q is not positive definite. The same argument shows $a_{ij} \in \{-1, 0, 1\}$ for $i \neq j$. In particular there are only finitely many possibilities for q. Now the next row of Q shows

$$(a_{12}, a_{13}, a_{23}) \in \{(-1, -1, 0), (-1, 0, -1), (0, -1, -1)\}.$$

The same holds for the following triples

$$(a_{16}, a_{17}, a_{67}),\ (a_{46}, a_{48}, a_{68}),\ (a_{57}, a_{59}, a_{79}),\ (a_{25}, a_{28}, a_{58}),\ (a_{34}, a_{39}, a_{49}).$$

Finally the last row of Q shows that the remaining entries add up to 4:

$$a_{14} + a_{15} + a_{18} + a_{19} + a_{24} + a_{26} + a_{27} + a_{29} + a_{35} + a_{36} + a_{37} + a_{38} + a_{45} + a_{47} + a_{56} + a_{69} + a_{78} + a_{89} = 4.$$

These are too many possibilities to check by hand. So we try to find a positive definite form q with GAP. To decrease the computational effort, we enumerate all positive definite 7×7 left upper submatrices of A first. There are 140428 of them, but none can be completed to a positive definite 9×9 matrix with the given constraints.

Nevertheless, we show that there is no corresponding decomposition matrix for C with more than 16 rows. For this let B be a block with Cartan matrix equivalent to C. (By [27] the $k(B)$-conjecture already holds for B. We give an independent argument for this.) We enumerate the possible decomposition matrices Q and count their rows. Since $Q \in \mathbb{Z}^{k(B)\times 9}$, every column of Q has the form $(\pm 1, \pm 1, \pm 1, \pm 1, 0, \ldots, 0)^{\mathrm{T}}$ for a suitable arrangement. Let us assume that the first two columns of Q have the form

$$\begin{pmatrix} 1 & 1 & 1 & 1 & . & \cdots & . \\ 1 & 1 & 1 & -1 & . & \cdots & . \end{pmatrix}^{\mathrm{T}}.$$

Then the entries of C show that there is no possibility for the fifth column of Q. Thus, we may assume that the first two columns of Q are

$$\begin{pmatrix} 1 & 1 & 1 & 1 & . & . & . & \cdots & . \\ . & . & 1 & 1 & 1 & 1 & . & \cdots & . \end{pmatrix}^{\mathrm{T}}.$$

Now we use a backtracking algorithm with GAP to show that Q has at most 16 rows (and at least 9).

Unfortunately, this method does not carry over to major subsections. For if we multiply C by a 2-power (namely the order of a 2-element), the corresponding (generalized) decomposition matrices can be entirely different.

4 2-Blocks with defect 5

In order to prove Brauer's $k(B)$-conjecture for 2-blocks of defect 5, we discuss central extensions of groups of order 16 by cyclic groups. We start with the abelian (and nonmetacyclic) groups of order 16. In the next proposition we have to exclude one case, as the last section has shown. Moreover, we use the work of Kessar, Koshitani and Linckelmann [13] (and thus the classification) in the proof. We have not checked if it is possible to avoid the classification by considering more (virtually impossible) cases. For this reason, we will also freely use the method of Usami and Puig (see [29, 30, 24]), although there is no explicit proof in the case $p = 2$ and $e(B) = 3$.

Proposition 1. *Let B be a block with a defect group which is a central extension of an elementary abelian group of order 16 by a cyclic group. If $9 \nmid e(B)$, then Brauer's $k(B)$-conjecture holds for B.*

Proof. Let D be the defect group of B. We choose $u \in \mathrm{Z}(D)$ such that $D/\langle u\rangle$ is elementary abelian of order 16. Let (u, b) be a B-subsection. Then it is easy to see that $e(b)$ is a divisor of $e(B)$. By hypothesis $e(b) \in \{1, 3, 5, 7, 15, 21\}$. As in the proof of Theorem 1, we replace b by B for simplicity. In order to prove the proposition, we determine the Cartan matrix C of B up to equivalence. If this is done, it will be immediately clear that a suitable inequality as in Lemma 1 is satisfied.

The case $e(B) = 1$ is clear. We consider the remaining cases separately.

Case 1: $e(B) = 3$.
In this case we may use the method of Usami and Puig (see [29, 30, 24]). Thus, we can replace G by $D \rtimes C_3$ via a perfect isometry (observe that there are two essentially different actions of C_3 on D). Then C has the form

$$\begin{pmatrix} 8 & 4 & 4 \\ 4 & 8 & 4 \\ 4 & 4 & 8 \end{pmatrix} \text{ or } \begin{pmatrix} 6 & 5 & 5 \\ 5 & 6 & 5 \\ 5 & 5 & 6 \end{pmatrix}$$

up to equivalence.

