

On centers of blocks with one simple module

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Abstract

Let G be a finite group, and let B be a non-nilpotent block of G with respect to an algebraically closed field of characteristic 2. Suppose that B has an elementary abelian defect group of order 16 and only one simple module. The main result of this paper describes the algebra structure of the center of B . This is motivated by a similar analysis of a certain 3-block of defect 2 in [Kessar, 2012].

Keywords: center of block algebra, one Brauer character, abelian defect

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1 Introduction

This paper is concerned with the algebra structure of the center of a p -block B of a finite group G . In order to make statements precise let (K, \mathcal{O}, F) be a p -modular system where \mathcal{O} is a complete discrete valuation ring of characteristic 0, K is the field of fractions of \mathcal{O} , and $F = \mathcal{O}/J(\mathcal{O}) = \mathcal{O}/(\pi)$ is an algebraically closed field of prime characteristic p . As usual, we assume that K is a splitting field for G .

A well-known result by Broué-Puig [8] asserts that if B is nilpotent, then the number of irreducible Brauer characters in B equals $l(B) = 1$. Since the algebra structure of nilpotent blocks is well understood by work of Puig [26], it is natural to study non-nilpotent blocks with only one irreducible Brauer character. These blocks are necessarily non-principal by Brauer's result (see [24, Corollary 6.13]) and maybe the first example was given by Kiyota [17]. Here, $p = 3$ and B has an elementary abelian defect group of order 9. More generally, a theorem by Puig-Watanabe [28] states that if the defect group of B is abelian, then B has a Brauer correspondent with more than one simple module. Ten years later, Benson-Green [2] and others [13, 16] have developed a general theory of these blocks by making use of quantum complete intersections. Applying this machinery, Kessar [15] was able to describe the algebra structure of Kiyota's example explicitly. Her arguments were simplified recently in [21]. We also mention two more recent papers dealing with these blocks. Malle-Navarro-Späth [23] have shown that the unique irreducible Brauer character in B is the restriction of an ordinary irreducible character. Finally, Benson-Kessar-Linckelmann [3] studied Hochschild cohomology in order to obtain results on blocks of defect 2 with only one irreducible Brauer character.

In the present paper we deal with the second smallest example in terms of defect groups. Here, $p = 2$ and B has elementary abelian defect group D of order 16. In [22] the numerical invariants of B have been determined. In particular, it is known that the number of irreducible ordinary characters (of height 0) of B is $k(B) = k_0(B) = 8$. Moreover, the inertial quotient $I(B)$ of B is elementary abelian of order 9. Examples for B are given by the non-principal blocks of $G = \text{SmallGroup}(432, 526) \cong D \rtimes 3_+^{1+2}$ where 3_+^{1+2} denotes the extraspecial group of order 27 and exponent 3. Here, the center of 3_+^{1+2} acts trivially on D and $G/Z(G) \cong A_4 \times A_4$ where A_4 is the alternating group of degree 4. Since the algebra structure of B seems too difficult to describe at the moment, we are content with studying the center $Z(B)$ as an algebra over F . As a consequence of Broué's Abelian Defect Group Conjecture, the isomorphism type of $Z(B)$ should be independent of G . In fact, our main theorem is the following.

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Theorem 1.1. *Let B be a non-nilpotent 2-block with elementary abelian defect group of order 16 and only one irreducible Brauer character. Then*

$$Z(B) \cong F[X, Y, Z_1, \dots, Z_4] / \langle X^2 + 1, Y^2 + 1, (X + 1)Z_i, (Y + 1)Z_i, Z_i Z_j \rangle.$$

In particular, $Z(B)$ has Loewy length 3.

The paper is organized as follows. In the second section we consider the generalized decomposition matrix Q of B . Up to certain choices there are essentially three different possibilities for Q . A result by Puig [27] (cf. [9, Theorem 5.1]) describes the isomorphism type of $Z(B)$ (regarded over \mathcal{O}) in terms of Q . In this way we prove that there are at most two isomorphism types for $Z(B)$. In the two subsequent sections we apply ring-theoretical arguments to the basic algebra of B in order to exclude one possibility for $Z(B)$. Finally, we give some concluding remarks in the last section. Our notation is standard and can be found in [24, 29].

2 The generalized decomposition matrix

From now on we will always assume that B is given as in Theorem 1.1 with defect group D .

Since a Sylow 3-subgroup of $\text{Aut}(D) \cong \text{GL}(4, 2) \cong A_8$ has order 9, the action of $I(B)$ on D is essentially unique. In particular, the $I(B)$ -conjugacy classes of D have lengths 1, 3, 3 and 9. Let $\mathcal{R} = \{1, x, y, xy\}$ be a set of representatives for these classes. For $u \in \mathcal{R}$ we fix a B -subsection (u, b_u) . Recall that b_u is a Brauer correspondent of B in $C_G(u)$ with defect group D . Moreover, the inertial quotient of b_u is given by $I(b_u) \cong C_{I(B)}(u)$. Since D has exponent 2, the generalized decomposition numbers $d_{\chi\varphi}^u$ for $\chi \in \text{Irr}(B)$ and $\varphi \in \text{IBr}(b_u)$ are (rational) integers. We set $Q_u := (d_{\chi\varphi}^u : \chi \in \text{Irr}(B), \varphi \in \text{IBr}(b_u))$ for $u \in \mathcal{R}$. Then $C_u := Q_u^T Q_u$ is the Cartan matrix of b_u where Q_u^T denotes the transpose of Q_u . On the other hand, the orthogonality relation implies $Q_u^T Q_v = 0 \in \mathbb{Z}^{l(b_u) \times l(b_v)}$ for $u \neq v \in \mathcal{R}$. A *basic set* for b_u is a basis for the \mathbb{Z} -module of class functions on the 2-regular elements of $C_G(u)$ spanned by $\text{IBr}(b_u)$. If we change the underlying basic set, the matrix Q_u transforms into $Q_u S$ where $S \in \text{GL}(l(b_u), \mathbb{Z})$. Similarly, C_u becomes $S^T C_u S$. By [27, Remark 1.8] the isomorphism type of $Z(B)$ does not depend on the chosen basic sets. Following Brauer [4], we define the *contribution matrix* of b_u by

$$M^u := (m_{\chi\psi}^u)_{\chi, \psi \in \text{Irr}(B)} := Q_u C_u^{-1} Q_u^T \in \mathbb{Q}^{8 \times 8}.$$

Observe that M^u does not depend on the choice of the basic set, but on the order of $\text{Irr}(B)$. Since the largest elementary divisor of C_u equals 16, it follows that $16M^u \in \mathbb{Z}^{8 \times 8}$. Moreover, all entries of $16M^u$ are odd, because all irreducible characters of B have height 0 (see [29, Proposition 1.36]).

We may assume that $l(b_x) = l(b_y) = 3$ and $l(b_{xy}) = 1$. Then the Cartan matrices of b_x and b_y are given by

$$C_x = C_y = 4 \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

up to basic sets (see e.g. [30, Proposition 16]). It is well-known that the entries of Q_1 are positive. Since $C_1 = C_{xy} = (16)$, we may choose the order of $\text{Irr}(B)$ such that

$$Q_1 = (3, 1, 1, 1, 1, 1, 1, 1)^T.$$

Now we do some computations with the $*$ -construction introduced in [7]. Observe that the following generalized characters of D are $I(B)$ -stable:

	1	x	y	xy
λ_1	4	4	.	.
λ_2	4	.	4	.
λ_3	.	4	.	4

Since

$$\sum_{u \in \mathcal{R}} \lambda_i(u) m_{\chi\psi}^u = (\chi * \lambda_i, \psi)_G \in \mathbb{Z} \quad (\chi, \psi \in \text{Irr}(B))$$

for $i = 1, 2, 3$, we obtain the following relations between the contribution matrices:

$$16M^1 + 16M^x \equiv 16M^1 + 16M^y \equiv 16M^x + 16M^{xy} \equiv 0_8 \pmod{4}. \quad (2.1)$$

For the trivial character λ we obtain $\sum_{u \in \mathcal{R}} M^u = 1_8$. Therefore, $d_{11}^{xy} = \pm 1$. After changing the basic set for b_{xy} (i. e. multiplying $\varphi \in \text{IBr}(b_{xy})$ by a sign), we may assume that $d_{11}^{xy} = 1$. Now (2.1) implies

$$Q_{xy} = (1, 3, -1, -1, -1, -1, -1, -1)^T$$

for a suitable order of $\text{Irr}(B)$. Observe that the orthogonality relation is satisfied.

The matrices Q_x and Q_y are (integral) solutions of the matrix equation

$$X^T X = C_x. \quad (2.2)$$

We solve (2.2) by using an algorithm of Plesken [25]. In the first step we compute all possible rows $r = (r_1, r_2, r_3) \in \mathbb{Z}^3$ of X . These rows satisfy $r C_x^{-1} r^T \leq 1$ where $C_x^{-1} = \frac{1}{16}(-1 + 4\delta_{ij})$ and δ_{ij} is the Kronecker delta. Since in our case the numbers $r C_x^{-1} r^T$ are contributions, we get the additional constraint $16r C_x^{-1} r^T \equiv 1 \pmod{2}$. It follows that

$$r_1^2 + r_2^2 + r_3^2 + (r_1 - r_2)^2 + (r_1 - r_3)^2 + (r_2 - r_3)^2 \leq 15. \quad (2.3)$$

Thus, up to permutations of r_i and signs we have the following solutions for r :

$$(1, 0, 0), (1, 1, 1), (0, 1, 2), (1, 1, -1), (1, 2, 2).$$

Observe that the first two solutions give a contribution of $3/16$ while the other three solutions give $11/16$. By [25, Proposition 2.2], the matrix X contains five rows contributing $3/16$ and three rows contributing $11/16$ in the sense above. If we change the basic set of b_x according to the transformation matrix

$$S := \begin{pmatrix} 1 & . & . \\ . & 1 & . \\ -1 & -1 & -1 \end{pmatrix},$$

then C_x does not change (in fact, C_x is the Gram matrix of the A_3 lattice and its automorphism group is $S_4 \times C_2$). Doing so, we may assume that the first row of X is $(2, 2, 1)$. Now we need to discuss the possibilities for the other rows where we will ignore their signs. We may assume that the second and third row also contribute $11/16$. It is easy to see that the rows $(1, 2, 2)$, $(2, 1, 2)$, $(2, 2, 1)$, $(1, 2, 0)$, $(2, 1, 0)$ and $(1, 1, -1)$ are excluded. Now suppose that the second row is $(2, 0, 1)$. Then we may certainly assume that the third row is $(0, 1, 2)$ or $(0, 2, 1)$. In both cases the remaining rows are essentially determined (up to signs and order) as

$$(I) : \begin{pmatrix} 2 & 2 & 1 \\ 2 & . & 1 \\ . & 2 & 1 \\ . & . & 1 \\ . & . & 1 \\ . & . & 1 \\ . & . & 1 \end{pmatrix}, \quad (II) : \begin{pmatrix} 2 & 2 & 1 \\ 2 & . & 1 \\ . & 1 & 2 \\ . & 1 & . \\ . & 1 & . \\ . & . & 1 \\ . & . & 1 \end{pmatrix}.$$

Suppose next that the second row is $(0, 1, 2)$. If the third row is $(2, 0, 1)$, then we end up in case (II) (interchange the second and third row). Hence, the third row must be $(1, -1, 1)$. Again the remaining rows are essentially determined. In order to avoid negative entries, we give a slightly different representative

$$(III) : \begin{pmatrix} 2 & 1 & . \\ . & 2 & 1 \\ 1 & . & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & . & . \\ . & 1 & . \\ . & . & 1 \end{pmatrix}.$$

