ON DEFECTS OF CHARACTERS AND DECOMPOSITION NUMBERS

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Abstract. We propose upper bounds for the number of modular constituents of the restriction modulo p of a complex irreducible character of a finite group, and for its decomposition numbers, in certain cases.

1. Introduction

Let G be a finite group and let p be a prime. In his fundamental paper [1], Richard Brauer studied the irreducible complex characters $\chi \in \mathrm{Irr}(G)$ such that $\chi(1)_p = |G|_p/p$, where here n_p denotes the largest power of p dividing the integer n. This was the birth of what later became the cyclic defect theory, developed by E. C. Dade [6], building upon work of J. A. Green and J. G. Thompson on vertices and sources. Of course, Brauer and Nesbitt had studied before the *defect zero* characters (those $\chi \in \mathrm{Irr}(G)$ with $\chi(1)_p = |G|_p$) proving that they lift irreducible modular characters in characteristic p. A constant in Brauer's work was to analyse the decomposition of the complex irreducible characters χ into modular characters:

$$\chi^0 = \sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi\varphi}\varphi,$$

where here χ^0 is the restriction of χ to the elements of G of order prime to p, and where we have chosen a set $\mathrm{IBr}(G)$ of irreducible p-Brauer characters of G. To better understand the *decomposition numbers* $d_{\chi\varphi}$ remains one of the challenges in Representation Theory.

If $\chi \in \mathrm{Irr}(G)$ let us write $\mathrm{IBr}(\chi^0) = \{\varphi \in \mathrm{IBr}(G) \mid d_{\chi\varphi} \neq 0\}$, and recall that the *defect* of χ is the integer d_χ with

$$p^{d_\chi}\chi(1)_p = |G|_p\,.$$

For $\chi \in \mathrm{Irr}(G)$ of defect one Brauer proved in [1] that all decomposition numbers $d_{\chi\varphi}$ are less than or equal to 1, and implicitly, that there are less than p characters $\varphi \in \mathrm{IBr}(G)$ occurring with multiplicity $d_{\chi\varphi} \neq 0$.

In order to gain insight into decomposition numbers in general, it is natural from this perspective to next study characters of defect two. This step looks innocent, but it

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deepens things in such a way that at present we can only guess what might be happening in general:

Conjecture A. *Let G be a finite group, p a prime. Let $\chi \in \mathrm{Irr}(G)$ with $|G|_p = p^2 \cdot \chi(1)_p$. Then:*

(1) $|\mathrm{IBr}(\chi^0)| \leq p^2 - 1$ *and*
(2) $d_{\chi\varphi} \leq p$ *for all* $\varphi \in \mathrm{IBr}(G)$.

It is remarkable that, as we shall prove below (Theorem 3.2), Conjecture A follows from the Alperin–McKay conjecture together with the work of K. Erdmann, but only for the prime $p = 2$. We do prove below Conjecture A for p-solvable groups (see Theorem 2.5) and for certain classes of quasi-simple and almost simple groups (see Theorem 4.2, Propositions 4.3 and 4.4 and Theorem 5.4). On the other hand, Conjecture A is wide open for example for groups of Lie type in non-defining characteristic (see Example 5.5, but also Proposition 5.9). As far as we are aware, no bounds for the number $|\mathrm{IBr}(\chi^0)|$ have been proposed before, and therefore, we move into unexplored territory here.

If Conjecture A is true then both of our bounds are sharp. Of course, the irreducible characters of degree p in a non-abelian group of order p^3 have decomposition numbers equal to p. Also, the irreducible character χ of degree $p^2 - 1$ in the semidirect product of $C_p \times C_p$ with a cyclic group of order $p^2 - 1$ acting faithfully satisfies $|\mathrm{IBr}(\chi^0)| = p^2 - 1$.

In view of Brauer's analysis of characters of defect one and our Conjecture A, it is tempting to guess that whenever $\chi(1)_p = |G|_p/p^3$, then $d_{\chi\varphi} \leq p^2$ and $|\mathrm{IBr}(\chi^0)| \leq p^3 - 1$. While we are not aware of any p-solvable counter-examples, still this is not true in general: For $p = 2$, $G = 3.J_3$ has a character χ such that $\chi(1)_p = |G|_p/p^3$, having 8 irreducible Brauer constituents. For $p = 3$, $G = Co_3$ has an irreducible character such that $\chi(1)_p = |G|_p/p^3$ with 13 occurring as a decomposition number. Perhaps other bounds are possible.

While studying Conjecture A, we came across a remarkable inequality, to which we have not yet found a counterexample.

Conjecture B. *Let G be a finite group, p a prime, and $\chi \in \mathrm{Irr}(G)$. Then*

$$|\mathrm{IBr}(\chi^0)| \cdot \chi(1)_p \leq |G|_p.$$

For characters of defect 1, this is the cited result of Brauer, and for defect 2 it follows from Conjecture A(1).

We shall prove below that Conjecture B is satisfied in symmetric groups and for simple groups of Lie type in defining characteristic (see Proposition 4.1 and Corollary 5.2). Our bound seems exactly the right bound for these classes of groups, and somehow this makes us think that Conjecture B points in the right direction. On the other hand, we do not know what is happening in other important classes of groups, like solvable groups or groups of Lie type in non-defining characteristic, for instance. Whether Conjecture B is true or false, we believe that the relation between $|\mathrm{IBr}(\chi^0)|$ and $|G|_p/\chi(1)_p$ is worth exploring. Of course, if $|G|_p/\chi(1)_p = 1$, then $|\mathrm{IBr}(\chi^0)| = 1$ by the Brauer–Nesbitt theorem.