Case 2: $e(B) = 5$.
Then there are four subsections $(1, B)$, (u_1, b_1), (u_2, b_2) and (u_3, b_3) with $l(b_1) = l(b_2) = l(b_3) = 1$ up to conjugation. According to the fact that $|D| = 16$ is a sum of $k(B)$ squares, we have six possibilities:

(i) $k(B) = k_0(B) = 16$ and $l(B) = 13$,

(ii) $k(B) = k_0(B) = 8$ and $l(B) = 5$,

(iii) $k(B) = 13$, $k_0(B) = 12$, $k_1(B) = 1$ and $l(B) = 10$,

(iv) $k(B) = 10$, $k_0(B) = 8$, $k_1(B) = 2$ and $l(B) = 7$,

(v) $k(B) = 7$, $k_0(B) = 4$, $k_1(B) = 3$ and $l(B) = 4$,

(vi) $k(B) = 5$, $k_0(B) = 4$, $k_1(B) = 1$ and $l(B) = 2$.

(Brauer's height zero conjecture would contradict the last four possibilities.) In case (i) we have

$$C = \begin{pmatrix} 4 & 3 & 3 & 3 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\ 3 & 4 & 3 & 3 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\ 3 & 3 & 4 & 3 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\ 3 & 3 & 3 & 4 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 & . & . & . & . & . & . \\ 1 & 1 & 1 & 1 & 1 & 2 & 1 & . & . & . & . & . & . \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & . & . & . & . & . & . \\ 1 & 1 & 1 & 1 & . & . & . & 2 & 1 & 1 & . & . & . \\ 1 & 1 & 1 & 1 & . & . & . & 1 & 2 & 1 & . & . & . \\ 1 & 1 & 1 & 1 & . & . & . & 1 & 1 & 2 & . & . & . \\ -1 & -1 & -1 & -1 & . & . & . & . & . & . & 2 & 1 & 1 \\ -1 & -1 & -1 & -1 & . & . & . & . & . & . & 1 & 2 & 1 \\ -1 & -1 & -1 & -1 & . & . & . & . & . & . & 1 & 1 & 2 \end{pmatrix}$$

up to equivalence. In particular $\det C = 256$. However, this contradicts Corollary 1 in [9]. Now we assume that case (ii) occurs. Then there are several ways to arrange the generalized decomposition numbers corresponding to b_i for $i = 1, 2, 3$:

$$\begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & 3 \\ 1 & 3 & -1 \\ 3 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & 3 \\ 1 & 3 & -1 \\ 3 & -1 & -1 \end{pmatrix}.$$

In the last two cases the determinant of C would be 64. Thus, only the first case can occur. Then we have

$$C = \begin{pmatrix} 4 & 3 & 3 & 3 & 3 \\ 3 & 4 & 3 & 3 & 3 \\ 3 & 3 & 4 & 3 & 3 \\ 3 & 3 & 3 & 4 & 3 \\ 3 & 3 & 3 & 3 & 4 \end{pmatrix}$$

up to equivalence. Hence, we can consider the case (iii). Then the generalized decomposition numbers corresponding to b_i for $i = 1, 2, 3$ can be arranged in the form

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 2 \end{pmatrix}^{\mathrm{T}}.$$

However, in this case C would have determinant 256. In the same manner we see that also the case (iv) is not possible. Thus, assume case (v). Then the generalized decomposition numbers corresponding to b_i for $i = 1, 2, 3$ have the form

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 \\ -1 & -1 & -1 & -1 & 2 & 2 & -2 \\ 1 & 1 & 1 & 1 & 2 & -2 & -2 \end{pmatrix}^{\mathrm{T}}.$$

This gives

$$C = \begin{pmatrix} 5 & 4 & 4 & 5 \\ 4 & 5 & 4 & 5 \\ 4 & 4 & 5 & 5 \\ 5 & 5 & 5 & 7 \end{pmatrix},$$

and the claim follows. Finally let case (vi) occur. Then the generalized decomposition numbers corresponding to b_i for $i = 1, 2, 3$ have the form

$$\begin{pmatrix} 1 & 1 & 1 & 3 & 2 \\ 1 & 1 & -3 & -1 & 2 \\ 1 & -3 & 1 & -1 & 2 \end{pmatrix}^{\mathrm{T}}.$$

It follows that

$$C = \begin{pmatrix} 4 & 6 \\ 6 & 13 \end{pmatrix}.$$

Case 3: $e(B) = 7$.
There are again four subsections $(1, B)$, (u_1, b_1), (u_2, b_2) and (u_3, b_3) up to conjugation. But in this case $l(b_1) = l(b_2) = 1$ and $l(b_3) = 7$ by the Kessar-Koshitani-Linckelmann paper. Moreover, 2 appears six times as elementary divisor of the Cartan matrix of b_3. Using the theory of lower defect groups it follows that 2 occurs at least six times as elementary divisor of C. By [27] we have $k(B) \le 16$. This gives $k(B) = k_0(B) = 16$, $l(B) = 7$. The generalized decomposition matrix (without the ordinary part) can be arranged in the form

$$\begin{pmatrix}
1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1
\end{pmatrix}^{\mathrm{T}}.$$

Hence, C has the form

$$\begin{pmatrix} 4 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 4 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 4 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 4 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 4 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 4 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 4 \end{pmatrix}$$

up to equivalence (notice that this is also the Cartan matrix of b_3).

Case 4: $e(B) = 15$.
There are just two subsections $(1, B)$ and (u, b) with $l(b) = 1$ up to conjugation. It is easy to prove the claim using a similar case decision as in Case 2. We skip the details.