Finally, suppose that the second row is $(1, -1, 1)$. Observe that the third row cannot be $(1, 0, 2)$. If it is $(0, 1, 2)$, then we are in case (III). Therefore, we may assume that the third row is $(-1, 1, 1)$. Here a transformation similar to the matrix S above gives case (II). Summarizing we have seen that by ignoring the order and signs of the rows, there exists a matrix $S \in \text{GL}(3, \mathbb{Z})$ such that XS is exactly one of the possibilities (I), (II) or (III). The fact that these solutions are essentially different can be seen by computing the elementary divisors which are $(1, 2, 2)$, $(1, 1, 2)$ and $(1, 1, 1)$ respectively. In the following we will refer to (I), (II) or (III) whenever Q_x belongs to (I), (II) or (III) respectively. Then the corresponding contribution matrices (multiplied by 16) are given as follows

$$\begin{pmatrix} 11 & 5 & 5 & -1 & -1 & -1 & -1 & -1 \\ 5 & 11 & -5 & 1 & 1 & 1 & 1 & 1 \\ 5 & -5 & 11 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 3 & 3 & 3 & 3 & 3 \\ -1 & 1 & 1 & 3 & 3 & 3 & 3 & 3 \\ -1 & 1 & 1 & 3 & 3 & 3 & 3 & 3 \\ -1 & 1 & 1 & 3 & 3 & 3 & 3 & 3 \\ -1 & 1 & 1 & 3 & 3 & 3 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 11 & 5 & 1 & 3 & 3 & 3 & -1 & -1 \\ 5 & 11 & -1 & -3 & -3 & -3 & 1 & 1 \\ 1 & -1 & 11 & 1 & 1 & 1 & 5 & 5 \\ 3 & -3 & 1 & 3 & 3 & 3 & -1 & -1 \\ 3 & -3 & 1 & 3 & 3 & 3 & -1 & -1 \\ 3 & -3 & 1 & 3 & 3 & 3 & -1 & -1 \\ -1 & 1 & 5 & -1 & -1 & -1 & 3 & 3 \\ -1 & 1 & 5 & -1 & -1 & -1 & 3 & 3 \end{pmatrix},$$

$$\begin{pmatrix} 11 & -1 & -1 & 3 & 3 & 5 & 1 & -3 \\ -1 & 11 & -1 & 3 & 3 & -3 & 5 & 1 \\ -1 & -1 & 11 & 3 & 3 & 1 & -3 & 5 \\ 3 & 3 & 3 & 3 & 3 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 & 1 & 1 & 1 \\ 5 & -3 & 1 & 1 & 1 & 3 & -1 & -1 \\ 1 & 5 & -3 & 1 & 1 & -1 & 3 & -1 \\ -3 & 1 & 5 & 1 & 1 & -1 & -1 & 3 \end{pmatrix}.$$

Note that the order of the rows does not correspond to the order of $\text{Irr}(B)$ chosen above.

Suppose that case (I) occurs. Then, using (2.1), we may choose a basic set for b_x and the order of the last six characters of $\text{Irr}(B)$ such that

$$Q_x = \begin{pmatrix} \cdot & \cdot & 1 \\ \cdot & \cdot & -1 \\ -2 & -2 & -1 \\ 2 & \cdot & 1 \\ \cdot & 2 & 1 \\ \cdot & \cdot & -1 \\ \cdot & \cdot & -1 \\ \cdot & \cdot & -1 \end{pmatrix}.$$

Since $M^1 + M^x + M^y + M^{xy} = 1_8$, we obtain

$$16M^y = \begin{pmatrix} 3 & -3 & -3 & -3 & -3 & 1 & 1 & 1 \\ -3 & 3 & 3 & 3 & 3 & -1 & -1 & -1 \\ -3 & 3 & 3 & 3 & 3 & -1 & -1 & -1 \\ -3 & 3 & 3 & 3 & 3 & -1 & -1 & -1 \\ -3 & 3 & 3 & 3 & 3 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 & -1 & 11 & -5 & -5 \\ 1 & -1 & -1 & -1 & -1 & -5 & 11 & -5 \\ 1 & -1 & -1 & -1 & -1 & -5 & -5 & 11 \end{pmatrix}.$$

Thus, also Q_y corresponds to the first solution above. After choosing an order of the last three characters in

$\text{Irr}(B)$, we get

$$Q_y = \begin{pmatrix} \cdot & \cdot & -1 \\ \cdot & \cdot & 1 \\ \cdot & \cdot & 1 \\ \cdot & \cdot & 1 \\ \cdot & \cdot & 1 \\ 2 & 2 & 1 \\ -2 & \cdot & -1 \\ \cdot & -2 & -1 \end{pmatrix}.$$

Hence, the generalized decomposition matrix of B in case (I) is given by:

$$(I) : \begin{pmatrix} 3 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & -1 \\ 1 & 3 & \cdot & \cdot & -1 & \cdot & \cdot & 1 \\ 1 & -1 & -2 & -2 & -1 & \cdot & \cdot & 1 \\ 1 & -1 & 2 & \cdot & 1 & \cdot & \cdot & 1 \\ 1 & -1 & \cdot & 2 & 1 & \cdot & \cdot & 1 \\ 1 & -1 & \cdot & \cdot & -1 & 2 & 2 & 1 \\ 1 & -1 & \cdot & \cdot & -1 & -2 & \cdot & -1 \\ 1 & -1 & \cdot & \cdot & -1 & \cdot & -2 & -1 \end{pmatrix}.$$

Now we consider case (II). Here, at first sight it is not clear if the first row of Q_x is $(0, 0, 1)$ or $(0, 1, 0)$. Suppose that it is $(0, 0, 1)$. Then we may assume that $16m_{13}^x = 5$. This gives $16(m_{13}^1 + m_{13}^x + m_{13}^{xy}) = 7$. However, $16m_{13}^y$ can never be -7 . Therefore, we may assume that the first row of Q_x is $(0, 1, 0)$. Now it is straight forward to obtain the generalized decomposition matrix of B as

$$(II) : \begin{pmatrix} 3 & 1 & \cdot & -1 & \cdot & \cdot & -1 & \cdot \\ 1 & 3 & \cdot & 1 & \cdot & \cdot & 1 & \cdot \\ 1 & -1 & 2 & 2 & 1 & \cdot & \cdot & 1 \\ 1 & -1 & -2 & \cdot & -1 & \cdot & \cdot & 1 \\ 1 & -1 & \cdot & -1 & -2 & \cdot & 1 & \cdot \\ 1 & -1 & \cdot & 1 & \cdot & \cdot & -1 & -2 \\ 1 & -1 & \cdot & \cdot & 1 & -2 & \cdot & -1 \\ 1 & -1 & \cdot & \cdot & 1 & 2 & 2 & 1 \end{pmatrix}.$$

Similarly, in case (III) we compute

$$(III) : \begin{pmatrix} 3 & 1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 2 & 1 & \cdot & \cdot & 1 & \cdot \\ 1 & -1 & \cdot & 2 & 1 & \cdot & \cdot & 1 \\ 1 & -1 & 1 & \cdot & 2 & 1 & \cdot & \cdot \\ 1 & -1 & -1 & \cdot & \cdot & -1 & \cdot & -2 \\ 1 & -1 & \cdot & -1 & \cdot & -2 & -1 & \cdot \\ 1 & -1 & \cdot & \cdot & -1 & \cdot & -2 & -1 \end{pmatrix}.$$

Now let $Q = (q_{ij})$ be the transpose of one of these three generalized decomposition matrices. Let e be the block idempotent of B in $\mathcal{O}G$. Then [27] gives an isomorphism

$$Z(\mathcal{O}Ge) \cong D_8(K) \cap Q^{-1}\mathcal{O}^{8 \times 8}Q = D_8(\mathcal{O}) \cap Q^{-1}\mathcal{O}^{8 \times 8}Q =: Z$$

where $D_8(K)$ (respectively $D_8(\mathcal{O})$) is the ring of 8×8 diagonal matrices over K (respectively \mathcal{O}). For a matrix $A = (a_{ij}) \in \mathcal{O}^{8 \times 8}$ the condition $Q^{-1}AQ \in D_8(K)$ transforms into a homogeneous linear system in a_{ij} with $8^2 - 8 = 56$ equations of the form

$$\sum_{i,j=1}^8 q'_{ri}q_{js}a_{ij} = 0 \quad (r \neq s).$$

After multiplying with a common denominator, we may assume that the coefficients of this system are (rational) integers. (Even if Q were not rational, one could get an integral coefficient matrix by using the Galois action of a suitable cyclotomic field.) Using the Smith normal form, it is easy to construct an \mathcal{O} -basis β_1, \dots, β_8 of Z consisting of integral matrices (this can be done conveniently in GAP [11]). For instance, in case (I) such a basis is given by

$$(I) : \begin{pmatrix} 1 & -1 & -1 & . & . & -3 & . & -4 \\ 1 & 7 & 3 & . & . & 9 & . & 12 \\ 1 & 3 & -1 & -8 & . & 9 & . & 12 \\ 1 & 3 & 7 & . & 8 & 9 & . & 12 \\ 1 & 3 & 7 & 8 & 8 & 9 & . & 12 \\ 1 & 3 & 3 & . & . & 13 & 8 & 12 \\ 1 & -5 & -5 & . & -8 & -11 & . & -12 \\ 1 & -5 & -5 & . & -8 & -11 & -8 & -12 \end{pmatrix}$$

where each column is the diagonal of a basis vector. The canonical ring epimorphism $Z(\mathcal{O}G) \rightarrow Z(FG)$ sending class sums to class sums restricts to an epimorphism $Z(\mathcal{O}Ge) = Z(\mathcal{O}G)e \rightarrow Z(B)$ with kernel $Z(\mathcal{O}G)\pi \cap Z(\mathcal{O}Ge) = Z(\mathcal{O}Ge)\pi$. This gives an isomorphism of F -algebras

$$Z(B) \cong Z(\mathcal{O}Ge)/Z(\mathcal{O}Ge)\pi \cong Z/\pi Z.$$

Obviously, the elements $\beta_i + \pi Z$ form an F -basis of $Z/\pi Z$. Thus, in order to obtain a presentation for $Z(B)$ it suffices to reduce the structure constants coming from β_i modulo 2. An even nicer presentation can be achieved by replacing the generators with some \mathbb{F}_2 -linear combinations. Eventually, this proves the following result.

Proposition 2.1. *We have*

$$Z(B) \cong \begin{cases} F[X, Y, Z_1, \dots, Z_4]/\langle X^2 + 1, Y^2 + 1, (X + 1)Z_i, (Y + 1)Z_i, Z_i Z_j \rangle & \text{case (I) or (II),} \\ F[X, Z_1, \dots, Z_6]/\langle X^2 + 1, XZ_{2i} + Z_{2i-1}, Z_i Z_j \rangle & \text{case (III).} \end{cases}$$

These two algebras are non-isomorphic, since $\dim_F J(Z(B))^2$ differs.

In the following two sections we will see that the second alternative in Proposition 2.1 does not occur.

3 Tools from ring theory

In this section we will gather some well known facts about local symmetric F -algebras and applications thereof to our block B . We start with some basic lemmas:

Lemma 3.1 ([15, Lemma 2.1]). *Let A be a local symmetric F -algebra. Then the following hold:*

- (i) $\dim_F \text{soc}(A) = 1$.
- (ii) $\text{soc}(A) \subseteq \text{soc}(Z(A))$.
- (iii) $\text{soc}(A) \cap [A, A] = 0$.
- (iv) $\dim_F A = \dim_F Z(A) + \dim_F [A, A]$.
- (v) $Z(A)$ is local and $J(A) \cap Z(A) = J(Z(A))$.
- (vi) *If n is the least non-negative integer such that $J^{n+1}(A) = 0$, then $J^n(A) = \text{soc}(A)$.*

Lemma 3.2 ([19, slight modification of Lemma E]). *Let A be an F -algebra, let I be a two-sided ideal in A and let $n \in \mathbb{N}$. Suppose*

$$I^n = F\{x_{i_1} \dots x_{i_n} \mid i = 1, \dots, d\} + I^{n+1}$$

with elements $x_{ij} \in I$. Then we have

$$I^{n+1} = F\{x_{j_1} x_{i_1} \dots x_{i_n} \mid i, j = 1, \dots, d\} + I^{n+2},$$

and also

$$I^{n+1} = F\{x_{i_1} \dots x_{i_n} x_{j_n} \mid i, j = 1, \dots, d\} + I^{n+2}.$$

The proof of the last statement of this lemma goes exactly as in [19]. We just have to do everything from the opposing side.

Lemma 3.3 ([19, Lemma G]). *Let A be a local symmetric F -algebra and let $n \in \mathbb{N}$ with $\dim_F(\mathbb{J}^n(A)/\mathbb{J}^{n+1}(A)) = 1$. Then $\mathbb{J}^{n-1}(A) \subseteq \mathbb{Z}(A)$.*

Finally we have the following.

Lemma 3.4. *Let A be a local symmetric F -algebra. Then $[A, A] \subseteq \mathbb{J}^2(A)$.*

Proof. This is an easy consequence since $[A, A] = [F1 + \mathbb{J}(A), F1 + \mathbb{J}(A)] = [\mathbb{J}(A), \mathbb{J}(A)] \subseteq \mathbb{J}^2(A)$. \square

We recall the definition of the *Külshammer spaces* from [18]. Let A be a finite dimensional F -algebra and $n \in \mathbb{N}_0$. Then we define

$$T_n(A) := \{a \in A \mid a^{2^n} \in [A, A]\}$$

and

$$T(A) := \{a \in A \mid a^{2^n} \in [A, A] \text{ for some } n \in \mathbb{N}\}.$$

It is well known (see [12, Section 2]) that $T(A) = \mathbb{J}(A) + [A, A]$, and that there is a chain of inclusions $[A, A] = T_0(A) \subseteq T_1(A) \subseteq T_2(A) \subseteq \dots \subseteq T(A)$. From this and [18, Satz J] we can deduce the following.