So far we have not mentioned blocks, heights, or defect groups. These are, of course, related to Conjectures A and B. If $\chi \in \mathrm{Irr}(G)$, then $\chi \in \mathrm{Irr}(B)$ for a unique p-block B of

G, and $\chi(1)_p = p^{a-d+h}$, where $|G|_p = p^a$, $|D| = p^d$ is the order of a defect group D for B, d is the defect of the block B, and $h \geq 0$ is the height of χ. Hence, the defect d_χ of χ is

$$d_\chi = d - h\,.$$

Brauer's famous $k(B)$-conjecture asserts that $k(B) := |\mathrm{Irr}(B)| \leq |D| = p^d$. Since obviously $|\mathrm{IBr}(\chi^0)| \leq |\mathrm{IBr}(B)| =: l(B) < |\mathrm{Irr}(B)|$ (if $d > 0$), it follows that for characters of height zero both Conjectures A(1) and B are implied by Brauer's $k(B)$-conjecture. (In particular, by the Kessar–Malle solution of one implication of the Height Zero Conjecture [20], Conjectures A(1) and B follow from the $k(B)$-conjecture for characters in blocks with abelian defect groups.) It is worth mentioning that $l(B)$ is bounded by $\frac{1}{4}p^{2d} + 1$, using a well-known bound of $k(B)$ by Brauer and Feit.

A recent conjecture, formulated by Malle and Robinson in [25], is also related to the present work. Malle and Robinson have proposed that $l(B) \leq p^{s(D)}$, where $s(D)$ is the so called p-sectional rank of the group D. Hence Conjecture B follows from the Malle–Robinson conjecture for those irreducible characters whose height h is such that

$$s(D) + h \leq d. \tag{1}$$

Observe that the stronger inequality $l(B)\cdot\chi(1)_p \leq |G|_p$ is not always true. For example, let G be the central product of n copies of $\mathrm{SL}_2(3)$ where the centres of order 2 are identified. Then the principal 2-block B of G has a normal defect group and satisfies $l(B) = 3^n$, but there is an irreducible character $\chi \in \mathrm{Irr}(B)$ (deflated from the direct product of $\mathrm{SL}_2(3)$'s) such that $|G|_2/\chi(1)_2 = 2^{n+1}$. Other examples are the principal 2-block of J_3 or the principal 3-block of $\mathrm{SL}_6(2)$.

Finally, we come back to characters of defect 2 for small primes, but from the perspective of their relationship with their blocks and defect groups. Here we prove:

Theorem C. (a) *If $p = 2$ then the Alperin–McKay conjecture implies Conjecture A.*
(b) *If $p = 3$ then Robinson's ordinary weight conjecture implies $l(B) \leq 10$ for every block B containing a character χ as in Conjecture A.*

If there is an upper bound for $l(B)$ in Theorem C for arbitrary primes, we have not been able to find it. In the situation of Theorem C for p-solvable groups, we shall prove below that either $|D| \leq p^3$ or $|D| = p^4$ and $p \leq 3$, and the possible defect groups are classified. For non p-solvable groups, however, $|D|$ is unbounded, as shown by $\mathrm{SL}_2(q)$ with q odd and $p = 2$.

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2. p-solvable groups

We start with the proof of Conjecture A for p-solvable groups. In fact, we prove something more general (which will include certain p-constrained groups). Our notation for complex characters follows [18], and for Brauer characters [28]. If G is a finite group, $N \triangleleft G$, and $\theta \in \mathrm{Irr}(N)$, then $\mathrm{Irr}(G|\theta)$ is the set of complex irreducible characters $\chi \in \mathrm{Irr}(G)$ such that the restriction χ_N contains θ as a constituent. Notice that if $\psi^G = \chi \in \mathrm{Irr}(G)$ for ψ an irreducible character of the inertia group T of θ, then $|G : T|_p\psi(1)_p = \chi(1)_p$, and therefore $|G|_p/\chi(1)_p = |T|_p/\psi(1)_p$. Hence, $d_\chi = d_\psi$.

First we collect some rather well-known results.

Lemma 2.1. *Let p be a prime, and let U be a subgroup of* $\mathrm{GL}_2(p)$.

(a) *Assume that U has order divisible by p. Then either U has a normal Sylow p-subgroup or* $\mathrm{SL}_2(p) \subseteq U$.
(b) *If* $W \subseteq \mathrm{SL}_2(p) \subseteq U \subseteq \mathrm{GL}_2(p)$, *then W and U have trivial Schur multiplier.*
(c) *If* $W \subseteq \mathrm{SL}_2(p)$ *is a p'-subgroup, then either W is cyclic of order a divisor of $p-1$ or $p+1$, W has a normal cyclic subgroup of index 2, or* $W = \mathrm{SL}_2(3)$, $\texttt{SmallGroup}(48, 28)$ *or* $\mathrm{SL}_2(5)$.
(d) *Suppose that* $G = (C_p \times C_p) \rtimes U$ *in natural action, where* $\mathrm{SL}_2(p) \subseteq U \subseteq \mathrm{GL}_2(p)$. *If* $\chi \in \mathrm{Irr}(G)$, *and p divides $\chi(1)$, then $\chi(1) = p$.*
(e) *Suppose that L is an extra-special group of order p^3 and exponent p. Then we have that* $\mathrm{Aut}(L)/\mathrm{Inn}(L) \cong \mathrm{GL}_2(p)$. *If fact, if* $Z = Z(L)$, $A = \mathrm{Aut}(L)$ *and* $I = \mathrm{Inn}(L)$, *then* $A = C_A(Z) \rtimes \langle\sigma\rangle$, *where σ has order $p-1$, and* $C_A(Z)/I = \mathrm{SL}_2(p)$.

Proof. (a) This part follows from [21, 8.6.7].