Case 5: $e(B) = 21$.
There are four subsections $(1, B)$, (u_1, b_1), (u_2, b_2) and (u_3, b_3) up to conjugation. We have $l(b_1) = l(b_2) = 3$ and $l(b_3) = 5$ by the Kessar-Koshitani-Linckelmann paper. With the notations of [15], B is a centrally controlled

block. In particular $l(B) \geq l(b_3) = 5$ (see Theorem 1.1 in [15]). Since the $k(B)$-conjecture holds for B, we have $k(B) = 16$ and $l(B) = 5$. The Cartan matrix of b_3 is given by

$$2\begin{pmatrix} 2 & . & . & . & 1 \\ . & 2 & . & . & 1 \\ . & . & 2 & . & 1 \\ . & . & . & 2 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{pmatrix}$$

(see the proof of Theorem 3 in [27]). Using this, it is easy to deduce that the generalized decomposition numbers corresponding to (u_3, b_3) can be arranged in the form

$$\begin{pmatrix} 1 & 1 & 1 & 1 & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & 1 & 1 & 1 & 1 & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & 1 & 1 & 1 & 1 & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & 1 & 1 & 1 & 1 \\ . & . & 1 & 1 & . & . & 1 & 1 & . & . & 1 & 1 & . & . & 1 & 1 \end{pmatrix}^{\mathrm{T}}.$$

It is also easy to see that the columns of generalized decomposition numbers corresponding to b_1 and b_2 consist of eight entries ± 1 and eight entries 0. The theory of lower defect groups shows that 2 occurs as elementary divisor of C. Now we use GAP to enumerate all possible arrangements of these columns. It turns out that C is equivalent to the Cartan matrix of b_3. $\square$

Proposition 2. *Brauer's $k(B)$-conjecture holds for defect groups which are central extensions of $C_4 \times C_2^2$ by a cyclic group.*

Proof. Let B be a block with defect group $D \cong C_4 \times C_2^2$. We may assume $e(B) = 3$. Then we can use the method of Usami and Puig (see [29, 30, 24]). This means it suffices to consider the case $G = D \rtimes C_3$ and $B = B_0(G)$. An easy calculation shows that the Cartan matrix of B is equivalent to

$$\begin{pmatrix} 8 & 4 & 4 \\ 4 & 8 & 4 \\ 4 & 4 & 8 \end{pmatrix}.$$

Hence, the result follows from Lemma 1 as before. $\square$

Now we turn to the nonabelian (and nonmetacyclic) groups of order 16.

Proposition 3. *Let B be a nonnilpotent block with defect group $D_8 \times C_2$. Then $k(B) = 10$, $k_0(B) = 8$ and $k_1(B) = 2$. The ordinary irreducible characters are 2-rational. Moreover, $l(B) \in \{2, 3\}$ and the Cartan matrix of B is equivalent to*

$$\begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix} \quad or \quad \begin{pmatrix} 6 & 2 & 2 \\ 2 & 4 & 0 \\ 2 & 0 & 4 \end{pmatrix}.$$

In particular the $k(B)$-conjecture holds for defect groups which are central extensions of $D_8 \times C_2$ by a cyclic group.

Proof. First we remark that the proof and the result is very similar to the case where the defect group is D_8 (see [5]). Let $D := \langle x, y, z \mid x^4 = y^2 = z^2 = [x, z] = [y, z] = 1,\ yxy = x^{-1} \rangle \cong D_8 \times C_2$ and let (D, b_D) a Sylow subpair. It is easy to see that $\mathrm{Aut}(D)$ is a 2-group. Thus, $e(B) = 1$. We use the theory developed in [22]. One can show, that all self-centralizing proper subgroups of D are maximal and there are precisely three of them:

$$M_1 := \langle x^2, y, z \rangle \cong C_2^3,$$
$$M_2 := \langle x^2, xy, z \rangle \cong C_2^3,$$
$$M_3 := \langle x, z \rangle \cong C_4 \times C_2.$$

Now Lemma 1.7 in [20] yields $\mathrm{A}_0(D, b_D) = \{M_1, M_2, M_3, D\}$. Assume that M_1 and M_2 are conjugate in G. Then also the B-subpairs (M_1, b_{M_1}) and (M_2, b_{M_2}) are conjugate. By Alperin's fusion theorem they are already conjugate in $\mathrm{N}_G(D, b_D)$. Since $e(B) = 1$, this is impossible.

Now we determine a system of representatives for the conjugacy classes of B-subsections using (6C) in [6]. As usual, one gets four major subsections $(1, B)$, (x^2, b_{x^2}), (z, b_z), (x^2z, b_{x^2z}). Then b_{x^2} dominates a block with defect group $D/\langle x^2\rangle \cong C_2^3$. Since $e(B) = 1$, we get $l(b_{x^2}) = 1$. On the other hand, b_z and b_{x^2z} dominate blocks with defect group D_8.

Since $\mathrm{Aut}(M_3)$ is a 2-group, we have $\mathrm{N}_G(M_3, b_{M_3}) = D\,\mathrm{C}_G(M_3)$. This gives two subsections (x, b_x) and (xz, b_{xy}). Again we have $l(b_x) = l(b_{xz}) = 1$.