Lemma 3.5. *We have $T(B) = T_1(B)$. In particular, $a^2 \in [B, B]$ for every $a \in \mathbb{J}(B)$.*

There is a remarkable property of group algebras and their blocks considering the rate of growth of a minimal projective resolution of any of their finite dimensional modules. Let A be a finite dimensional F -algebra and M a finite dimensional A -module. Furthermore let

$$\dots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be a minimal projective resolution of M . If there is a smallest integer $c \in \mathbb{N}_0$ such that for some positive number λ we have $\dim_F P_n \leq \lambda n^{c-1}$ for every sufficiently large n , then we say that M has *complexity* c . If there is no such number, then we say that M has *infinite complexity*. Using [1, Corollary 4] we get the following.

Lemma 3.6. *The maximal complexity of any indecomposable finite dimensional B -module equals 4.*

We will conclude this section with a proposition which gives us a sufficient condition for a finite dimensional F -algebra A to have a module with infinite complexity. Although it might seem quite special at first, this condition will be crucial in the next section.

Proposition 3.7. *Let A be a local F -algebra and let $x, z \in \mathbb{J}(A)$ be such that $\{x + \mathbb{J}^2(A), z + \mathbb{J}^2(A)\}$ is F -linearly independent in $\mathbb{J}(A)/\mathbb{J}^2(A)$ and such that $xz = zx = z^2 = 0$ holds. Furthermore, we denote by $(f_i)_{i=-1}^{\infty}$ the shifted Fibonacci sequence given by $f_{-1} = 1 = f_0$, and $f_i = f_{i-1} + f_{i-2}$ for $i \in \mathbb{N}$. Then there are a minimal projective resolution*

$$\dots \longrightarrow P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} F \longrightarrow 0$$

of the trivial A -module $F \cong A/\mathbb{J}(A)$ and, for $i \in \mathbb{N}_0$, an A -basis $\{b_{i,1}, \dots, b_{i,n_i}\}$ of P_i with the following properties:

- $n_0 = 1 = f_0$ and $zb_{0,1}, xb_{0,1} \in K_0 := \text{Ker}(\varphi_0)$.
- For $i \in \mathbb{N}$ we have $n_i \geq f_i$ and $zb_{i,1}, \dots, zb_{i,f_i}, xb_{i,1}, \dots, xb_{i,f_{i-1}} \in K_i := \text{Ker}(\varphi_i)$.

In particular, the A -module F has infinite complexity.

Proof. The first claim is clear, since $P_0 = A$ and $\text{Ker}(\varphi_0) = J(A)$, so that we can choose $b_{0,1} = 1$. Let us now assume that for some $i \in \mathbb{N}_0$ we have already constructed P_0, \dots, P_i and $\varphi_0, \dots, \varphi_i$ with the properties from above. We will show that the claim also holds true for $i + 1$. First we notice that from $\varphi_i : P_i \rightarrow K_{i-1}$ being a projective cover we get $K_i = \text{Ker}(\varphi_i) \subseteq J(A)P_i$ and, therefore, $J(A)K_i \subseteq J^2(A)P_i$. Since $\{b_{i,1}, \dots, b_{i,f_i}\}$ is A -linearly independent in P_i , we see that

$$\{zb_{i,1} + J^2(A)P_i, \dots, zb_{i,f_i} + J^2(A)P_i, xb_{i,1} + J^2(A)P_i, \dots, xb_{i,f_{i-1}} + J^2(A)P_i\}$$

is an F -linearly independent set in $J(A)P_i/J^2(A)P_i$. Hence, the set $\{zb_{i,1} + J(A)K_i, \dots, zb_{i,f_i} + J(A)K_i, xb_{i,1} + J(A)K_i, \dots, xb_{i,f_{i-1}} + J(A)K_i\}$ is F -linearly independent in $K_i/J(A)K_i$.

Therefore, there is a projective cover $\varphi_{i+1} : P_{i+1} \rightarrow K_i$ together with an A -basis $\{b_{i+1,1}, \dots, b_{i+1,n_{i+1}}\}$ of P_{i+1} with the properties $n_{i+1} \geq f_i + f_{i-1} = f_{i+1}$ and $\varphi_{i+1}(b_{i+1,j}) = zb_{i,j}$ for $j = 1, \dots, f_i$, and $\varphi_{i+1}(b_{i+1,f_i+j}) = xb_{i,j}$ for $j = 1, \dots, f_{i-1}$. Since $xz = z^2 = 0$, we have $\varphi_{i+1}(zb_{i+1,j}) = z\varphi_{i+1}(b_{i+1,j}) = 0$ for $j \in \{1, \dots, f_{i+1}\}$ and since $xz = 0$, we have $\varphi_{i+1}(xb_{i+1,j}) = x\varphi_{i+1}(b_{i+1,j}) = 0$ for $j \in \{1, \dots, f_i\}$. We thus have constructed a projective cover $\varphi_{i+1} : P_{i+1} \rightarrow K_i$ with the claimed properties.

From the exponential growth of the Fibonacci sequence and the shown properties of a minimal projective resolution of the A -module F and the fact that A was assumed to be a local algebra, we deduce that $\dim_F P_i \geq f_i \dim_F A$, so that F has, in fact, infinite complexity. \square

We mention that another version of the proposition which is due to J.F. Carlson can be found in the upcoming paper [21, Proposition 7]. In that version it is proved that the trivial A -module F has infinite complexity provided $x, y, z \in J(A)$ with $\{x + J^2(A), y + J^2(A), z + J^2(A)\}$ is F -linearly independent in $J(A)/J^2(A)$ and $xz = zx = yz = zy = 0$. We will need this statement in our paper too.

4 Determining the isomorphism type of the center

In order to understand the structure of our fixed block B , we will now consider its basic algebra A over F . Since A and B are Morita equivalent, we can deduce a number of properties which are shared by these algebras. For example, the following lemma shows that A is a local symmetric algebra. So we are in a position to use the results from the previous section.

Lemma 4.1.

- (i) $\dim_F A = 16$, $\dim_F Z(A) = 8$ and $\dim_F [A, A] = 8$.
- (ii) A is a local symmetric F -algebra.
- (iii) $Z(A) \cong Z(B)$.
- (iv) For every $a \in J(A)$ we have $a^2 \in [A, A]$.
- (v) Every indecomposable A -module M has finite complexity.

Proof. Part (iii) is well-known. From the introduction we already know that $\dim_F Z(A) = \dim_F Z(B) = k(B) = 8$. Moreover, the dimension of A equals the order of a defect group of B (see [19, Section 1]). This proves the first part of (i). Since B has exactly one irreducible Brauer character, we infer that B , and therefore A , has just one isomorphism class of simple modules. Together with the property of A of being a basic F -algebra this yields $A/J(A) \cong F$, so that A is a local F -algebra. It is a well known fact that blocks of finite groups are symmetric algebras and that symmetry is a Morita invariant. Thus, also A is a symmetric F -algebra which shows (ii). The third part of (i), and (iv) follow at once by combining the results in [12, Corollary 5.3], Lemma 3.1(iv) and Lemma 3.5. Finally, since Morita equivalences preserve projectivity and also projective covers, (v) follows easily from Lemma 3.6. \square

From now on we will assume that

$$Z(B) \cong Z(A) \cong F[X, Z_1, \dots, Z_6] / \langle X^2 + 1, XZ_{2i} + Z_{2i-1}, Z_i Z_j \rangle$$

(see Proposition 2.1). We are seeking a contradiction. To avoid initial confusion about signs it is to be noted explicitly that we calculate over a field of characteristic 2. We introduce a new F -basis for $Z(A)$ by setting:

$$\begin{aligned} W_0 &:= 1, & W_1 &:= X + 1, & W_2 &:= Z_1, & W_3 &:= Z_3, \\ W_4 &:= Z_5, & W_5 &:= Z_1 + Z_2, & W_6 &:= Z_3 + Z_4, & W_7 &:= Z_5 + Z_6. \end{aligned}$$

The structure constants with respect to W_i are given as follows.

\cdot	1	W_1	W_2	W_3	W_4	W_5	W_6	W_7
1	1	W_1	W_2	W_3	W_4	W_5	W_6	W_7
W_1	W_1	\cdot	W_5	W_6	W_7	\cdot	\cdot	\cdot
W_2	W_2	W_5	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot
W_3	W_3	W_6	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot
W_4	W_4	W_7	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot
W_5	W_5	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot
W_6	W_6	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot
W_7	W_7	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot

By abuse of notation we will identify $Z(A)$ with $F\{W_0, \dots, W_7\}$. For every $z \in J(Z(A)) = F\{W_1, \dots, W_7\}$ we have $z^2 = 0$ since $\text{char}(F) = 2$. From Lemma 4.1(ii) we know that A is a symmetric F -algebra. Let

$$s : A \rightarrow F$$

be a symmetrizing form for A . Hence, s is F -linear, for every $a, b \in A$ we have $s(ab) = s(ba)$. Moreover, the kernel $\text{Ker}(s)$ includes no non-zero (one-sided) ideal of A . For a subspace $U \subseteq A$ we define the set

$$U^\perp := \{a \in A \mid s(aU) = 0\}.$$

It is well known that we always have $\dim_F A = \dim_F U + \dim_F(U^\perp)$ and $U^{\perp\perp} = U$. In particular, the identities $Z(A)^\perp = [A, A]$ and $\text{soc}(A)^\perp = J(A)$ are known to hold. Defining $\text{soc}^2(A) := \{a \in A \mid aJ^2(A) = 0\}$ we easily see

$$\begin{aligned} \text{soc}^2(A) &= \{a \in A \mid aJ^2(A) = 0\} = \{a \in A \mid s(aJ^2(A)) = 0\} = \{a \in A \mid s(J^2(A)a) = 0\} \\ &= \{a \in A \mid J^2(A)a = 0\} = (J^2(A))^\perp. \end{aligned}$$

In particular, $\text{soc}^2(A)$ is a two-sided ideal in A . We will now collect some basic facts about the F -algebra A .

Lemma 4.2.

- (i) $J(Z(A)) = F\{W_1, \dots, W_7\}$ and $\text{soc}(Z(A)) = J^2(Z(A)) = F\{W_5, W_6, W_7\}$. In particular, $\dim_F J(Z(A)) = 7$ and $\dim_F \text{soc}(Z(A)) = 3$.
- (ii) $\text{soc}(Z(A))^\perp = [A, A] + J(Z(A)) \cdot A = J(Z(A)) + J^2(A)$ and this is an ideal in A . In particular, $\text{soc}(Z(A))$ is an ideal in A .
- (iii) $J(A) \cdot \text{soc}(Z(A)) = \text{soc}(A)$.
- (iv) $\dim_F((J(Z(A)) + J^2(A))/J^2(A)) \leq 2$.
- (v) For any $a \in \text{soc}^2(A)$ and $b \in J(A)$ we have $ab, ba \in \text{soc}(A)$ and $ab = ba$.

Proof.

- (i) This can be read off immediately from the multiplication table of $Z(A)$.