(b) Checking $p = 3$ directly, we may assume $p \geq 5$. Suppose by way of contradiction that U has a proper covering group S with $1 \neq Z \leq Z(S) \cap S'$ and $S/Z \cong U$. Let $N \trianglelefteq S$ be the preimage of $\mathrm{SL}_2(p)$. It is a well-known fact that the Sylow subgroups of $\mathrm{SL}_2(p)$ are cyclic or quaternion groups. This implies that $\mathrm{SL}_2(p)$ (and any of its subgroups) has trivial Schur multiplier. In particular, N is not a covering group of $\mathrm{SL}_2(p)$. Since $\mathrm{SL}_2(p)$ is perfect for $p \geq 5$, we must have $N = N'Z$ and $Z \nsubseteq N'$. Since $S/N \cong U/\mathrm{SL}_2(p)$ is cyclic, it follows that $S' = N'$. But this gives the contradiction $Z \nsubseteq S'$.

(c) We follow the well-known classification of the subgroups of $\mathrm{L}_2(p)$. Notice that S has a unique involution, so the 2-subgroups of S are cyclic or quaternion. Set $Z := Z(S)$. Now, if WZ/Z is cyclic of order a divisor of $(p \pm 1)/2$, then it follows that WZ is abelian with cyclic Sylow subgroups. Thus WZ (and W) are cyclic of order dividing $p \pm 1$. Suppose now that H/Z is dihedral of order $2 \cdot (p \pm 1)$, where H is a p'-subgroup of S. Then H/Z has a cyclic subgroup of index 2. Thus H has a cyclic subgroup of index 2. Hence if W is a subgroup of H, it has a cyclic normal 2-complement Q, and a Sylow 2-subgroup P such that $|Q : C_Q(P)| \leq 2$. Since Q is cyclic, or generalised quaternion, it follows that W has a cyclic normal subgroup of index 2. Suppose next that $WZ/Z = \mathfrak{A}_4$. Then WZ has order 24, centre Z of order 2, and a unique involution. Hence WZ is $\mathrm{SL}_2(3)$. The proper subgroups of $\mathrm{SL}_2(3)$ are cyclic or quaternion of order 8. Suppose now that $WZ/Z = \mathfrak{S}_4$. Then WZ has order 48, centre Z of order 2 and a unique involution. Thus $WZ = \texttt{SmallGroup}(48, 28)$. The proper subgroups of this group are cyclic, have a cyclic normal subgroup of index 2, or are isomorphic to $\mathrm{SL}_2(3)$. Finally, if $WZ/Z = \mathfrak{A}_5$, then $WZ = \mathrm{SL}_2(5)$. The proper subgroups of $\mathrm{SL}_2(5)$ are already on our list.

(d) Write $V = C_p \times C_p$. Note that $\mathrm{Irr}(V)$ is then the dual of the natural module for U, and $\mathrm{SL}_2(p) \leq U$ acts transitively on $\mathrm{Irr}(V) \setminus \{1\}$. Let $\theta \in \mathrm{Irr}(V)$. If $\theta = 1$ then it extends to G and the constituents of θ^G are the inflations of characters of U. The ones of degree divisible by p are thus the extensions of the Steinberg character of $\mathrm{SL}_2(p)$ to U, all of degree p. Now assume that $\theta \neq 1$. Then up to conjugation the inertia group T of θ in U contains all elements $\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$ in U, hence T is metacyclic with abelian Sylow subgroups and normal Sylow p-subgroup. So θ extends to $V \rtimes T$, and the characters above θ have degrees prime to p. Since $|U : T|$ is prime to p, all characters in $\mathrm{Irr}(G|\theta)$ are of degree prime to p.

(e) Consider the canonical map $F : \mathrm{Aut}(L) \to \mathrm{Aut}(L/L')$. Then $\mathrm{Inn}(L)$ lies in the kernel of F. Assume conversely that f lies in $\ker(F)$. If L is generated by x and y, we have at most p^2 possibilities for $f(x)$ in xL' and $f(y)$ in yL'. On the other hand, $|\mathrm{Inn}(L)| = p^2$. Hence, $\ker(F) = \mathrm{Inn}(L)$ and F induces an embedding $\mathrm{Out}(L)$ in $\mathrm{GL}_2(p)$. Winter [38] shows that $|\mathrm{Out}(L)| = |\mathrm{GL}_2(p)|$. The rest follows from the main theorem in [38]. □

Lemma 2.2. *Let $S = \mathrm{SL}_2(p)$, where p is an odd prime, and let $\alpha, \beta \in \mathrm{Irr}(S)$ of degrees $\frac{p-1}{2}$ and $\frac{p+1}{2}$ such that $|\alpha(x) + \beta(x)|^2 = |C_V(x)|$ for all $x \in S$, where $V = C_p \times C_p$ is the natural module for S. Let $\Psi = \alpha + \beta$.*

(a) *We have that $\Psi(x) = \pm 1$ for p-regular non-trivial $x \in S$. Furthermore, the values of Ψ on p-regular elements do not depend of the choices of α and β.*

(b) *Let $\gamma \in \mathrm{Irr}(S)$ of degree p. Let $\Delta = \gamma\Psi$. Assume that $p \geq 7$. Then the irreducible p-Brauer constituents of Δ^0 appear with multiplicity less than or equal to $(p-1)/2$, and $|\mathrm{IBr}(\Delta^0)| \leq p$. This remains true when $p = 5$, except that the multiplicity of the unique $\mu \in \mathrm{IBr}(S)$ of degree 3 is 3.*

(c) *Suppose that $1 < W$ is a subgroup of S of order not divisible by p. Suppose that ψ is a character of W such that $\psi(1) = p$ and $\psi(x) = \pm 1$ for $1 \neq x \in W$. Let $\theta, \delta \in \mathrm{Irr}(W)$. Then*

$$[\psi, \theta\delta] \leq \frac{p}{2}$$

unless $W = Q_8$ and $\theta = \delta$ has degree 2, or $|W| = 2$. In these cases,

$$[\psi, \theta\delta] \leq \frac{p+1}{2}\,.$$

Proof. Part (a) follows immediately from the well-known character table of $\mathrm{SL}_2(p)$.