If $\mathrm{N}_G(M_1, b_{M_1}) = D\,\mathrm{C}_G(M_1)$ and $\mathrm{N}_G(M_2, b_{M_2}) = D\,\mathrm{C}_G(M_2)$, then B would be nilpotent. Thus, we may assume $\mathrm{N}_G(M_1, b_{M_1})/\,\mathrm{C}_G(M_1) \cong S_3$. Then the elements $\{y, x^2y, yz, x^2yz\}$ are conjugate to elements of $\mathrm{Z}(D)$ under $\mathrm{N}_G(M_1, b_{M_1})$. Hence, there are no subsections corresponding to the subpair (M_1, b_{M_1}) (cf. Lemma 2.10 in [21]). We distinguish two cases.

Case 1: $\mathrm{N}_G(M_2, b_{M_2}) = D\,\mathrm{C}_G(M_2)$.
Then the action of $\mathrm{N}_G(M_2, b_{M_2})$ gives the subsections (xy, b_{xy}) and (xyz, b_{xyz}). Moreover, $l(b_{xy}) = l(b_{xyz}) = 1$ holds. Since $\mathrm{N}_G(M_1, b_{M_1})$ fixes exactly one element of $\{z, x^2z\}$, we get $l(b_z) + l(b_{x^2z}) = 3$ (see Theorem 2 in [5]) Collecting all the subsections, we deduce $k(B) = l(B) + 8$. We may assume that $l(b_z) = 2$ (otherwise replace b_z with b_{x^2z}). Then the Cartan matrix of b_z is equivalent to $\left(\begin{smallmatrix}6&2\\2&6\end{smallmatrix}\right)$ (see pages 294/5 in [8]). This gives $k(B) \le 10$. Since 16 is not the sum of 9 positive squares, we must have $k(B) = 10$. Then $k_0(B) = 8$, $k_1(B) = 2$ and $l(B) = 2$. In order to determine the Cartan matrix, we investigate the generalized decomposition numbers $d^u_{\chi\varphi}$ first. For $u \in D$ with $l(b_u) = 1$ we write $\mathrm{IBr}(b_u) = \{\varphi_u\}$. Then the numbers $\{d^{x^2}_{\chi\varphi_{x^2}} : \chi \in \mathrm{Irr}(B)\}$ can be arranged in the form

$$(1, 1, 1, 1, 1, 1, 1, 1, 2, 2)^{\mathrm{T}},$$

where the last two characters have height 1. It is easy to see that the subsections (x, b_x) and (x^{-1}, b_x) are conjugate by y. This shows that the numbers $d^x_{\chi\varphi_x}$ are integral. The same holds for $d^{xz}_{\chi\varphi_{xz}}$. Hence, all irreducible characters are 2-rational. For every character χ of height 0 we have $d^x_{\chi\varphi_x} \ne 0 \ne d^{xz}_{\chi\varphi_{xz}}$. Hence, we get three columns of the generalized decomposition matrix:

$$\begin{pmatrix}1&1&1&1&1&1&1&1&2&2\\1&1&1&1&-1&-1&-1&-1&.&.\\1&1&-1&-1&1&1&-1&-1&.&.\end{pmatrix}^{\mathrm{T}}.$$

Adding the columns $\{d^{xy}_{\chi\varphi_{xy}} : \chi \in \mathrm{Irr}(B)\}$ and $\{d^{xyz}_{\chi\varphi_{xyz}} : \chi \in \mathrm{Irr}(B)\}$ gives:

$$\begin{pmatrix}1&1&1&1&1&1&1&1&2&2\\1&1&1&1&-1&-1&-1&-1&.&.\\1&1&-1&-1&1&1&-1&-1&.&.\\1&-1&1&-1&1&-1&1&-1&.&.\\1&-1&1&-1&-1&1&-1&1&.&.\end{pmatrix}^{\mathrm{T}}$$

(To achieve this form, one may have to interchange the third row with the fifth and the fourth with the sixth as well as the second column with the third.) Since (x^2z, b_{x^2z}) is a major subsection, the column $\{d^{x^2z}_{\chi\varphi_{x^2z}} : \chi \in \mathrm{Irr}(B)\}$ consists of eight entries ±1 and two entries ±2. However, there are three essentially different ways to add this column to the previous ones:

$$\begin{pmatrix}1&1&1&1&1&1&1&1&2&2\\1&1&1&1&-1&-1&-1&-1&.&.\\1&1&-1&-1&1&1&-1&-1&.&.\\1&-1&1&-1&1&-1&1&-1&.&.\\1&-1&1&-1&-1&1&-1&1&.&.\\1&1&1&1&1&1&1&1&-2&-2\end{pmatrix}^{\mathrm{T}}$$

or

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & . & . \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & . & . \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & . & . \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & . & . \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 2 & -2 \end{pmatrix}^{\mathrm{T}}$$

or

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & . & . \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & . & . \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & . & . \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & . & . \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 2 & -2 \end{pmatrix}^{\mathrm{T}}.$$