(ii) For an element $z \in Z(A)$ we have

$$z \in \text{soc}(Z(A)) \Leftrightarrow zJ(Z(A)) = 0 \Leftrightarrow s((zJ(Z(A)))) \cdot A = 0 \Leftrightarrow z \in (J(Z(A)) \cdot A)^\perp.$$

Hence, $\text{soc}(Z(A)) = Z(A) \cap (J(Z(A)) \cdot A)^\perp$ and therefore, by going over to the orthogonal spaces,

$$\text{soc}(Z(A))^\perp = [A, A] + J(Z(A)) \cdot A.$$

This shows the first equality in (ii). From this and (i) we also get $\dim_F([A, A] + J(Z(A)) \cdot A) = 13$. Now since A is a local symmetric F -algebra we have $[A, A] \subseteq J^2(A)$ by Lemma 3.4 and from $A = F1 \oplus J(A)$ and Lemma 3.1(v) we get $J(Z(A)) \cdot A \subseteq J(Z(A)) + J^2(A)$. Hence, we obtain

$$[A, A] + J(Z(A)) \cdot A \subseteq J(Z(A)) + J^2(A).$$

If we had $[A, A] + J(Z(A)) \cdot A \neq J(Z(A)) + J^2(A)$, it would follow that $\dim_F(J(Z(A)) + J^2(A)) \geq 14$, so that $\dim_F(J(A)/(J(Z(A)) + J^2(A))) \leq 1$. But then we could find subsets $\mathcal{B}_1 \subseteq J(A)$ and $\mathcal{B}_2 \subseteq J(Z(A))$ with $|\mathcal{B}_1| \leq 1$ such that $\{1\} \cup \mathcal{B}_1 \cup \mathcal{B}_2$ generated A as an algebra. Since $|\mathcal{B}_1| \leq 1$, however, all the generators would commute with each other and so A would be a commutative algebra, a contradiction. Hence, $[A, A] + J(Z(A)) \cdot A = J(Z(A)) + J^2(A)$ and we have shown the second equality. Finally we note that, since A is a local algebra, every subspace of $J(A)$ containing $J^2(A)$ automatically is an ideal in A . Using this fact on $J(Z(A)) + J^2(A)$ we see that $\text{soc}(Z(A))^\perp$, and therefore also $\text{soc}(Z(A))$, is an ideal in A .

- (iii) From (ii) we have $\text{soc}(Z(A))^\perp = J(Z(A)) + J^2(A)$, so that $s(J^2(A) \cdot \text{soc}(Z(A))) = 0$. Since $J^2(A) \cdot \text{soc}(Z(A))$ is an ideal in A and s is non-degenerate, we get $J^2(A) \cdot \text{soc}(Z(A)) = 0$. But this implies $J(A) \cdot \text{soc}(Z(A)) \subseteq J(A)^\perp = \text{soc}(A)$. If we even had $J(A) \cdot \text{soc}(Z(A)) = 0$, then $\text{soc}(Z(A)) \subseteq J(A)^\perp = \text{soc}(A)$, a contradiction. Hence, the claim follows.
- (iv) Let us assume to the contrary that $\dim_F((J(Z(A)) + J^2(A))/J^2(A)) \geq 3$. Then we can find elements $z_1, z_2, z_3 \in J(Z(A))$ such that the set $\{z_1 + J^2(A), z_2 + J^2(A), z_3 + J^2(A)\}$ is F -linearly independent in $J(A)/J^2(A)$. We write $z_i = \alpha_i W_1 + b_i$ with $\alpha_i \in F$ and $b_i \in F\{W_2, \dots, W_7\}$ for $i = 1, 2, 3$. We can assume that $\alpha_1 = \alpha_2 = 0$. For if $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$ we may say for instance that $\alpha_1 \neq 0$ (after possibly swapping z_1 and z_2). By defining $z'_1 := z_2 - \frac{\alpha_2}{\alpha_1} z_1$, $z'_2 := z_3 - \frac{\alpha_3}{\alpha_1} z_1$ and $z'_3 := z_1$ we obtain elements $z'_1, z'_2, z'_3 \in J(Z(A))$ such that $\{z'_1 + J^2(A), z'_2 + J^2(A), z'_3 + J^2(A)\}$ is again F -linearly independent in $J(A)/J^2(A)$ and such that $z'_1, z'_2 \in F\{W_2, \dots, W_7\}$. After renaming z'_i into z_i for $i = 1, 2, 3$ we get $\alpha_1 = \alpha_2 = 0$ as claimed. But from this we get $z_1 z_2 = z_2 z_1 = z_2^2 = 0$ (see the multiplication table for $Z(A)$) which implies that the simple A -module F has infinite complexity by Proposition 3.7. This, however, contradicts Lemma 4.1(v). Hence, (iv) holds true.
- (v) For the proof of the first part of the statement we note that $\text{soc}^2(A) \cdot J^2(A) = J^2(A) \cdot \text{soc}^2(A) = 0$. Hence $\text{soc}^2(A) \cdot J(A) \subseteq (J(A))^\perp = \text{soc}(A)$ and $J(A) \cdot \text{soc}^2(A) \subseteq (J(A))^\perp = \text{soc}(A)$, and this is exactly the first part of the claim. From this the second part follows at once since for $a \in \text{soc}^2(A)$ and $b \in J(A)$ we have $ab - ba \in [A, A] \cap \text{soc}(A) = 0$, by Lemma 3.1(iii). \square

Corollary 4.3. *One of the following three cases occurs:*

- (I) *There are $x, y \in J(A)$ with $xy \neq yx$ and $A = F1 \oplus Fx \oplus Fy \oplus J^2(A)$. In particular, $\dim_F J^2(A) = 13$.*
- (II) *There are $x, y \in J(A)$ and $z \in J(Z(A))$ with $xy \neq yx$ and $A = F1 \oplus Fx \oplus Fy \oplus Fz \oplus J^2(A)$. In particular, $\dim_F J^2(A) = 12$.*
- (III) *There are $x, y \in J(A)$ and $z_1, z_2 \in J(Z(A))$ with $xy \neq yx$, $z_1 z_2 \neq 0$, and $A = F1 \oplus Fx \oplus Fy \oplus Fz_1 \oplus Fz_2 \oplus J^2(A)$. In particular, $\dim_F J^2(A) = 11$.*

Proof. By Lemma 4.2(iv) we have $\dim_F((J(Z(A)) + J^2(A))/J^2(A)) \leq 2$.

Now if $\dim_F((J(Z(A)) + J^2(A))/J^2(A)) = 0$, then $J(Z(A)) + J^2(A) = J^2(A)$ and by Lemma 4.2(i,ii) we obtain $\dim_F(J(Z(A)) + J^2(A)) = 13$. Therefore, since A is local and $\dim_F A = 16$, there are $x, y \in J(A)$ such that $A = F1 \oplus Fx \oplus Fy \oplus (J(Z(A)) + J^2(A)) = F1 \oplus Fx \oplus Fy \oplus J^2(A)$. By a similar argument as used in the proof of Lemma 4.2(ii) we must have $xy \neq yx$ since A is non-commutative. This gives case (I).

If $\dim_F((J(Z(A)) + J^2(A))/J^2(A)) = 1$, then $J(Z(A)) + J^2(A) = Fz \oplus J^2(A)$ for some $z \in J(Z(A))$ and, again, by Lemma 4.2(i,ii) we obtain $\dim_F(J(Z(A)) + J^2(A)) = 13$. Now there are $x, y \in J(A)$ such that $A =$

$F1 \oplus Fx \oplus Fy \oplus (J(Z(A)) + J^2(A)) = F1 \oplus Fx \oplus Fy \oplus Fz \oplus J^2(A)$. Since $z \in J(Z(A))$ we have $xz = zx$ and $yz = zy$, so that we must have $xy \neq yx$ since A is non-commutative. This gives case (II).

Finally, if $\dim_F((J(Z(A)) + J^2(A))/J^2(A)) = 2$, then $J(Z(A)) + J^2(A) = Fz_1 \oplus Fz_2 \oplus J^2(A)$ for some $z_1, z_2 \in J(Z(A))$. For the same reason as before there are $x, y \in J(A)$ with $A = F1 \oplus Fx \oplus Fy \oplus (J(Z(A)) + J^2(A)) = F1 \oplus Fx \oplus Fy \oplus Fz_1 \oplus Fz_2 \oplus J^2(A)$ and $xy \neq yx$. Because of Proposition 3.7 and Lemma 4.1(v) we must have $z_1 z_2 \neq 0$ since $z_1^2 = z_2^2 = 0$ (see the multiplication table for $Z(A)$). This gives case (III). \square

The aim for the remainder of this section is to show that none of the cases (I), (II) or (III) of Corollary 4.3 can actually occur. This will give the desired contradiction. Before we start to exclude the three cases one by one, we need two more crucial lemmas.

Lemma 4.4. *We have $\dim_F(([A, A] + J^3(A))/J^3(A)) = 1$. Moreover, there is an $a \in J(A)$ with $a^2 \notin J^3(A)$. In particular, $a \notin Z(A)$.*

Proof. In all the cases from Corollary 4.3 we have $A = F1 \oplus Fx \oplus Fy \oplus (J(Z(A)) + J^2(A))$ with $xy \neq yx$. Therefore, we get

$$[A, A] = [Fx + Fy + Z(A) + J^2(A), Fx + Fy + Z(A) + J^2(A)] \subseteq F[x, y] + J^3(A).$$

Hence, the coset of $[x, y] = xy + yx$ in $J^3(A)$ spans $([A, A] + J^3(A))/J^3(A)$ over F and so $\dim_F(([A, A] + J^3(A))/J^3(A)) \leq 1$. If we show that there is an $a \in J(A)$ with $a \notin J^3(A)$, we will get $\dim_F(([A, A] + J^3(A))/J^3(A)) \geq 1$ using Lemma 4.1(iv), so that all the remaining claims will follow at once from this (note that for every $w \in J(Z(A))$ we have $w^2 = 0$, so that $w^2 \in J^3(A)$).

In order to show that there is such an a , we will now assume to the contrary that $a^2 \in J^3(A)$ for every $a \in J(A)$. For arbitrary $a, b \in J(A)$ this implies that $[a, b] = ab + ba = (a + b)^2 + a^2 + b^2 \in J^3(A)$, so that $ab + J^3(A) = ba + J^3(A)$ holds true for every $a, b \in J(A)$. We will now separately deduce a contradiction for every case.

Let A be as in case (I) from Corollary 4.3. Then $J(A) = F\{x, y\} + J^2(A)$. Using Lemma 3.2 and our assumption we get $J^2(A) = F\{x^2, xy, yx, y^2\} + J^3(A) = F\{xy\} + J^3(A)$ since $x^2, y^2, [x, y] \in J^3(A)$. Again by Lemma 3.2 we get $J^3(A) = F\{x^2y\} + J^4(A) = J^4(A)$ since $x^2 \in J^3(A)$ and so $x^2y \in J^4(A)$. Therefore, $J^3(A) = 0$ by Nakayama's Lemma. But then $A = F\{1, x, y, xy\}$ and hence $\dim_F A \leq 4$ which contradicts $\dim_F A = 16$.

Next let A be as in case (II). Then $J(A) = F\{x, y, z\} + J^2(A)$ and $z \in J(Z(A))$. Using the same facts as before we successively obtain

$$\begin{aligned} J^2(A) &= F\{x^2, xy, xz, yx, y^2, yz, zx, zy, z^2\} + J^3(A) = F\{xy, xz, yz\} + J^3(A), \\ J^3(A) &= F\{x^2y, x^2z, xyz, yxy, yxz, y^2z\} + J^4(A) = F\{xyz\} + J^4(A), \\ J^4(A) &= F\{x^2yz\} + J^5(A) = J^5(A). \end{aligned}$$

Again by Nakayama's Lemma we have $J^4(A) = 0$ and $A = F\{1, x, y, z, xy, xz, yz, xyz\}$. This yields the contradiction $\dim_F A \leq 8$.

Finally let A be as in case (III). Then $J(A) = F\{x, y, z_1, z_2\} + J^2(A)$ with $z_1, z_2 \in J(Z(A))$. As before:

$$\begin{aligned} J^2(A) &= F\{x^2, xy, xz_1, xz_2, yx, y^2, yz_1, yz_2, z_1x, z_1y, z_1^2, z_1z_2, z_2x, z_2y, z_2z_1, z_2^2\} + J^3(A) \\ &= F\{xy, xz_1, xz_2, yz_1, yz_2, z_1z_2\} + J^3(A), \\ J^3(A) &= F\{x^2y, x^2z_1, x^2z_2, xyx, xyx, xz_1z_2, yxy, yxz_1, yxz_2, y^2z_1, y^2z_2, yz_1z_2, \dots \\ &\quad \dots, z_1xy, z_1xz_1, z_1xz_2, z_1yz_1, z_1yz_2, z_1^2z_2\} + J^4(A) \\ &= F\{xyz_1, xyz_2, xz_1z_2, yz_1z_2\} + J^4(A), \\ J^4(A) &= F\{x^2yz_1, x^2yz_2, x^2z_1z_2, xyz_1z_2, yxyz_1, yxyz_2, yxz_1z_2, y^2z_1z_2\} + J^5(A) \\ &= F\{xyz_1z_2\} + J^5(A) = J^5(A). \end{aligned}$$

The last equality is a consequence of Lemma 4.2(i,iii). For, we have

$$xyz_1z_2 \in J^2(A) \cdot J^2(Z(A)) = J^2(A) \cdot \text{soc}(Z(A)) = J(A) \cdot \text{soc}(A) = 0.$$

Now $J^4(A) = 0$ by Nakayama and

$$A = F\{1, x, y, z_1, z_2, xy, xz_1, xz_2, yz_1, yz_2, z_1z_2, xyz_1, xyz_2, xz_1z_2, yz_1z_2\},$$

so that $\dim_F A \leq 15$, a contradiction. This completes the proof. \square

Lemma 4.5. *With the notation of Corollary 4.3 we may assume the following:*

- $x^2 \notin J^3(A)$,
- *There is an $\alpha \in F \setminus \{0\}$ such that $xy \equiv yx + \alpha x^2 \pmod{J^3(A)}$,*
- $y^2 \in J^3(A)$.