(b) From the ordinary character table it can be worked out easily that $\gamma\Psi$ is multiplicity free (and does not involve the trivial character nor the characters of degree $\frac{p-1}{2}$). The Brauer trees of the p-blocks of S of positive defect have an exceptional node of multiplicity 2, so any Brauer character occurs in at most three ordinary characters. This shows the first claim for $p \geq 7$. For $p = 5$, direct calculation suffices. The second assertion is immediate, as $l(S) = p$.

(c) If W has order 2, our assertion easily follows because $\psi(1) = p$, and $\psi(w) = \pm 1$ if $1 \neq w \in W$. Suppose that W is a p'-subgroup of S with order $|W| > 2$. We have checked (c) with GAP for primes $3 \leq p \leq 23$. Hence, we may assume that $p > 23$, if necessary.

Now W is one of the groups in Lemma 2.1(c). Let $\theta, \delta \in \mathrm{Irr}(W)$. Then

$$[\psi, \theta\delta] = \frac{1}{|W|} \sum_{w \in W} \psi(w)\theta(w^{-1})\delta(w^{-1}) \leq \frac{\theta(1)\delta(1)(|W| - 1 + p)}{|W|}\,,$$

using that $|\theta(w)| \leq \theta(1)$ for $\theta \in \mathrm{Irr}(W)$.

Suppose first that θ and δ are linear. Then we see that $[\psi, \theta\delta] \leq p/2$ if $p > 3$. Thus we may assume that W is non-abelian.

Suppose now that W has a normal abelian subgroup of index 2. If $\theta, \delta \in \mathrm{Irr}(W)$, then $\theta(1)\delta(1) \leq 4$. We see, assuming that $p \geq 23$, that $[\psi_W, \theta\delta] \leq p/2$ if $|W| \geq 12$. There is only one non-abelian group of order less than 12 with a unique involution, which is Q_8. If $\theta(1)\delta(1) \leq 2$, then $[\psi, \theta\delta] \leq p/2$. So we assume that $\theta = \delta$ has degree 2. Let $Z = Z(W)$. Then using that θ is zero off Z, we have that

$$[\psi, \theta\delta] = \frac{1}{8}(4p \pm 4) = \frac{p \pm 1}{2}\,.$$

Suppose now that $W = \mathrm{SL}_2(3)$. The largest character degree of W is 3, and there is a unique character θ with that degree. Assuming that $p \geq 23$, we have that $[\psi, \theta\delta] \leq p/2$, if $\theta(1)\delta(1) \leq 6$. Assume now that $\theta = \delta$ has degree 3. This character has Z in its kernel, and otherwise takes value 0 except on the unique conjugacy class of elements of order 4. On these six elements, θ has value -1. Hence

$$[\psi, \theta\delta] \leq \frac{1}{24}(9p + 9 + 6) \leq p/2$$

(if $p \geq 5$, which we are assuming).

Suppose now that $W = \texttt{SmallGroup}(48, 28)$. This group has a unique character $\theta \in \mathrm{Irr}(W)$ of degree 4. By using the values of this character, we have that

$$[\psi, \theta^2] \leq \frac{1}{48}(16p + 32) \leq p/2$$

for $p \geq 5$. If $\theta(1)\delta(1) = 12$, then $\theta\delta$ is zero except on $Z(W)$, and the inequality is clear. If $\theta(1)\delta(1) = 9$, we again use the character values to check the inequality. Finally, if $\theta(1)\delta(1) \leq 6$, then we do not need to use the character values. The case $W = \mathrm{SL}_2(5)$ is done similarly. $\square$

The character Ψ in Lemma 2.2 is relevant in the character theory of fully ramified sections, as we shall see.

Lemma 2.3. *Suppose that $L \lhd G$ is an extra-special group of order p^3 and exponent p, where $p \geq 5$ is odd. Let $Z = Z(L) \subseteq Z(G)$, and assume that $G/L \cong \mathrm{SL}_2(p)$ and that $C_G(L) = Z$. Let $\alpha, \beta \in \mathrm{Irr}(G/L)$ of degrees $\frac{p-1}{2}$ and $\frac{p+1}{2}$ such that $|\alpha(x)+\beta(x)|^2 = |C_V(x)|$ for all $x \in S$, where V is the natural module for S. Let $1 \neq \lambda \in \mathrm{Irr}(Z)$, and write $\lambda^L = p\eta$, where $\eta \in \mathrm{Irr}(L)$. Then η has a unique extension $\hat{\eta} \in \mathrm{Irr}(G)$ and $\hat{\eta}^0 = \alpha^0 + \beta^0$.*

Proof. Let $\Psi = \alpha + \beta$. If K/L is the centre of G/L, and $Q \in \mathrm{Syl}_2(K)$, then N/Z is the unique (up to G-conjugacy) complement of L/Z in G/Z, where $N = N_G(Q)$. This follows by the Frattini argument and the fact that Q acts on L/Z with no non-trivial fixed points. Now, N is a central extension of $\mathrm{SL}_2(p)$ so $N = Z \times N'$, where $N' \cong \mathrm{SL}_2(p)$. Since the Schur multiplier of G/L is trivial (by Lemma 2.1(b)), it follows that η extends to G by [18, Thm. 11.7]. Since G/L is perfect, there is a unique extension $\hat{\eta} \in \mathrm{Irr}(G)$ by Gallagher's theorem. Now, by [17, Thm. 9.1], there is a character ψ of degree p of G/L such that

$$\hat{\eta}_N = \psi\nu$$

for some linear character ν of N over λ. Since N' is perfect, we have that $\nu = \lambda \times 1_{N'}$. By the proof of [17, Thm. 4.8], we have that $\psi = \Psi$, and therefore $\hat{\eta}^0 = \Psi^0$. (Notice that there are two choices of Ψ, but Ψ^0 is uniquely determined by Lemma 2.2(a).) $\square$