We use GAP to enumerate the remaining columns corresponding to the subsection (z, b_z). In all cases the Cartan matrix of B is equivalent to

$$\begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix}.$$

Case 2: $\mathrm{N}_G(M_2, b_{M_2})/\,\mathrm{C}_G(M_2) \cong S_3$.
Then one can see by the same argument as for (M_1, b_{M_1}) that there are no subsections corresponding to the subpair (M_2, b_{M_2}). Since $\mathrm{N}_G(M_1, b_{M_1})$ and $\mathrm{N}_G(M_2, b_{M_2})$ fix exactly one element of $\{z, x^2z\}$ (not necessarily the same), we have $l(b_z) + l(b_{x^2z}) = 4$ (the cases $l(b_z) = l(b_{x^2z}) = 2$, $l(b_z) = 3$, $l(b_{x^2z}) = 1$ and $l(b_z) = 1$, $l(b_{x^2z}) = 3$ are possible). We deduce $k(B) = l(B) + 7$. If $l(b_z) = 2$, then we get $k(B) \le 10$ as in Case 1. Assume $l(b_z) = 3$. Then the Cartan matrix of b_z is equivalent to

$$2\begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

Thus, also in this case we have $k(B) \le 10$. A consideration of the lower defect groups shows that 2 occurs as elementary divisor of the Cartan matrix C of B. In particular $l(B) \ge 2$ and $k(B) \ge 9$. Since 16 is not the sum of 9 positive squares, it follows that $k(B) = 10$, $k_0(B) = 8$, $k_1(B) = 2$ and $l(B) = 3$. An investigation of the generalized decomposition numbers similar as in the first case reveals that C is equivalent to

$$\begin{pmatrix} 4 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 4 \end{pmatrix}.$$

This proves the proposition. $\square$

It is easy to see that both cases ($l(B) \in \{2, 3\}$) in Proposition 3 occur for the principal blocks of $S_4 \times C_2$ and $\mathrm{GL}(3, 2) \times C_2$ respectively.

Proposition 4. *Let B be a nonnilpotent block with defect group $Q_8 \times C_2$. Then $k(B) = 14$, $k_0(B) = 8$, $k_1(B) = 6$ and $l(B) = 3$. The ordinary irreducible characters are 2-rational. The Cartan matrix of B is equivalent to*

$$\begin{pmatrix} 8 & 4 & 4 \\ 4 & 8 & 4 \\ 4 & 4 & 8 \end{pmatrix}.$$

In particular the $k(B)$-conjecture holds for defect groups which are central extensions of $Q_8 \times C_2$ by a cyclic group.

Proof. Let $D := \langle x, y, z \mid x^2 = y^2,\ xyx^{-1} = y^{-1},\ z^2 = [x, z] = [y, z] = 1\rangle \cong Q_8 \times C_2$ and let (D, b_D) a Sylow subpair. Since $|\mathrm{Z}(D) : \Phi(D)| = 2$, we have $e(B) \in \{1, 3\}$. As in the proof of Proposition 3 there are precisely three self-centralizing proper subgroups of D:

$$\begin{aligned} M_1 &:= \langle x, z\rangle \cong C_4 \times C_2,\\ M_2 &:= \langle y, z\rangle \cong C_4 \times C_2,\\ M_3 &:= \langle xy, z\rangle \cong C_4 \times C_2. \end{aligned}$$

It follows from Lemma 1.7 in [20] that $\mathrm{A}_0(D, b_D) = \{M_1, M_2, M_3, D\}$. Since $\mathrm{Aut}(M_i)$ is a 2-group for $i = 1, 2, 3$, B would be nilpotent if $e(B) = 1$. Thus, we may assume that $e(B) = 3$. Then M_1, M_2 and M_3 are conjugate in G. We describe a system of representatives for the conjugacy classes of B-subsections. As usual, there are four major subsections $(1, B)$, (x^2, b_{x^2}), (z, b_z) and (x^2z, b_{x^2z}). Moreover, the subpair (M, b_M) gives the subsections (x, b_x) and (xz, b_{xz}). The blocks b_z and b_{x^2z} dominate blocks with defect group $D/\langle z\rangle \cong D/\langle x^2z\rangle \cong Q_8$. Since $\mathrm{N}_G(D, b_D)$ centralizes $\mathrm{Z}(D)$, these blocks with defect group Q_8 have inertial index 3. Now Theorem 3.17 in [20] gives $l(b_z) = l(b_{x^2z}) = 3$. The block b_{x^2} covers a block with defect group $D/\langle x^2\rangle \cong C_2^3$ and inertial index 3. Thus, we also have $l(b_{x^2}) = 3$. Finally the blocks b_x and b_{xz} have defect group M_1. Hence, they are nilpotent, and we have $l(b_x) = l(b_{xz}) = 1$. This yields $k(B) = 11 + l(B)$. Since B is a centrally controlled block, we get $l(B) \geq l(b_z) = 3$ and $k(B) \geq 14$. The Cartan matrix of b_{x^2}, b_{x^2z} and b_z is equivalent to

$$\begin{pmatrix} 8 & 4 & 4\\ 4 & 8 & 4\\ 4 & 4 & 8 \end{pmatrix}$$

(see page 305 in [8]). Let $Q \in \mathbb{Z}^{k(B)\times 3}$ be the part of the generalized decomposition matrix corresponding to b_z. Then the columns of Q have one of the following forms: $(\pm2, \pm2, 0, \ldots, 0)$, $(\pm2, \pm1, \pm1, \pm1, \pm1, 0, \ldots, 0)$ or $(\pm1, \ldots, \pm1, 0, \ldots, 0)$. Since $k(B) \geq 14$, at least one column has the last form. A similar argument shows that no column has the first form. It follows that at least two columns have the form $(\pm1, \ldots, \pm1, 0, \ldots, 0)$. Hence, there are four possibilities for Q:

$$\underset{(a)}{\begin{pmatrix} 1&.&.\\ 1&.&.\\ 1&.&.\\ 1&.&.\\ 1&1&2\\ 1&1&1\\ 1&1&1\\ 1&1&.\\ .&1&.\\ .&1&.\\ .&1&.\\ .&1&.\\ .&.&1\\ .&.&1 \end{pmatrix}} \quad \underset{(b)}{\begin{pmatrix} 1&.&.\\ 1&.&.\\ 1&.&.\\ 1&.&.\\ 1&1&1\\ 1&1&1\\ 1&1&1\\ 1&1&1\\ .&1&1\\ .&1&-1\\ .&1&.\\ .&1&.\\ .&.&1\\ .&.&1 \end{pmatrix}} \quad \underset{(c)}{\begin{pmatrix} 1&.&.\\ 1&.&.\\ 1&.&1\\ 1&.&1\\ 1&1&1\\ 1&1&1\\ 1&1&.\\ 1&1&.\\ .&1&1\\ .&1&1\\ .&1&.\\ .&1&.\\ .&.&1\\ .&.&1 \end{pmatrix}} \quad \underset{(d)}{\begin{pmatrix} 1&.&.\\ 1&.&.\\ 1&.&.\\ 1&.&.\\ 1&1&1\\ 1&1&1\\ 1&1&1\\ 1&1&1\\ .&1&.\\ .&1&.\\ .&1&.\\ .&1&.\\ .&.&1\\ .&.&1\\ .&.&1\\ .&.&1 \end{pmatrix}}$$

In particular $k(B) \in \{14, 16\}$ and $l(B) \in \{3, 5\}$.

By way of contradiction, we assume $k(B) = 16$. Then Q is given as in case (d). Let $M_z = (m_{\chi\psi}^{(z,b_z)})$ be the matrix of contributions corresponding to (z, b_z). We denote the three irreducible Brauer characters of b_z by φ_1, φ_2 and φ_3. Then for $\chi \in \mathrm{Irr}(B)$ we have

$$\begin{aligned} 16m_{\chi\chi}^{(z,b_z)} &= 3\big((d^z_{\chi\varphi_1})^2 + (d^z_{\chi\varphi_2})^2 + (d^z_{\chi\varphi_3})^2\big) - 2d^z_{\chi\varphi_1}d^z_{\chi\varphi_2} - 2d^z_{\chi\varphi_1}d^z_{\chi\varphi_3} - 2d^z_{\chi\varphi_2}d^z_{\chi\varphi_3}\\ &\equiv d^z_{\chi\varphi_1} + d^z_{\chi\varphi_2} + d^z_{\chi\varphi_3} \pmod 2. \end{aligned}$$

In particular the numbers $16m_{\chi\chi}^{(z,b_z)}$ are odd for all $\chi \in \mathrm{Irr}(B)$. Now (5G) in [4] implies $k(B) = k_0(B)$. By Proposition 1 in [7] we get $d^x_{\chi\varphi_x} \neq 0$ for all $\chi \in \mathrm{Irr}(B)$. However, $\sum_{\chi\in\mathrm{Irr}(B)} |d^x_{\chi\varphi_x}|^2 = |M_1| = 8$.

This contradiction yields $k(B) = 14$ and $l(B) = 3$. The last argument gives also $k_0(B) \le 8$. Now a similar analysis of the contributions reveals that Q has the form (c) (see above) and $k_0(B) = 8$. Again (5G) in [4] implies $k_1(B) = 6$ (this follows also from Corollary 1.4 in [17]). Since the subsections (x, b_x) and (x^{-1}, b_x) are conjugate in G, the generalized decomposition numbers $d^x_{\chi\varphi_x}$ and $d^{xz}_{\chi\varphi_{xz}}$ are integral. Thus, they must consist of eight entries ± 1 (for the characters of height 0) and six entries 0. In particular all characters are 2-rational. Now we enumerate all possible decomposition matrices with GAP. In all cases the Cartan matrix of B has the stated form. □

The principal block of $\mathrm{SL}(2, 3) \times C_2$ gives an example for the last proposition.

Proposition 5. *Let B be a nonnilpotent block with defect group $D_8 * C_4$ (central product). Then $k(B) = 14$, $k_0(B) = 8$, $k_1(B) = 6$ and $l(B) = 3$. Moreover, the irreducible characters of height 0 are 2-rational and the characters of height 1 consist of three pairs of 2-conjugate characters. The Cartan matrix of B is equivalent to*

$$\begin{pmatrix} 8 & 4 & 4 \\ 4 & 8 & 4 \\ 4 & 4 & 8 \end{pmatrix}.$$

In particular the $k(B)$-conjecture holds for defect groups which are central extensions of $D_8 * C_4$ by a cyclic group.