Moreover, with the α from the second item above we have for any $m \in \mathbb{N}$:

- $x^{m+1} \equiv \frac{1}{\alpha}[x, x^{m-1}y] \pmod{J^{m+2}(A)}$,
- $x^{2m}y \equiv \frac{1}{\alpha}[y, x^{2m-1}y] \pmod{J^{2m+2}(A)}$,
- $x^{4m-1}y \equiv \frac{1}{\alpha}(x^{2m-1}y)^2 \pmod{J^{4m+1}(A)}$,
- $x^{m+1}w \equiv \frac{1}{\alpha}[x^{m-1}y, xw] \pmod{J^{m+3}(A)}$,

where the last item is to be omitted in case (I), $w = z$ in case (II), and $w \in \{z_1, z_2\}$ in case (III). In particular:

- $x^n \in [A, A] + J^{n+1}(A)$ for $n \geq 2$,
- $x^{n-1}y \in [A, A] + J^{n+1}(A)$ for $n \geq 3$ being odd or $n \geq 4$ being divisible by 4,
- $x^{n-1}z \in [A, A] + J^{n+1}(A)$ for $n \geq 3$ in case (II),
- $x^{n-1}z_1, x^{n-1}z_2 \in [A, A] + J^{n+1}(A)$ for $n \geq 3$ in case (III).

Proof. By Lemma 4.4 we can find an $a \in J(A)$ with $a^2 \notin J^3(A)$. From this we deduce $a \notin J^2(A)$. Since the square of any element from $J(Z(A)) + J^2(A)$ is in $J^3(A)$, we get $a \notin J(Z(A)) + J^2(A)$. Hence, $a + (J(Z(A)) + J^2(A)) \neq 0$ in $J(A)/(J(Z(A)) + J^2(A))$ and we may therefore assume without loss of generality that $x = a$ (after possibly swapping x and y). This shows the first item.

Again, by Lemma 4.4, we have $\dim_F(([A, A] + J^3(A))/J^3(A)) = 1$. Since in any of the cases (I), (II) and (III) we have

$$[A, A] = [Fx + Fy + Z(A) + J^2(A), Fx + Fy + Z(A) + J^2(A)] \subseteq F[x, y] + J^3(A)$$

and, by the first item and Lemma 4.1(iv), we have $[A, A] \subseteq Fx^2 + J^3(A)$, we conclude that $\{[x, y] + J^3(A)\}$ and $\{x^2 + J^3(A)\}$ are two F -bases for $([A, A] + J^3(A))/J^3(A)$. Hence, there is an $\alpha \in F \setminus \{0\}$ such that $xy + yx = [x, y] \equiv \alpha x^2 \pmod{J^3(A)}$. From this the second item follows at once.

Now by Lemma 4.1(iv) we have $y \in [A, A]$, so that there is a $\beta \in F$ with $y^2 \equiv \beta x^2 \pmod{J^3(A)}$. Let $\zeta \in F$ be a zero of the polynomial $p(X) = X^2 + \alpha X + \beta$. Replacing y by $y' := y + \zeta x$ we obtain $A = F1 \oplus Fx \oplus Fy' \oplus J^2(A)$ and

$$\begin{aligned} [x, y'] &= [x, y + \zeta x] = [x, y], \\ (y')^2 &= (y + \zeta x)^2 = y^2 + \zeta(xy + yx) + \zeta^2 x^2 \\ &\equiv (\zeta^2 + \alpha\zeta + \beta)x^2 \equiv 0 \pmod{J^3(A)}. \end{aligned}$$

Renaming y' into y we obtain the third item.

Now we just have to show the four desired congruences and from those the other claims follow at once together with Lemma 4.1(iv). Let $m \in \mathbb{N}$. Then we have

$$\frac{1}{\alpha}[x, x^{m-1}y] = \frac{1}{\alpha}(x^m y + x^{m-1}yx) \equiv \frac{1}{\alpha}(2 \cdot x^m y + \alpha x^{m+1}) \equiv x^{m+1} \pmod{J^{m+2}(A)}$$

by applying $xy \equiv yx + \alpha x^2 \pmod{J^3(A)}$ once. Moreover we obtain

$$\frac{1}{\alpha}[y, x^{2m-1}y] = \frac{1}{\alpha}(yx^{2m-1}y + x^{2m-1}y^2) \equiv \frac{1}{\alpha}(2 \cdot x^{2m-1}y^2 + (2m-1) \cdot \alpha x^{2m}y) \equiv x^{2m}y \pmod{J^{2m+2}(A)}$$

by repeatedly ($2m-1$ times to be more exact) applying $xy \equiv yx + \alpha x^2 \pmod{J^3(A)}$. Doing the same thing we also get

$$\frac{1}{\alpha}(x^{2m-1}y)^2 = \frac{1}{\alpha}(x^{2m-1}yx^{2m-1}y) \equiv \frac{1}{\alpha}(x^{4m-2}y^2 + (2m-1) \cdot \alpha x^{4m-1}y) \equiv x^{4m-1}y \pmod{J^{4m+1}(A)}$$

keeping in mind that $y^2 \in J^3(A)$. Finally by the same arguments and using $w \in Z(A)$ we get

$$\frac{1}{\alpha}[x^{m-1}y, xw] = \frac{1}{\alpha}(x^{m-1}yxw + x^m yw) \equiv \frac{1}{\alpha}(2 \cdot x^m yw + \alpha x^{m+1}w) \equiv x^{m+1}w \pmod{J^{m+3}(A)}$$

which finishes the proof. \square

In the following we will always assume that A fulfills all the properties stated in Lemma 4.5 and we will use them without further mentioning. We have everything we need in order to show that none of the cases (I), (II) or (III) from Corollary 4.3 can occur for the F -algebra A under consideration.

Proposition 4.6. *The case (I) of Corollary 4.3 cannot occur.*

Proof. In case (I) the algebra A has the decomposition $A = F1 \oplus Fx \oplus Fy \oplus J^2(A)$. Using Lemma 3.2 and $J(A) = F\{x, y\} + J^2(A)$ we get $J^2(A) = F\{x^2, xy, yx, y^2\} + J^3(A) = F\{x^2, xy\} + J^3(A)$. From here we get $J^n(A) = F\{x^n, x^{n-1}y\} + J^{n+1}(A)$ for every integer $n \geq 2$ by inductively applying Lemma 3.2. Therefore, we get $\dim_F(J^n(A)/J^{n+1}(A)) \leq 2$ for every $n \in \mathbb{N}$. Also by Lemma 3.2 we see that if $\dim_F(J^m(A)/J^{m+1}(A)) = 1$ for some $m \in \mathbb{N}$, then $\dim_F(J^n(A)/J^{n+1}(A)) \leq 1$ for every $n \geq m$. Since there is always such an m by Lemma 3.1(vi) and since $\dim_F J(Z(A)) = 7$, we obtain the following three possibilities, denoted by (I.1), (I.2) and (I.3), for the dimensions of the Loewy layers of A by keeping in mind Lemma 3.3:

Loewy layer	spanned by	dimensions		
$A/J(A)$	1	1	1	1
$J(A)/J^2(A)$	x, y	2	2	2
$J^2(A)/J^3(A)$	x^2, xy	2	2	2
$J^3(A)/J^4(A)$	x^3, x^2y	2	2	2
$J^4(A)/J^5(A)$	x^4, x^3y	2	2	2
$J^5(A)/J^6(A)$	x^5, x^4y	2	2	2
$J^6(A)/J^7(A)$	x^6, x^5y	2	2	1
$J^7(A)/J^8(A)$	x^7, x^6y	2	1	1
$J^8(A)/J^9(A)$	x^8, x^7y	1	1	1
$J^9(A)/J^{10}(A)$	x^9, x^8y		1	1
$J^{10}(A)/J^{11}(A)$	x^{10}, x^9y			1
		(I.1)	(I.2)	(I.3)

In case (I.1) we have $\text{soc}(A) = F\{x^8, x^7y\}$. On the other hand, since $J^9(A) = 0$, Lemma 4.5 yields $x^8, x^7y \in [A, A]$. Hence, $\text{soc}(A) \cap [A, A] \neq 0$, a contradiction.

In case (I.2) we have $\text{soc}(A) = F\{x^9, x^8y\}$. Again, by Lemma 4.5 and $J^{10}(A) = 0$, we have $x^9, x^8y \in [A, A]$ and hence a contradiction.

Finally in case (I.3) we have $J^{11}(A) = 0$ and $J^{10}(A) = F\{x^{10}, x^9y\} \neq 0$, so that $x^9 \notin J^{10}(A)$. Hence, $J^9(A) = F\{x^9\} + J^{10}(A)$ and $\text{soc}(A) = J^{10}(A) = F\{x^{10}\}$. But on the other hand $x^{10} \in [A, A]$, since $J^{11}(A) = 0$, and therefore $\text{soc}(A) \cap [A, A] \neq 0$, again a contradiction. This shows that neither of the cases (I.1), (I.2) or (I.3) can occur and so the proposition is proven. \square

Proposition 4.7. *The case (II) of Corollary 4.3 cannot occur.*

Proof. In case (II) the algebra A decomposes into $A = F1 \oplus Fx \oplus Fy \oplus Fz \oplus J^2(A)$. Using this and Lemma 3.2 and $z^2 = 0$ we easily see

$$\begin{aligned} J(A) &= F\{x, y, z\} + J^2(A), \\ J^2(A) &= F\{x^2, xy, xz, yz\} + J^3(A), \\ J^3(A) &= F\{x^3, x^2y, x^2z, xyz\} + J^4(A), \end{aligned}$$

and inductively

$$J^n(A) = F\{x^n, x^{n-1}y, x^{n-1}z, x^{n-2}yz\} + J^{n+1}(A)$$

for any integer $n \geq 3$. Now we will distinguish between the different cases that can occur for $\dim_F(J^2(A)/J^3(A))$. We note that $2 \leq \dim_F(J^2(A)/J^3(A)) \leq 4$. The upper bound is clear by the preceding discussion, and if $\dim_F(J^2(A)/J^3(A)) = 1$, then $J^2(A) \subseteq Z(A)$ by Lemma 3.3 which is a contradiction to $\dim_F Z(A) = 8$. The case $\dim_F(J^2(A)/J^3(A)) = 0$ leads to $J^2(A) = 0$ by Nakayama's Lemma and this is clearly false.

Case (II.1): $\dim_F(J^2(A)/J^3(A)) = 2$.

Since $x^2 \notin J^3(A)$ we proceed by distinguishing three subcases for an F -basis of $J^2(A)/J^3(A)$. More specifically there is always a basis of $J^2(A)/J^3(A)$ given by $\{x^2 + J^3(A), d + J^3(A)\}$ for some $d \in \{xy, xz, yz\}$.

(1): $J^2(A) = F\{x^2, xy\} + J^3(A)$. We inductively obtain $J^n(A) = F\{x^n, x^{n-1}y\} + J^{n+1}(A)$ for every $n \geq 2$. With the same arguments as in the proof of Proposition 4.6 we see that there are the following two possibilities for the dimensions of the Loewy layers of A :

Loewy layer	spanned by	dimensions	
$A/J(A)$	1	1	1
$J(A)/J^2(A)$	x, y, z	3	3
$J^2(A)/J^3(A)$	x^2, xy	2	2
$J^3(A)/J^4(A)$	x^3, x^2y	2	2
$J^4(A)/J^5(A)$	x^4, x^3y	2	2
$J^5(A)/J^6(A)$	x^5, x^4y	2	2
$J^6(A)/J^7(A)$	x^6, x^5y	2	1
$J^7(A)/J^8(A)$	x^7, x^6y	1	1
$J^8(A)/J^9(A)$	x^8, x^7y	1	1
$J^9(A)/J^{10}(A)$	x^9, x^8y		1
		(II.1.1)	(II.1.2)

In case (II.1.1) we have $\text{soc}(A) = F\{x^8, x^7y\}$ and $x^8, x^7y \in [A, A]$, a contradiction. Similarly, in case (II.1.2) we have $\text{soc}(A) = F\{x^9, x^8y\}$ and $x^9, x^8y \in [A, A]$, again a contradiction.