Lemma 2.4. *Suppose that $H \leq G$ are finite groups. Let $\chi \in \mathrm{Irr}(G)$ and $\theta \in \mathrm{Irr}(H)$. Then we have $[\chi_H, \theta]^2 \leq |G : H|$.*

Proof. Write $\chi_H = e\theta + \Delta$, where Δ is a character of H or zero, and $[\Delta, \theta] = 0$. Then $\chi(1) \geq e\theta(1)$. By Frobenius reciprocity, we have that $\theta^G = e\chi + \Xi$, where Ξ is a character of G or zero. Hence $|G : H|\theta(1) \geq e\chi(1) \geq e^2\theta(1)$, and the proof is complete. □

Theorem 2.5. *Suppose that G is a finite group and let $\chi \in \mathrm{Irr}(G)$ be such that $\chi(1)_p = |G|_p/p^2$. Let $N = \mathbf{O}_{p'}(G)$ and $L = \mathbf{O}_p(G)$. Suppose that $N \subseteq Z(G)$, and that $C_G(L) \subseteq LN$. Then $|\mathrm{IBr}(\chi^0)| \leq p^2 - 1$ and $d_{\chi\varphi} \leq p$ for all $\varphi \in \mathrm{IBr}(G)$.*

Proof. If $\chi^0 \in \mathrm{IBr}(G)$, then $d_{\chi\varphi} \leq 1$ and $|\mathrm{IBr}(\chi^0)| = 1$. We may clearly assume that $\chi(1) > p$. Let $\theta \in \mathrm{Irr}(N)$ be the irreducible constituent of the restriction χ_N.

By hypothesis, we have that $C_G(L) = Z(L) \times N$. In particular, $L > 1$ (because otherwise G is a p'-group and $\chi^0 \in \mathrm{IBr}(G)$). Also, we have that $G/C_G(L) \cong U \subseteq \mathrm{Aut}(L)$. Since N is central, then $\mathbf{O}_p(G/N) = LN/N$, and thus $\mathbf{O}_p(U) \cong L/Z(L)$.

Let $\eta \in \mathrm{Irr}(L)$ be under χ. Then $\chi(1)_p/\eta(1)$ divides $|G|_p/|L|$ by [18, Cor. 11.29]. Using the hypothesis, we have that $|L|/\eta(1) \leq p^2$. Since $\eta(1) < |L|^{1/2}$ because L is a non-trivial p-group, we deduce that $|L| \leq p^3$. Also, $\eta(1) \leq p$.

If G/LN is a p-group, then $G = N \times L$. Then $\chi = \theta \times \eta$. Thus $\chi^0 = \eta(1)\theta$. Hence $d_{\chi\varphi} = \eta(1) \leq p$, and $|\mathrm{IBr}(\chi^0)| = 1$, so the theorem is true in this case too.

Suppose that θ extends to some $\gamma \in \mathrm{Irr}(G)$. Notice that $\gamma^0 \in \mathrm{IBr}(G)$ also extends θ. By Gallagher's Corollary 6.17 of [18], we know that $\chi = \beta\gamma$ for some $\beta \in \mathrm{Irr}(G/N)$. If $\beta^0 = d_1\tau_1 + \ldots + d_s\tau_s$, where $\tau_i \in \mathrm{IBr}(G/N)$ are distinct, then we have that

$$\chi^0 = d_1\tau_1\gamma^0 + \ldots + d_s\tau_s\gamma^0$$

and that the $\tau_i\gamma^0$ are also distinct and irreducible (using that γ^0 is linear). In particular, we see that if θ extends to G, then the theorem holds for G if it holds for G/N.

(a) Assume first that $\eta(1) = 1$. We have then that $|L| \leq p^2$, by the third paragraph of this proof. In particular, L is abelian. In this case, $G/LN \cong U \subseteq \mathrm{Aut}(L)$ and $\mathbf{O}_p(U) = 1$.

(a1) If L is a Sylow p-subgroup of G, then by [28, Thm. 10.20], we have that $\mathrm{Irr}(G|\theta)$ is a block of G, which has defect group L. By the $k(GV)$-theorem ([11]), we have that $|\mathrm{Irr}(G|\theta)| \leq p^2$. Hence $|\mathrm{IBr}(G|\theta)| \leq p^2 - 1$ by [28, Thm. 3.18]. Thus $|\mathrm{IBr}(\chi^0)| \leq p^2 - 1$. Also, notice that the decomposition numbers are

$$d = [\chi_H, \mu]$$

for $\mu \in \mathrm{Irr}(H)$, where H is a p-complement of G. By Lemma 2.4, $d^2 \leq |G : H| = p^2$, and the theorem is true in this case.

(a2) We assume that L is not a Sylow p-subgroup of G. Suppose first that L is cyclic. Notice that L cannot have order p, since p^2 divides the order of G (and $|\mathrm{Aut}(C_p)|$ is not divisible by p). Suppose that $L = C_{p^2}$. Then G/LN is cyclic. Since $\mathbf{O}_p(G/LN) = 1$, we conclude that G/LN is cyclic of order dividing $p - 1$. Thus L is a normal Sylow p-subgroup of G, a contradiction.