Proof. The proof (and the result) is very similar to that of Proposition 4. Let $D := \langle x, y, z \mid x^4 = y^2 = [x, z] = [y, z] = 1,\ yxy = x^{-1},\ x^2 = z^2 \rangle \cong D_8 * C_4$. We have $e(B) \in \{1, 3\}$ and $\mathrm{A}_0(D, b_D) = \{M_1, M_2, M_3, D\}$ with

$$\begin{aligned} M_1 &:= \langle x, z \rangle \cong C_4 \times C_2, \\ M_2 &:= \langle y, z \rangle \cong C_4 \times C_2, \\ M_3 &:= \langle xy, z \rangle \cong C_4 \times C_2. \end{aligned}$$

Hence, we may assume $e(B) = 3$. Then M_1, M_2 and M_3 are conjugate in G. There are four major subsections $(1, B)$, (z, b_z), $(z^{-1}, b_{z^{-1}})$ and (x^2, b_{x^2}). The subpair (M_1, b_{M_1}) gives two nonmajor subsections (x, b_x) and (xz, b_{xz}) up to conjugation. As usual, we have $l(b_x) = l(b_{xz}) = 1$. The blocks b_z and $b_{z^{-1}}$ dominate blocks with defect groups $D/\langle z \rangle \cong C_2^2$ and inertial index 3. Hence, we have $l(b_z) = l(b_{z^{-1}}) = 3$. The block b_{x^2} dominates a block with defect group C_2^3 and inertial index 3. Thus, again we have $l(b_{x^2}) = 3$. Collecting these numbers gives $k(B) = 11 + l(B)$. The Cartan matrix of the blocks b_z, $b_{z^{-1}}$ and b_{x^2} is

$$\begin{pmatrix} 8 & 4 & 4 \\ 4 & 8 & 4 \\ 4 & 4 & 8 \end{pmatrix}$$

up to equivalence. Now an analysis of the generalized decomposition numbers $d^{x^2}_{\chi\varphi}$ as in the proof of Proposition 4 reveals $k(B) = 14$, $k_0(B) = 8$, $k_1(B) = 6$ and $l(B) = 3$. Next we study the other generalized decomposition numbers. Again as in the proof of Proposition 4 the numbers $d^x_{\chi\varphi}$ and $d^{xz}_{\chi\varphi}$ are integral. Thus, they consist of eight entries ± 1 and six entries 0. However, in contrast to Proposition 4 the numbers $d^z_{\chi\varphi}$ and $d^{z^{-1}}_{\chi\varphi}$ are not always real (see (6B) in [4]). Let Q be the part of the generalized decomposition matrix corresponding to (z, b_z). By Brauer's Permutation Lemma, eight of the ordinary irreducible characters are 2-rational. The remaining ones split in three pairs of 2-conjugate characters (see Theorem 11 in [3]). This shows that Q has exactly eight real-valued rows. Let q_j be the j-th column of Q for $j = 1, 2, 3$. Then we can write $q_j = a_j + b_j i$ with $i := \sqrt{-1}$ and $a_j, b_j \in \mathbb{Z}^{14}$. The orthogonality relations show that a_j has four entries ± 1 and ten entries 0 (for $j = 1, 2, 3$). The same holds for b_j. Moreover, we have $4 = (q_1 \mid q_2) = (a_1 \mid a_2) + (b_1 \mid b_2)$ and $0 = (q_1 \mid \overline{q_2}) = (a_1 \mid a_2) - (b_1 \mid b_2)$, where $(. \mid .)$ denotes the standard scalar product of $\mathbb{C}^{14}$. This shows $(a_1 \mid a_2) = (b_1 \mid b_2) = 2$ and similarly $(a_1 \mid a_3) = (a_2 \mid a_3) = (b_1 \mid b_3) = (b_2 \mid b_3) = 2$. Using this, we see that Q has the form

$$Q = \begin{pmatrix} 1 & 1 & 1 & 1 & . & . & . & . & i & -i & i & -i & . & . \\ 1 & 1 & . & . & 1 & 1 & . & . & i & -i & . & . & i & -i \\ 1 & 1 & . & . & . & . & 1 & 1 & . & . & i & -i & i & -i \end{pmatrix}^{\mathrm{T}}.$$

The theory of contributions reveals that the eight characters of height 0 are 2-rational. As in the proof of the previous propositions we enumerate the possible generalized decomposition matrices with GAP, and obtain the Cartan matrix of B. □

We collect the previous propositions in the next theorem.

Theorem 3. *Let B be a block with a defect group which is a central extension of a group Q of order 16 by a cyclic group. If $Q \not\cong C_2^4$ or $9 \nmid e(B)$, then Brauer's $k(B)$-conjecture holds for B.*

Proof. For convenience of the reader, we list the 14 groups of order 16:

- the metacyclic groups: C_{16}, $C_8 \times C_2$, C_4^2, $C_4 \rtimes C_4$, D_{16}, Q_{16}, SD_{16} (semidihedral), M_{16} (modular),
- the minimal nonabelian group: $\langle x, y \mid x^4 = y^2 = [x, y]^2 = [x, x, y] = [y, x, y] = 1\rangle$,
- the nonmetacyclic abelian groups: $C_4 \times C_2^2$, C_2^4,
- $D_8 \times C_2$,
- $Q_8 \times C_2$,
- $D_8 * C_4$. □

Corollary 2. *Let B be a block with defect group D of order 32. If D is not extraspecial of type $D_8 * D_8$ or if $9 \nmid e(B)$, then Brauer's $k(B)$-conjecture holds for B.*

Proof. By Theorem 3 we may assume that $9 \mid e(B)$. In particular $9 \mid \operatorname{Aut}(D)$. Now one can show (for example with GAP) that there are just three possibilities for D, namely C_2^5, $Q_8 \times C_2^2$ and the extraspecial group $D_8 * D_8$. In the case $D \cong Q_8 \times C_2^2$ we can choose a major subsection (u, b) such that $D/\langle u\rangle \cong Q_8 \times C_2$.