(2): $J^2(A) = F\{x^2, xz\} + J^3(A)$. We can assume that $xy \in F\{x^2\} + J^3(A)$ since otherwise we are in the first subcase. Let $xy \equiv \gamma x^2 \pmod{J^3(A)}$. Using this we obtain

$$x^3 \equiv \frac{1}{\alpha}[x, xy] \equiv \frac{\gamma}{\alpha}[x, x^2] = 0 \pmod{J^4(A)}.$$

This, however, implies $J^3(A) = F\{x^3, x^2z\} + J^4(A) = F\{x^2z\} + J^4(A)$ and $J^4(A) = F\{x^3z\} + J^5(A) = J^5(A)$. Hence, $J^4(A) = 0$ by Nakayama's Lemma and therefore $\dim_F A \leq 1 + 3 + 2 + 1 = 7$, a contradiction.

(3): $J^2(A) = F\{x^2, yz\} + J^3(A) = F\{x^2, zy\} + J^3(A)$. We may assume that $xy, xz \in F\{x^2\} + J^3(A)$ since otherwise we are in one of the previous two subcases. Using this we obtain $J^3(A) = F\{x^3, zxy, x^2z, z^2y\} + J^4(A) = F\{x^3, xyz, x^2z\} + J^4(A) = F\{x^3\} + J^4(A)$. Hence, $J^2(A) \subseteq Z(A)$ by Lemma 3.3, and so $\dim_F Z(A) \geq \dim_F J^2(A) = 12$, a contradiction. We have thus shown that $\dim_F(J^2(A)/J^3(A)) \neq 2$.

Case (II.2): $\dim_F(J^2(A)/J^3(A)) = 3$.

Again, since $x^2 \notin J^3(A)$, there is always an F -basis of $J^2(A)/J^3(A)$ of the form $\{x^2 + J^3(A), d_1 + J^3(A), d_2 + J^3(A)\}$ for some $d_1, d_2 \in \{xy, xz, yz\}$. Hence, we can proceed by distinguishing three subcases for a basis of $J^2(A)/J^3(A)$.

(1): $J^2(A) = F\{x^2, xy, xz\} + J^3(A)$. We have $J^n(A) = F\{x^n, x^{n-1}y, x^{n-1}z\} + J^{n-1}(A)$ for every $n \geq 2$. We obtain the following possibilities for the dimensions of the Loewy layers of A :

Loewy layer	spanned by	dimensions					
$A/J(A)$	1	1	1	1	1	1	1
$J(A)/J^2(A)$	x, y, z	3	3	3	3	3	3
$J^2(A)/J^3(A)$	x^2, xy, xz	3	3	3	3	3	3
$J^3(A)/J^4(A)$	x^3, x^2y, x^2z	3	3	3	3	2	2
$J^4(A)/J^5(A)$	x^4, x^3y, x^3z	3	3	2	2	2	2
$J^5(A)/J^6(A)$	x^5, x^4y, x^4z	2	1	2	1	2	2
$J^6(A)/J^7(A)$	x^6, x^5y, x^5z	1	1	1	1	2	1
$J^7(A)/J^8(A)$	x^7, x^6y, x^6z		1	1	1	1	1
$J^8(A)/J^9(A)$	x^8, x^7y, x^7z			1	1	1	1
		(II.2.1)	(II.2.2)	(II.2.3)	(II.2.4)	(II.2.5)	(II.2.6)

In cases (II.2.2), (II.2.3) and (II.2.5) we have $\text{soc}(A) = F\{x^7, x^6y, x^6z\}$, but $x^7, x^6y, x^6z \in [A, A]$, a contradiction. Similarly, in cases (II.2.4) and (II.2.6) we have $\text{soc}(A) = F\{x^8, x^7y, x^7z\}$ and $x^8, x^7y, x^7z \in [A, A]$, again a contradiction.

Finally let us consider case (II.2.1). By Lemma 4.5 we obtain that $\dim_F((([A, A] \cap J^2(A)) + J^3(A)) / J^3(A)) = 1$, since this space is spanned by $\{x^2 + J^3(A)\}$. Moreover $\dim_F((([A, A] \cap J^3(A)) + J^4(A)) / J^4(A)) = 3$, since this space is spanned by $\{x^3 + J^4(A), x^2y + J^4(A), x^2z + J^4(A)\}$. Analogously $\dim_F((([A, A] \cap J^4(A)) + J^5(A)) / J^5(A)) = 3$ and $\dim_F((([A, A] \cap J^5(A)) + J^6(A)) / J^6(A)) = 2$. Using the canonical isomorphism

$$((([A, A] \cap J^n(A)) + J^{n+1}(A)) / J^{n+1}(A)) \cong (([A, A] \cap J^n(A)) / ([A, A] \cap J^{n+1}(A)))$$

for $n \in \mathbb{N}$ we obtain

$$8 = \dim_F[A, A] \geq \sum_{n=2}^5 \dim_F((([A, A] \cap J^n(A)) / ([A, A] \cap J^{n+1}(A)))) = 1 + 3 + 3 + 2 = 9,$$

a contradiction.

(2): $J^2(A) = F\{x^2, xy, yz\} + J^3(A) = F\{x^2, xy, zy\} + J^3(A)$. Here we can assume $xz \in F\{x^2, xy\} + J^3(A)$ since otherwise we are in the subcase $J^2(A) = F\{x^2, xy, xz\} + J^3(A)$ again. We obtain

$$J^3(A) = F\{x^3, x^2y, xzy, zx^2, zxy, zxz\} + J^4(A) = F\{x^3, x^2y\} + J^4(A).$$

Hence, we get the following two possibilities for the dimensions of the Loewy layers of A :

Loewy layer	spanned by	dimensions	
$A/J(A)$	1	1	1
$J(A)/J^2(A)$	x, y, z	3	3
$J^2(A)/J^3(A)$	x^2, xy, yz	3	3
$J^3(A)/J^4(A)$	x^3, x^2y	2	2
$J^4(A)/J^5(A)$	x^4, x^3y	2	2
$J^5(A)/J^6(A)$	x^5, x^4y	2	2
$J^6(A)/J^7(A)$	x^6, x^5y	2	1
$J^7(A)/J^8(A)$	x^7, x^6y	1	1
$J^8(A)/J^9(A)$	x^8, x^7y	1	1

Since $x^7, x^6y \in [A, A] + J^8(A)$ and $x^8, x^7y \in [A, A] + J^9(A)$, similar arguments as used before show that both cases lead to a contradiction.

(3): $J^2(A) = F\{x^2, xz, yz\} + J^3(A)$. Here we can assume $xy \in Fx^2 + J^3(A)$, since we are in one of the previous two subcases otherwise. Hence,

$$J^3(A) = F\{x^3, xzx, yzx, x^2z, xz^2, yz^2\} + J^4(A) = F\{x^3, x^2z\} + J^4(A).$$

Inductively we get $J^n(A) = F\{x^n, x^{n-1}z\} + J^{n+1}(A)$ for $n \geq 3$. But together with Lemma 4.5 this implies $J^3(A) \subseteq [A, A]$ which is a contradiction. This shows that $\dim_F(J^2(A)/J^3(A)) \neq 3$.

Case (II.3): $\dim_F(J^2(A)/J^3(A)) = 4$.

In this case we have $J^2(A) = F\{x^2, xy, xz, yz\} + J^3(A)$ and the cosets of the elements in $\{x^2, xy, xz, yx\}$ in $J^3(A)$ form an F -basis of $J^2(A)/J^3(A)$. Inductively we get $J^n(A) = F\{x^n, x^{n-1}y, x^{n-1}z, x^{n-2}yz\} + J^{n+1}(A)$ for $n \geq 2$ (cf. the beginning of this proof). Arguing as before we see that there are the following possible cases for the dimensions of the Loewy layers of A :

Loewy layer	dimensions						
$A/J(A)$	1	1	1	1	1	1	1
$J(A)/J^2(A)$	3	3	3	3	3	3	3
$J^2(A)/J^3(A)$	4	4	4	4	4	4	4
$J^3(A)/J^4(A)$	4	4	3	3	3	2	2
$J^4(A)/J^5(A)$	3	2	3	2	2	2	2
$J^5(A)/J^6(A)$	1	1	1	2	1	2	1
$J^6(A)/J^7(A)$		1	1	1	1	1	1
$J^7(A)/J^8(A)$					1	1	1
$J^8(A)/J^9(A)$							1
	(II.3.1)	(II.3.2)	(II.3.3)	(II.3.4)	(II.3.5)	(II.3.6)	(II.3.7)

In case (II.3.1) we have $\text{soc}(A) = J^5(A) = F\{x^5, x^4y, x^4z, x^3yz\}$. By Lemma 4.5 we obtain $x^5, x^4y, x^4z \in [A, A]$. Since $x^3y \in [A, A] + J^5(A)$ and $Z(A) \cdot [A, A] \subseteq [A, A]$, we also get $x^3yz \in [A, A]$. But this contradicts $\text{soc}(A) \cap [A, A] = 0$.

In cases (II.3.2) and (II.3.3) we have $\text{soc}(A) = J^6(A) = F\{x^6, x^5y, x^5z, x^4yz\}$ and $J^4(A) \subseteq Z(A)$ by Lemma 3.3. Therefore, $x^3y \in Z(A)$. Now we have $x^6, x^5z \in [A, A]$ and, using $Z(A) \cdot [A, A] \subseteq [A, A]$ again, we also obtain $x^5y, x^4yz \in [A, A]$ since $x^2 \in [A, A]$, $x^4z \in [A, A] + J^6(A)$, and $x^3y, z \in Z(A)$. Therefore, $\text{soc}(A) \cap [A, A] \neq 0$, a contradiction.

In cases (II.3.5) and (II.3.6) we have $\text{soc}(A) = J^7(A) = F\{x^7, x^6y, x^6z, x^5yz\}$ and $J^5(A) \subseteq Z(A)$. Since $x^2 \in [A, A]$ and $x^3yz \in Z(A)$ we get $x^5yz \in [A, A]$. Moreover $x^7, x^6y, x^6z \in [A, A]$ and therefore $\text{soc}(A) \subseteq [A, A]$, a contradiction. In case (II.3.7) we have $\text{soc}(A) = F\{x^8, x^7y, x^7z, x^6yz\}$. As before we obtain a contradiction using $x^8, x^7y, x^7z \in [A, A]$, $x^6y \in [A, A] + J^8(A)$, and $z \in Z(A)$.

There remains case (II.3.4) and excluding this one requires some additional arguments. We have $J^7(A) = 0$, $\text{soc}(A) = J^6(A) = F\{x^6, x^5y, x^5z, x^4yz\}$, and $J^5(A) \subseteq Z(A)$. Since $x^6, x^5z \in [A, A]$, $x^4y \in [A, A] + J^6(A)$, and $z \in Z(A)$, we obtain $x^6, x^5z, x^4yz \in \text{soc}(A) \cap [A, A] = 0$. Hence, $x^6 = x^5z = x^4yz = 0$ and $\text{soc}(A) = F\{x^5y\}$. This also yields $x^3 \notin J^4(A)$ and $x^4 \notin J^5(A)$. If $x^2y \in F\{x^3\} + J^4(A)$, then $x^5y \in F\{x^6\} + J^7(A) = 0$, a contradiction. Hence, $\{x^3 + J^4(A), x^2y + J^4(A)\}$ is F -linearly independent in $J^3(A)/J^4(A)$. With similar arguments one gets that $\{x^4 + J^5(A), x^3y + J^5(A)\}$ is F -linearly independent in $J^4(A)/J^5(A)$. Therefore, there is a pair $(\lambda_1, \lambda_2) \in F^2 \setminus \{(0, 0)\}$ such that

$$\{1, x, y, z, x^2, xy, xz, yz, x^3, x^2y, \lambda_1 x^2z + \lambda_2 xyz, x^4, x^3y, x^5, x^4y, x^5y\}$$

is an F -basis of A . We will proceed by showing in two steps that x^2z must be zero. The first step will be to show $x^2z \in J^4(A)$. In order to do this, assume that $x^2z \notin J^4(A)$ and define the subspace

$$T := F\{x^2, xy, xz, yz, x^3, x^2y, x^3y\}$$

of $J^2(A)$. We will show that $T \cap Z(A) = 0$. This will imply the inequality

$$12 = \dim_F J^2(A) \geq \dim_F T + \dim_F (Z(A) \cap J^2(A)) = 7 + 6 = 13$$

which is certainly false, so that x^2z must be in $J^4(A)$. Let

$$w = \delta_1 x^2 + \delta_2 xy + \delta_3 xz + \delta_4 yz + \delta_5 x^3 + \delta_6 x^2y + \delta_7 x^3y \in T \cap Z(A)$$

with $\delta_i \in F$ for $i = 1, \dots, 7$ be arbitrary. We have to show $w = 0$. Considering $x^6 = x^5z = x^4yz = 0$, $J^7(A) = 0$, $w \in Z(A)$, and $x^4 = (x^2)^2 \in [A, A]$, we obtain $\delta_2x^5y = x^4w \in \text{soc}(A) \cap [A, A] = 0$, so that $\delta_2 = 0$ and $w = \delta_1x^2 + \delta_3xz + \delta_4yz + \delta_5x^3 + \delta_6x^2y + \delta_7x^3y$. Using $w \in Z(A)$ again, we obtain

$$0 = xw + wx \equiv \delta_1(x^3 + x^3) + \delta_3(x^2z + x^2z) + \delta_4(xy + yx)z \equiv \delta_4(\alpha x^2z) \pmod{J^4(A)}$$

and

$$0 = yw + wy \equiv \delta_1(yx^2 + x^2y) + \delta_3(yx + xy)z + \delta_4(y^2z + y^2z)z \equiv \delta_3(\alpha x^2z) \pmod{J^4(A)}.$$