Assume now that $L = C_p \times C_p$. In this case $G/LN \cong U \subseteq \mathrm{GL}_2(p)$. Thus $|U|_p = p$. Recall that $\mathbf{O}_p(U) = 1$. By Lemma 2.1(a), we conclude that $\mathrm{SL}_2(p) \subseteq U \subseteq \mathrm{GL}_2(p)$. Therefore U has trivial Schur multiplier by Lemma 2.1(b). Now, consider $\hat{\theta} = 1_L \times \theta$. By [18, Thm. 11.7], we have that $\hat{\theta}$ extends to G. In particular, so does θ. Again, by the fifth paragraph of this proof, we may assume in this case that $N = 1$. If $p = 2$, then

$G = \mathfrak{S}_4$ and $\chi(1) = 2$, so the theorem is true in this case. If p is odd, let $Z/L \subseteq Z(G/L)$ of order 2, and let $Q \in \mathrm{Syl}_2(Z)$. Since Q acts on L as the minus identity matrix, we have that $C_L(Q) = 1$. Therefore G is the semidirect product of L with $N_G(Q)$. In this case, by Lemma 2.1(d), we have that $\chi(1) = p$, and we are also done in this case.

(b) Assume now that $\eta(1) = p$. Hence L is extraspecial of order p^3 and exponent p or p^2. Write $Z = Z(L)$. We have that $G/ZN \cong U \subseteq \mathrm{Aut}(L)$ and $\mathbf{O}_p(U) = C_p \times C_p$. Also, write $\eta_Z = \eta(1)\lambda$, where $1 \neq \lambda \in \mathrm{Irr}(Z)$.

(b1) Suppose first that $p = 2$. Then $L = D_8$ or Q_8. If $L = D_8$, then $\mathrm{Aut}(L)$ is a 2-group, and then $G = L \times N$, and the theorem is true in this case. If $L = Q_8$, then $\mathrm{Aut}(Q_8)$ is $\mathfrak{S}_4$. Then $G/(L \times N)$ is a subgroup of $\mathfrak{S}_3$. Then $\theta \times 1_L$ (and therefore θ) extends to G. By the fifth paragraph of this proof, we may therefore assume that $N = 1$. Thus $|G| \leq 48$. In this case, $G = \mathrm{SL}_2(3), \mathrm{GL}_2(3)$ or the *fake* $\mathrm{GL}_2(3)$ (`SmallGroup`(48,28)). If $G = \mathrm{SL}_2(3)$, then $\chi(1) = 2$, and we are done. In the remaining cases, $\chi(1) = 4$, $|\mathrm{IBr}(\chi^0)| = 2$ and $d_{\chi\varphi} = 1$ or 2. Hence, we may assume that p is odd.

(b2) Suppose first that L is extra-special of exponent p^2, p odd. By [38], we have that $\mathrm{Aut}(L) = X \rtimes C_{p-1}$, were $|X| = p^3$ has as a normal subgroup $\mathrm{Inn}(L)$, and C_{p-1} acts Frobenius on Z. Thus $G/(L \times N)$ has cyclic Sylow subgroups. It follows that $1_L \times \theta$ (and therefore θ) extends to G. Recall that $\mathbf{O}_p(G/Z) = L/Z$. Since G/L is a subgroup of $C_p \rtimes C_{p-1}$ and this group has a normal Sylow p-subgroup, it follows that G/L is cyclic of order dividing $p-1$. Now, G is the semidirect product of L with a cyclic group C of order h dividing $p-1$ that acts Frobenius on Z. Since C acts Frobenius on Z, it follows that the stabiliser $I_G(\lambda) = L$. Since $\lambda^L = p\eta$, it follows that $I_G(\eta) = L$. Hence $\chi = \eta^G$. Then $\chi_C = p\rho$, where ρ is the regular character of C. Thus $d_{\chi\varphi} = p$ for every $\varphi \in \mathrm{IBr}(G)$ and $|\mathrm{IBr}(\chi^0)| = h \leq p-1$. The theorem follows in this case too.

(b3) So finally assume that L is extra-special of exponent p, p odd. We have that $\mathrm{Aut}(L)/\mathrm{Inn}(L) = \mathrm{GL}_2(p)$ by Lemma 2.1(e). Thus G/LN is isomorphic to a subgroup W of $\mathrm{GL}_2(p)$ with $\mathbf{O}_p(W) = 1$. Write $C = C_G(Z)$. Notice that C/LN maps into $\mathrm{SL}_2(p)$ by [38], and therefore this group has at most one involution. Also, G/C is a cyclic p'-group (because $\mathrm{GL}_2(p)/\mathrm{SL}_2(p)$ is) that acts Frobenius on Z. Thus, our group W has a normal subgroup that has at most one involution and with cyclic p'-quotient. Also notice that C is the stabiliser of λ (and of η) in G. Furthermore, we claim that the stabiliser of any $\gamma \in \mathrm{Irr}(C/LN)$ in G has index at most 2 in G. This is because $\mathrm{GL}_2(p)$ has a centre of order $p-1$ that intersects with $\mathrm{SL}_2(p)$ in its unique subgroup of order 2. Write $\chi = \mu^G$, where $\mu \in \mathrm{Irr}(C)$.

(b.3.1) If L is a Sylow p-subgroup of G, we claim that $|\mathrm{IBr}(\chi^0)| \leq p^2 - 1$. Let H be a p-complement of G. Then H/N is isomorphic to a p'-subgroup of $\mathrm{GL}_2(p)$. By the $k(GV)$-theorem applied to $\Gamma = (C_p \times C_p) \rtimes H/N$, we have that $k(\Gamma) \leq p^2$. Since Γ has a unique p-block, it follows that $k(H/N) \leq p^2 - 1$. Now, $|\mathrm{IBr}(\chi^0)| \leq |\mathrm{Irr}(H|\theta)| \leq k(H/N)$, and the claim is proved. (The last inequality follows from Problem 11.10 of [18].)