Hence, by hypothesis we may assume that D is elementary abelian. By Corollary 1.2(ii) in [25] we may also assume that the inertial group $I(B)$ of B is nonabelian. In particular 9 is a proper divisor of $e(B)$. In general $e(B)$ is a divisor of $3^2 \cdot 5 \cdot 7 \cdot 31$ (this is the odd part of $|\operatorname{Aut}(D)| = |\mathrm{GL}(5, 2)|$).

Assume that $e(B)$ is also divisible by 31. Since the normalizer of a Sylow 31-subgroup of $\operatorname{Aut}(D) \cong \mathrm{GL}(5, 2)$ has order $5 \cdot 31$, $I(B)$ does not contain a normal Sylow 31-subgroup. Thus, by Sylow's theorem we also have $7 \mid e(B)$. However, all groups of order $3^2 \cdot 7 \cdot 31$ and $3^2 \cdot 5 \cdot 7 \cdot 31$ have a normal Sylow 31-subgroup. This shows $31 \nmid e(B)$.

Now suppose that $5 \cdot 7 \mid e(B)$. Since the normalizer of a Sylow 7-subgroup of $\mathrm{GL}(5, 2)$ has order $2 \cdot 3^2 \cdot 7$, $I(B)$ does not contain a normal Sylow 7-subgroup. However, all groups of order $3^2 \cdot 5 \cdot 7$ have a normal Sylow 7-subgroup. Hence, $5 \cdot 7 \nmid e(B)$.

Next we consider the case $e(B) = 3^2 \cdot 7$. Then the action of $I(B)$ on D induces an orbit of length 21. If we choose the major subsection (u, b) such that u lies in this orbit, then the inertial index of b is 3. Thus, the claim follows in this case.

Finally in the case $e(B) = 3^2 \cdot 5$, the inertial group $I(B)$ would be abelian. Hence, the proof is complete. □

5 2-Blocks with minimal nonmetacyclic defect groups

Since the block invariants of 2-blocks with metacyclic defect groups are known (see [28]), it seems natural to consider minimal nonmetacyclic defect groups. The groups C_2^3, $Q_8 \times C_2$ and $D_8 * C_4$ are minimal nonmetacyclic. Apart from these there is only one more minimal nonmetacyclic 2-group (see Theorem 66.1 in [2]). We consider this defect group. The next proposition shows that the corresponding blocks are nilpotent. We use the notion of fusion systems (see [18] for definitions and results).

Proposition 6. *Every fusion system on $P := \langle x, y, z \mid x^4 = y^4 = [x, y] = 1,\ z^2 = x^2,\ zxz^{-1} = xy^2,\ zyz^{-1} = x^2y\rangle$ is nilpotent.*

Proof. Let $\mathcal{F}$ be a fusion system on P, and let $Q < P$ be an $\mathcal{F}$-essential subgroup. Since Q is metacyclic and $\mathrm{Aut}(Q)$ is not a 2-group, we have $Q \cong Q_8$ or $Q \cong C_{2^r}^2$ for some $r \in \mathbb{N}$ (see Lemma 1 in [19]). By Proposition 10.17 and Proposition 1.8 in [1] it follows that $Q \cong C_4^2$. Now Theorem 66.1 in [2] implies $Q = \langle x, y\rangle$. As usual, $\mathrm{Aut}_{\mathcal{F}}(Q) \cong S_3$ acts nontrivially on $\Omega_1(Q)$. However, P acts trivially on $\Omega_1(Q) = \mathrm{Z}(P)$. This is not possible, since P/Q is a Sylow 2-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$. Thus, we have shown that P does not contain $\mathcal{F}$-essential subgroups. By Alperin's fusion theorem, P controls $\mathcal{F}$. Finally one can show (with GAP) that $\mathrm{Aut}(P)$ is a 2-group. □

The group in the last proposition has order 32. As a byproduct of the last section we deduce the following corollary.

Corollary 3. *Let B be a 2-block with minimal nonmetacyclic defect group D. Then one of the following holds:*

(i) *B is nilpotent. Then $k_i(B)$ is the number of ordinary characters of D of degree 2^i. In particular $k(B)$ is the number of conjugacy classes of D and $k_0(B) = |D : D'|$. Moreover, $l(B) = 1$.*

(ii) *$D \cong C_2^3$. Then $k(B) = k_0(B) = 8$ and $l(B) \in \{3, 5, 7\}$ (all cases occur).*

(iii) *$D \cong Q_8 \times C_2$ or $D \cong D_8 * C_4$. Then $k(B) = 14$, $k_0(B) = 8$, $k_1(B) = 6$ and $l(B) = 3$.*

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