Thus, $\delta_3 = \delta_4 = 0$ since $\alpha \neq 0$ and we assumed $x^2z \notin J^4(A)$. Hence, $w = \delta_1x^2 + \delta_5x^3 + \delta_6x^2y + \delta_7x^3y$, and using $x^3y \in [A, A] + J^5(A)$ and $w \in Z(A)$ we get $\delta_1x^5y = wx^3y \in \text{soc}(A) \cap [A, A] = 0$. Therefore, $\delta_1 = 0$ and $w = \delta_5x^3 + \delta_6x^2y + \delta_7x^3y$. Using $x^3 \in [A, A] + J^4(A)$ and $x^6 = 0$ we get $\delta_6x^5y = x^3w \in \text{soc}(A) \cap [A, A] = 0$, so that $\delta_6 = 0$ and $w = \delta_5x^3 + \delta_7x^3y$. With $x^2y \in [A, A] + J^4(A)$ we conclude $\delta_5x^5y = wx^2y = 0$ and hence $w = \delta_7x^3y$. Now, again, $\delta_7x^5y = x^2w = 0$, so that $w = 0$. Therefore, we have shown $T \cap Z(A) = 0$ and by the argument above we obtain a contradiction. We have thus shown that x^2z can be assumed to be in $J^4(A)$. This also implies that the following elements form an F -basis of A :

$$1, x, y, z, x^2, xy, xz, yz, x^3, x^2y, xyz, x^4, x^3y, x^5, x^4y, x^5y.$$

In the second step we will show $x^2z = 0$. Since $x^2z \in J^4(A)$, there are $\varepsilon_1, \dots, \varepsilon_5 \in F$ such that

$$x^2z = \varepsilon_1x^4 + \varepsilon_2x^3y + \varepsilon_3x^5 + \varepsilon_4x^4y + \varepsilon_5x^5y.$$

Since $J^4(A) = F\{x^4, x^3y, x^5, x^4y, x^5y\}$, we observe that from $x^3, x^2y \in [A, A] + J^4(A)$, and $x^4, x^3y \in [A, A] + J^5(A)$, and $x^5, x^4y \in [A, A] + J^6(A)$ it follows that $x^3, x^2y, x^3y \in [A, A] + J^6(A)$. Now as before, $x^4 \in [A, A]$ and therefore $\varepsilon_2x^5y = x^2 \cdot x^2z = x^4z \in \text{soc}(A) \cap [A, A] = 0$, so that $\varepsilon_2 = 0$ and $x^2z = \varepsilon_1x^4 + \varepsilon_3x^5 + \varepsilon_4x^4y + \varepsilon_5x^5y$. Since $x^3y \in [A, A] + J^6(A)$, we get $\varepsilon_1x^5y = x^2z \cdot xy = (x^3y)z \in \text{soc}(A) \cap [A, A] = 0$, so that $\varepsilon_1 = 0$ and $x^2z = \varepsilon_3x^5 + \varepsilon_4x^4y + \varepsilon_5x^5y$. Using $x^3 \in [A, A] + J^6(A)$ next, we obtain $\varepsilon_4x^5y = x \cdot x^2z = x^3z \in \text{soc}(A) \cap [A, A] = 0$, so that $\varepsilon_4 = 0$ and $x^2z = \varepsilon_3x^5 + \varepsilon_5x^5y$. Similarly, we get $\varepsilon_3x^5y = x^2z \cdot y = (x^2y)z = 0$, so that $\varepsilon_3 = 0$ and $x^2z = \varepsilon_5x^5y$. But now $x^2z \in [A, A]$ and $x^5y \in \text{soc}(A)$ imply that $x^2z = \varepsilon_5x^5y = 0$. Hence, we have shown $x^2z = 0$ as claimed. Since the three elements x, y, z generate A as an F -algebra, and since $z \in Z(A)$, we observe that the center $Z(A)$ consists exactly of all elements $w \in A$ which commute with both x and y . Using this and the fact $x^2z = 0$ one can easily show that in our case the elements $1, z, xz, yz, xyz, x^3y, x^5, x^4y, x^5y$ are central in A . Since they are also F -linearly independent and $\dim_F Z(A) = 8$, we obtain $Z(A) = F\{1, z, xz, yz, xyz, x^5, x^4y, x^5y\}$ and, in particular, $J(Z(A)) = F\{z, xz, yz, xyz, x^5, x^4y, x^5y\}$. But then for any $w_1, w_2 \in J(Z(A))$ we get $w_1 \cdot w_2 = 0$ and this contradicts Lemma 4.1(iii) and the subsequent multiplication table for $Z(A)$. This finishes the proof. \square

Proposition 4.8. *The case (III) of Corollary 4.3 cannot occur.*

Proof. In this final case (III) the algebra A has a decomposition $A = F1 \oplus Fx \oplus Fy \oplus Fz_1 \oplus Fz_2 \oplus J^2(A)$ with $z_1, z_2 \in J(Z(A))$ and $\dim_F J^2(A) = 11$. In the following we will frequently make use of

$$J(A) \cdot J^2(Z(A)) = \text{soc}(A)$$

(see Lemma 4.2(i,iii)) without further mentioning it. From $J(A) = F\{x, y, z_1, z_2\} + J^2(A)$ we get $J^2(A) = F\{x^2, xy, xz_1, xz_2, yz_1, yz_2, z_1z_2\} + J^3(A)$ and this implies $J^3(A) \neq \text{soc}(A)$ because of dimension reasons. Hence, $J(A) \cdot J^2(Z(A)) = \text{soc}(A) \subseteq J^4(A)$ and so $J^3(A) = F\{x^3, x^2y, x^2z_1, x^2z_2, xyz_1, xyz_2\} + J^4(A)$.

Now since $[A, A]$ does not contain any non-zero ideal of A and since $J^3(A) \neq 0$, we get $[A, A] \neq [A, A] + J^3(A)$ and therefore

$$\dim_F(A/J^3(A)) = \dim_F(A/([A, A] + J^3(A))) + \dim_F((([A, A] + J^3(A))/J^3(A)) \leq 7 + 1 = 8.$$

Hence, $\dim_F J^3(A) \geq 8$ and together with $\dim_F J^2(A) = 11$ and $J(A) \not\subseteq Z(A)$ we get $\dim_F J^3(A) \in \{8, 9\}$ by Lemma 3.3. We thus have to distinguish two cases.

Case (III.1): $\dim_F(J^2(A)/J^3(A)) = 2$.

We will again consider several subcases corresponding to possible choices of an F -basis of $J^2(A)/J^3(A)$. Since $x^2 \notin J^3(A)$ we can fix $x + J^3(A)$ as a basis element of $J^2(A)/J^3(A)$. Then there always is an F -basis $\{x^2 + J^3(A), d + J^3(A)\}$ of $J^2(A)/J^3(A)$ for some $d \in \{xy, xz_1, xz_2, yz_1, yz_2, z_1z_2\}$.

(1): $J^2(A) = F\{x^2, xy\} + J^3(A)$. As in the proof of Proposition 4.7 we get the following possibilities for the dimensions of the Loewy layers of A :

Loewy layer	spanned by	dimensions	
$A/J(A)$	1	1	1
$J(A)/J^2(A)$	x, y, z_1, z_2	4	4
$J^2(A)/J^3(A)$	x^2, xy	2	2
$J^3(A)/J^4(A)$	x^3, x^2y	2	2
$J^4(A)/J^5(A)$	x^4, x^3y	2	2
$J^5(A)/J^6(A)$	x^5, x^4y	2	2
$J^6(A)/J^7(A)$	x^6, x^5y	2	1
$J^7(A)/J^8(A)$	x^7, x^6y	1	1
$J^8(A)/J^9(A)$	x^8, x^7y		1
		(III.1.1)	(III.1.2)

Case (III.1.1) cannot occur since $\text{soc}(A) = F\{x^7, x^6y\} \subseteq [A, A]$ and case (III.1.2) cannot occur since $\text{soc}(A) = F\{x^8, x^7y\} \subseteq [A, A]$.

(2): $J^2(A) = F\{x^2, xz_i\} + J^3(A)$ for some $i \in \{1, 2\}$. We may assume that $xy \in F\{x^2\} + J^3(A)$ since otherwise we are in the situation we have just considered. Let $xy \equiv \gamma x^2 \pmod{J^3(A)}$. Then we get

$$x^3 \equiv \frac{1}{\alpha}[x, xy] \equiv \frac{\gamma}{\alpha}[x, x^2] = 0 \pmod{J^4(A)}$$

which implies $J^3(A) = F\{x^3, x^2z_i\} + J^4(A)$ and $J^4(A) = F\{x^3z_i\} + J^5(A) = J^5(A)$, so that $J^4(A) = 0$ by Nakayama's Lemma, a contradiction.

(3): $J^2(A) = F\{x^2, yz_i\} + J^3(A) = F\{x^2, z_iy\} + J^3(A)$ for some $i \in \{1, 2\}$. We may, as before, assume that $xy, xz_1, z_2 \in F\{x^2\} + J^3(A)$. We obtain

$$J^3(A) = F\{x^3, xyz_i, x^2z_i, yz_i^2\} + J^4(A) = F\{x^3\} + J^4(A).$$

This implies $J^2(A) \subseteq Z(A)$ by Lemma 3.3, a contradiction.

(4): $J^2(A) = F\{x^2, z_1z_2\} + J^3(A)$. Here we may assume that $xy, xz_1, xz_2, yz_1, yz_2 \in F\{x^2\} + J^3(A)$. As before we obtain a contradiction by

$$J^3(A) = F\{x^3, xz_1z_2, x^2z_1, z_1^2z_2\} + J^4(A) = F\{x^3\} + J^4(A).$$

This completes case (III.1).

Case (III.2): $\dim_F(J^2(A)/J^3(A)) = 3$.

We will again distinguish the different cases for a possible basis of $J^2(A)/J^3(A)$. Since $x^2 \notin J^3(A)$, we may fix $x^2 + J^3(A)$ as a basis element of $J^2(A)/J^3(A)$ and we have to look through the possibilities for the remaining two basis elements. Since those two elements can be chosen from $\{xy, xz_1, xz_2, yz_1, yz_2, z_1z_2\}$, there are essentially 8 different cases.

(1): $J^2(A) = F\{x^2, xy, xz_i\} + J^3(A)$ for some $i \in \{1, 2\}$. We get the following four possibilities for the dimensions of the Loewy layers of A :

Loewy layer	spanned by	dimensions			
$A/J(A)$	1	1	1	1	1
$J(A)/J^2(A)$	x, y, z_1, z_2	4	4	4	4
$J^2(A)/J^3(A)$	x^2, xy, xz_i	3	3	3	3
$J^3(A)/J^4(A)$	x^3, x^2y, x^2z_i	3	3	3	2
$J^4(A)/J^5(A)$	x^4, x^3y, x^3z_i	3	2	2	2
$J^5(A)/J^6(A)$	x^5, x^4y, x^4z_i	1	2	1	2
$J^6(A)/J^7(A)$	x^6, x^5y, x^5z_i	1	1	1	1
$J^7(A)/J^8(A)$	x^7, x^6y, x^6z_i			1	1
		(III.2.1)	(III.2.2)	(III.2.3)	(III.2.4)

Cases (III.2.3) and (III.2.4) can be excluded immediately since there we have $\text{soc}(A) = F\{x^7, x^6y, x^6z_i\} \subseteq [A, A]$, a contradiction. Similarly, in case (III.2.1) we have $x^6, x^5z_i \in [A, A]$ and by Lemma 3.3 we have $x^3y \in Z(A)$. Since $x^2 \in [A, A]$ we conclude $x^5y = x^2 \cdot x^3y \in [A, A]$ and thus $\text{soc}(A) \cap [A, A] \neq 0$, a contradiction.