(b.3.2) If L is a Sylow p-subgroup of G, then $d_{\chi\varphi} \leq p$:

Let H be a p-complement of G. Write $C \cap H = Q$. Also, $U = HZ \cap C$ is the unique complement of L/Z in C/Z up to C-conjugacy. By Lemma 2.1(b) (and [18, Thm. 11.7]), we have that $1_L \times \theta$ has a (linear) extension $\tilde{\theta} \in \mathrm{Irr}(C)$. Therefore $\mu = \beta\tilde{\theta}$, where

$\beta \in \mathrm{Irr}(C/N)$. Notice that β lies over η. By [17, Thm. 9.1], we have that

$$\beta_U = \psi\beta_0$$

for some $\beta_0 \in \mathrm{Irr}(U/N)$, where ψ is a character of C/N of degree p. Since $C_{L/Z}(x)$ is trivial for non-trivial p-regular xN, it follows that $\psi(xN) = \pm 1$ for p-regular $Nx \neq N$. (The values of ψ are given in page 619 of [17], and it can be checked that ψ is the restriction of the character Ψ of $\mathrm{SL}_2(p)$ given in Lemma 2.2(a) under suitable identification.) Notice that $(\beta_0)_Q \in \mathrm{Irr}(Q/N)$ because Q is central in $U = ZQ$. Now, let $\nu \in \mathrm{Irr}(Q|\theta)$. By Gallagher, we have that $\nu = \tilde{\theta}_Q\tau$ for some $\tau \in \mathrm{Irr}(Q/N)$. Then

$$[\mu_Q, \nu] = [\beta_Q\tilde{\theta}_Q, \tilde{\theta}_Q\tau] = [\psi_Q(\beta_0)_Q, \tau]\,.$$

Now, by Lemma 2.2(c) applied in Q/N this number is less than $p/2$, except if $Q/N = Q_8$ and τ is the unique character of degree 2. In this case, this number is less than $\frac{p+1}{2}$, and τ extends to H because H/Q is cyclic. Hence, if $\rho \in \mathrm{Irr}(H|\theta)$, then

$$[\chi_H, \rho] = [\mu_Q, \rho_Q]\,.$$

If ρ_Q is irreducible, then we are done. Let $\tau \in \mathrm{Irr}(Q|\theta)$ be under ρ. We know that the stabiliser I of τ in H has index at most 2. If $I = H$, then ρ_Q is irreducible (because H/Q is cyclic). We conclude that $|H : I| = 2$. Using that I/Q is cyclic, we have that $\rho_Q = \tau + \tau^x$, where $x \in H \setminus I$. In this case,

$$[\chi_H, \rho] = [\mu_Q, \tau + \tau^x] \le p/2 + p/2 = p\,.$$

(b.3.3) We may assume that $p > 3$: Else we have that G/LN is a subgroup of $\mathrm{GL}_2(3)$. All subgroups of $\mathrm{GL}_2(3)$ have trivial Schur multiplier except for $C_2 \times C_2$, D_8 and D_{12}. By the requirements in (b3), only $C_2 \times C_2$ can occur. But then, L is a normal Sylow p-subgroup, and again the theorem holds in this case.

(b.3.4) Suppose finally that p divides $|W|$, and that $p > 3$. Then $\mathrm{SL}_2(p) \subseteq W \subseteq \mathrm{GL}_2(p)$ by Lemma 2.1(a). Now, by considering the character $1_L \times \theta$ and using Lemma 2.1(b), we may again assume that θ extends to G, and therefore that $N = 1$ in this case. Now, $C = C_G(Z)$ is such that $\mathrm{SL}_2(p) \cong C/L \lhd G/L$. By Lemma 2.3, we have that η has a unique extension $\hat{\eta} \in \mathrm{Irr}(C)$. Hence, μ, the Clifford correspondent of χ over η is such that $\mu = \gamma\hat{\eta}$ for a unique $\gamma \in \mathrm{Irr}(C/L)$ of degree p. Also, by Lemma 2.3, we know that $\hat{\eta}^0 = \Psi^0$. Therefore, by Lemma 2.2(b), we have that

$$\mu^0 = d_1\varphi_1 + \ldots + d_k\varphi_k$$

for some distinct $\varphi_i \in \mathrm{IBr}(C/L)$, with $k \le p$, and $d_i < p/2$, except in the case where $p = 5$. In this latter case, we still have that $k \le p$ and that $d_i \le p/2$, except for the unique irreducible p-Brauer character of degree 3, call it φ_1, which is such that $d_1 = 3$. Now, since G/C is cyclic of p'-order and $|Z(\mathrm{GL}_2(p))| = p - 1$, it follows that the stabiliser T_i of φ_i in G has index at most 2. Also φ_i extends to T_i ([28, Cor. 8.12]) and the irreducible constituents of $(\varphi_i)^{T_i}$ all appear with multiplicity 1, by [28, Thm. 8.7] and Gallagher's theorem for Brauer characters [28, Thm. 8.20]. Also, they all induce irreducibly to G by the Clifford correspondence for Brauer characters. Hence, the number of Brauer irreducible constituents of $\chi^0 = (\mu^0)^G$ is less than or equal to $k \cdot (p-1) \le p^2 - p$, and the decomposition numbers are at most $2d_i \le p$, except if $p = 5$ and $i = 1$. In this case, φ_1 is G-invariant, because it is the unique irreducible p-Brauer character of $\mathrm{SL}_2(5)$ of degree 3. Hence, for $j > 1$ the Brauer character φ_j^G does not contain any irreducible constituent of φ_1^G. So the irreducible constituents of φ_1^G appear in χ^0 with multiplicity $3 < 5 = p$. □

Now, we can finally prove the p-solvable case of Conjecture A.