Case (III.2.2) needs some more calculation to exclude. After interchanging z_1 and z_2 if necessary, we may assume that $i = 1$. Since $x^6, x^5z_1 \in \text{soc}(A) \cap [A, A] = 0$ we have $x^6 = x^5z_1 = 0$ and $\text{soc}(A) = J^6(A) = F\{x^5y\}$. In particular, we have $x^4 \notin J^5(A)$ and $x^5 \notin J^6(A)$. As in the last subcase of the proof of Proposition 4.7 we get $J^4(A) = F\{x^4, x^3y\} + J^5(A)$ and $J^5(A) = F\{x^5, x^4y\} + J^6(A)$ from this. Hence, the following elements form an F -basis of A :

$$1, x, y, z_1, z_2, x^2, xy, xz_1, x^3, x^2y, x^2z_1, x^4, x^3y, x^5, x^4y, x^5y.$$

Now we define the subspace

$$T := F\{1, x, y, z_1, x^2, xy, xz_1, x^3, x^2y, x^4, x^3y\}$$

of A . We will first show that $T \cap \text{soc}^2(A) = 0$. In order to do so we remark that $J(A) \cdot \text{soc}^2(A) \subseteq \text{soc}(A)$ by Lemma 4.2(v). In particular, $x \text{soc}^2(A) \subseteq \text{soc}(A)$. But now $xT = F\{x, x^2, xy, xz_1, x^3, x^2y, x^2z_1, x^4, x^3y, x^5, x^4y\}$ and therefore $xT \cap \text{soc}(A) = 0$. This implies $T \cap \text{soc}^2(A) = 0$. Since $\dim_F A = 16$, $\dim_F T = 11$, $\dim_F \text{soc}^2(A) = \dim_F((J^2(A))^\perp) = 5$, and $T \cap \text{soc}^2(A) = 0$, we obtain $A = T \oplus \text{soc}^2(A)$. Decomposing $z_2 \in A$ into its direct summands, we find $\lambda_0, \lambda_1, \lambda_2, \lambda_3 \in F$ and $u \in T \cap J^2(A)$ and $v \in \text{soc}^2(A)$ such that $z_2 = \lambda_0 1 + \lambda_1 x + \lambda_2 y + \lambda_3 z_1 + u + v$ or, equivalently,

$$v = z_2 + \lambda_0 1 + \lambda_1 x + \lambda_2 y + \lambda_3 z_1 + u \in \text{soc}^2(A).$$

Furthermore, λ_0 must vanish since otherwise $v J^2(A) \subseteq J^2(A) \setminus J^3(A)$ which would contradict $v \in \text{soc}^2(A)$. Hence, $v = z_2 + \lambda_1 x + \lambda_2 y + \lambda_3 z_1 + u \in \text{soc}^2(A)$. Replacing z_2 by v we obtain a new set of elements $\{x, y, z_1, v\}$ in $J(A)$ such that $\{x + J^2(A), y + J^2(A), z_1 + J^2(A), v + J^2(A)\}$ is an F -basis of $J(A)/J^2(A)$. Since $v \in \text{soc}^2(A)$, we also have $xv, vx, yv, vy, z_1v, vz_1 \in \text{soc}(A)$ and $xv = vx, yv = vy, z_1v = vz_1$ by Lemma 4.2(v). Now if $z_1v = 0$, then $z_1v = vz_1 = z_1^2 = 0$ and so, by Proposition 3.7, the A -module F would have infinite complexity in contradiction to Lemma 4.1(v). Hence, we may assume that $z_1v \neq 0$. Since $\dim_F \text{soc}(A) = 1$, $xv, yv, z_1v \in \text{soc}(A)$, and $z_1v \neq 0$, there are $\gamma_1, \gamma_2 \in F$ such that $(x - \gamma_1 z_1)v = 0 = (y - \gamma_2 z_1)v$. Replacing x by $x' := x - \gamma_1 z_1$ and y by $y' := y - \gamma_2 z_1$, we obtain a set of elements $\{x', y', z_1, v\}$ in $J(A)$ such that $\{x' + J^2(A), y' + J^2(A), z_1 + J^2(A), v + J^2(A)\}$ is an F -basis in $J(A)/J^2(A)$ with $x'v = 0 = y'v$. By Lemma 4.2(v) we also have $x'v = vx' = y'v = vy' = 0$ and this implies that the A -module F has infinite complexity by the remark after Proposition 3.7 in contradiction to Lemma 4.1(v). This finishes the first subcase.

(2): $J^2(A) = F\{x^2, xy, yz_i\} + J^3(A) = F\{x^2, xy, z_i y\} + J^3(A)$ for some $i \in \{1, 2\}$. We may assume that $xz_1, xz_2 \in F\{x^2, xy\} + J^3(A)$ for otherwise we are in the first subcase again. Using this we get

$$J^3(A) = F\{x^3, x^2y, xz_i y, x^2z_i, yz_i^2\} + J^4(A) = F\{x^3, x^2y\} + J^4(A).$$

Now there is only one possibility for the dimensions of the Loewy layers of A (see the table above in case (III.2.4)), namely $\dim_F(J^3(A)/J^4(A)) = \dim_F(J^4(A)/J^5(A)) = \dim_F(J^5(A)/J^6(A)) = 2$ and $\dim_F(J^6(A)/J^7(A)) = \dim_F J^7(A) = 1$. But this implies $\text{soc}(A) = F\{x^7, x^6y\} \subseteq [A, A]$, a contradiction.

(3): $J^2(A) = F\{x^2, xy, z_1 z_2\} + J^3(A)$. We may assume that $xz_1, xz_2, yz_1, yz_2 \in F\{x^2, xy\} + J^3(A)$ for otherwise we are in one of the previous subcases. Consequently,

$$J^3(A) = F\{x^3, x^2y, xz_1 z_2, x^2 z_1, xy z_1, z_1^2 z_2\} + J^4(A) = F\{x^3, x^2y\} + J^4(A).$$

From here on we get a contradiction exactly as in the previous subcase.

(4): $J^2(A) = F\{x^2, xz_1, xz_2\} + J^3(A)$. We may assume that $xy \in F\{x^2\} + J^3(A)$ for otherwise we are in one of the subcases considered before. We inductively obtain

$$J^n(A) = F\{x^n, x^{n-1} z_1, x^{n-1} z_2\} + J^{n+1}(A)$$

for every $n \geq 2$. But since $x^n, x^{n-1} z_1, x^{n-1} z_2 \in [A, A] + J^{n+1}(A)$ for any $n \geq 3$, we get the contradiction $\text{soc}(A) \cap [A, A] \neq 0$.

(5): $J^2(A) = F\{x^2, xz_i, yz_j\} + J^3(A) = F\{x^2, xz_i, z_j y\} + J^3(A)$ for some $i, j \in \{1, 2\}$. We may assume that $xy \in F\{x^2\} + J^3(A)$ and $xz_1, xz_2 \in F\{x^2, xz_i\} + J^3(A)$. Then we get

$$J^3(A) = F\{x^3, x^2 z_i, xy z_j, x^2 z_j, xz_i z_j, yz_j^2\} + J^4(A) = F\{x^3, x^2 z_i\} + J^4(A).$$

Hence, $J^n(A) = F\{x^n, x^{n-1}z_i\} + J^{n+1}(A)$ for any $n \geq 3$. But since $x^n, x^{n-1}z_i \in [A, A] + J^{n+1}(A)$ for any $n \geq 3$, this yields, as before, the contradiction $\text{soc}(A) \cap [A, A] \neq 0$.

(6): $J^2(A) = F\{x^2, xz_i, z_1z_2\} + J^3(A)$ for some $i \in \{1, 2\}$. We may assume that $xy \in F\{x^2\} + J^3(A)$ and $xz_1, xz_2, yz_1, yz_2 \in F\{x^2, xz_i\} + J^3(A)$. Hence,

$$J^3(A) = F\{x^3, x^2z_i, xz_1z_2, xz_1^2, z_1^2z_2\} + J^4(A) = F\{x^3, x^2z_i\} + J^4(A)$$

which leads to the same contradiction as the subcase before.

(7): $J^2(A) = F\{x^2, yz_1, yz_2\} + J^3(A) = F\{x^2, z_1y, z_2y\} + J^3(A)$. We may assume that $xy, xz_1, xz_2 \in F\{x^2\} + J^3(A)$. Hence,

$$J^3(A) = F\{x^3, xyz_1, xyz_2, x^2z_1, yz_1^2, yz_1z_2, x^2z_2, yz_2^2\} + J^4(A) = F\{x^3\} + J^4(A).$$

Therefore, $J^2(A) \subseteq Z(A)$ by Lemma 3.3, a contradiction.

(8): $J^2(A) = F\{x^2, yz_i, z_1z_2\} + J^3(A) = F\{x^2, z_izy, z_1z_2\} + J^3(A)$ for some $i \in \{1, 2\}$. We may assume that $xy, xz_1, xz_2 \in F\{x^2\} + J^3(A)$ and $yz_1, yz_2 \in F\{x^2, yz_i\} + J^3(A)$. Then

$$J^3(A) = F\{x^3, xyz_i, xz_1z_2, x^2z_i, yz_i^2, z_1z_2z_i\} + J^4(A) = F\{x^3\} + J^4(A)$$

which is a contradiction just as before. This finishes the proof of this proposition. \square

5 Concluding remarks

Coming back to the analysis of the generalized decomposition matrix Q in Section 2, we now know that only the possibilities (I) and (II) can occur for Q . In the example $G = D \rtimes 3_+^{1+2}$ mentioned in the introduction, one can show that case (I) occurs (for both of the two non-principal blocks of G). Thus, by Külshammer [20], case (I) occurs whenever D is normal (see also [29, Proposition 1.20]).

In view of [27, Remark 1.8], one might think that the generalized decomposition matrices Q_I and Q_{II} in case (I) and (II) respectively are linked via $PQ_I S = Q_{II}$ where $P, S \in \text{GL}(8, \mathbb{Z})$ and P is a signed permutation matrix (this is more general than changing basic sets). However, this is not the case. In fact, we conjecture that case (II) never occurs for Q .

By [14, Theorem 2], Q determines the perfect isometry class of B . Now we consider isotypies. Since the block b_{xy} is nilpotent, all its *ordinary* decomposition numbers equal 1. Let $Q_{b_x} \in \mathbb{Z}^{16 \times 3}$ be the ordinary decomposition matrix of b_x with respect to the basic set in Section 2. As usual, the trace of the contribution matrix $Q_{b_x} C_x^{-1} Q_{b_x}^T$ equals $l(b_x) = 3$ (see [25, Proposition 2.2]). Hence, its diagonal entries are all $3/16$ and the rows of Q_{b_x} have the form

$$(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)$$

(see (2.3)). It follows that

$$Q_{b_x} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^T$$

for a suitable order of $\text{Irr}(b_x)$. Exactly the same arguments apply for b_y . This shows that also the isotypy class of B is uniquely determined by Q (see [6, 5]). Note that Usami [31, Theorem 1.5] showed that there is only one such class provided $I(B) \cong C_3 \times C_3$ and $2 \neq p \neq 7$.

Generalizing our result, we note that the isomorphism type of $Z(B)$ is uniquely determined by local data whenever B has elementary abelian defect group of order 16 (not necessarily $l(B) = 1$). In fact, one can use the methods from the second section to construct the generalized decomposition matrix in the remaining cases (this has been done to some extent in [30, Proposition 16]). In particular, the character-theoretic version of Broué's Conjecture can be verified unless $l(B) = 1$. We omit the details. Even more, $Z(B)$ can be computed whenever B is any 2-block of defect at most 4. To see this one has to construct the generalized decomposition matrix for the non-abelian defect groups (see [29, Theorem 13.6]). Again, we do not go into the details.

We remark that it is also possible to determine the isomorphism type of $Z(B)$ as an algebra over \mathcal{O} . In fact, we may compute its structure constants as in Section 2 (these are integral).

Charles Eaton has communicated privately that he determined the Morita equivalence class of B by relying heavily on the classification of the finite simple groups (his methods are described in [10] where he handles the elementary abelian defect group of order 8). We believe that the methods of the present paper are of independent interest.

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