Corollary 2.6. *Suppose that G is a finite p-solvable group and let $\chi \in \mathrm{Irr}(G)$ be such that $\chi(1)_p = |G|_p/p^2$. Then $|\mathrm{IBr}(\chi^0)| \le p^2 - 1$ and $d_{\chi\varphi} \le p$ for all $\varphi \in \mathrm{IBr}(G)$.*

Proof. Write $\mathbf{O}_{p'}(G) = N$. We argue by induction on $|G : N|$. Let $\theta \in \mathrm{Irr}(N)$ be under χ.

Let T be the inertia group of θ in G and let $\psi \in \mathrm{Irr}(T|\theta)$ be the Clifford correspondent of χ over θ. Since $\psi^G = \chi$, then we know that $d_\psi = 2$. Now we apply the Fong–Reynolds Theorem 9.14 of [28] to conclude that we may assume that θ is G-invariant.

By using ordinary/modular character triples (see Problem 8.13 of [28]), we may replace (G, N, θ) by some other triple (Γ, M, λ), where $M = \mathbf{O}_{p'}(\Gamma) \subseteq Z(\Gamma)$. Hence, by working now in Γ, it is no loss to assume that $N \subseteq Z(G)$. Now, if $L = \mathbf{O}_p(G)$, then we have that $C_G(L) \subseteq LN$, and we may apply Theorem 2.5 to conclude. □

3. Proof of Theorem C

A well-known theorem by Taussky asserts that a non-abelian 2-group P has maximal (nilpotency) class if and only if $|P/P'| = 4$ (see [16, Satz III.11.9]). In this case P is a dihedral group, a semidihedral group or a quaternion group. Of course, $|P/P'|$ is the number of linear characters of P. Our next result indicates that Taussky's theorem holds for blocks, assuming the Alperin–McKay conjecture. For this we need:

Lemma 3.1. *Let B be a 2-block of G with defect group $D \trianglelefteq G$. Then $k_0(B) = 4$ if and only if D is dihedral (including Klein four), semidihedral, quaternion or cyclic of order 4.*

Proof. If D is one of the listed groups, then $k_0(B) = 4$ by work of Brauer and Olsson (see [33, Thm. 8.1]). Now suppose conversely that $k_0(B) = 4$. By Reynolds [30], we may also assume that D is a Sylow 2-subgroup of G. By [23, Thm. 6], B dominates a block $\overline{B}$ of G/D' with defect group $\overline{D} := D/D'$ and $k(\overline{B}) = k_0(B) = 4$. Using Taussky's theorem, it suffices to show $|\overline{D}| = 4$.

By way of contradiction, suppose that $2^d := |\overline{D}| > 4$. Let $(1, \overline{B}) = (x_1, b_1), \dots, (x_r, b_r)$ be a set of representatives for the conjugacy classes of $\overline{B}$-subsections. Then

$$\sum_{i=1}^{r} l(b_i) = k(\overline{B}) = 4.$$

By [22, Thm. A], we have $l(\overline{B}) \ge 2$ and $r \le 3$. Let I be the inertial quotient of $\overline{B}$. Then I has odd order and so is solvable by Feit–Thompson. The case $r = 3$ is impossible, since 2^d is the sum of I-orbit lengths. Hence, $r = 2$ and $\overline{D}$ is elementary abelian.

Since $\overline{D} \rtimes I$ is a solvable 2-transitive group, Huppert [15] implies that I lies in the semilinear group $\Gamma\mathrm{L}_1(2^d) \cong C_{2^d-1} \rtimes C_d$. Let $N \trianglelefteq I$ with $N \le C_{2^d-1}$ and $I/N \le C_d$. Then N acts semiregularly on $\overline{D} \setminus \{1\}$. Hence, $C_I(x_2)$ is cyclic (as a subgroup of I/N). On the other hand, $C_I(x_2)$ is the inertial quotient of b_2. It follows that $l(b_2) = |C_I(x)|$ is odd and therefore $l(b_2) = 1$. Consequently, I acts regularly on $\overline{D} \setminus \{1\}$. This implies that all Sylow subgroups of I are cyclic. Hence, by a theorem of Külshammer (see [33, Theorem 1.19]), $\overline{B}$ is Morita equivalent to the group algebra of $\overline{D} \rtimes I$. We obtain the contradiction $l(\overline{B}) = k(I) > 3$. □

Our next result includes the first part of Theorem C.

Theorem 3.2. *Let B be a 2-block of G satisfying the Alperin–McKay conjecture. If there exists a character $\chi \in \mathrm{Irr}(B)$ such that $\chi(1)_2 = |G|_2/4$, then $|\mathrm{IBr}(\chi^0)| \le l(B) \le 3$ and $d_{\chi\phi} \le 2$ for all $\phi \in \mathrm{IBr}(B)$.*

Proof. By a result of Landrock (see [33, Prop. 1.31]), $k_0(B) = 4$. Hence, Lemma 3.1 applies. If B has defect 2, then B is Morita equivalent to the principal block of a defect group D of B, of $\mathfrak{A}_4$ or $\mathfrak{A}_5$. The claim follows easily in this case. Thus, we may assume that B has defect at least 3. Then by work of Brauer and Olsson (see [33, Thm. 8.1]), $l(B) \le 3$. The claim about the decomposition numbers follows from the tables at the end of [9]. □

Remark 3.3. For every defect $d \ge 2$ there are 2-blocks with defect d containing an irreducible character χ such that $\chi(1)_2 = |G|_2/4$. This is clear for $d = 2$ and for $d \ge 3$ one can take the principal block of $\mathrm{SL}_2(q)$ where q is a suitable odd prime power. These blocks have quaternion defect groups. Similarly, the principal 2-block of $\mathrm{GL}_2(q)$ where $q \equiv 3 \pmod 4$ gives an example with semidihedral defect group. On the other hand, Brauer showed that there are no examples with dihedral defect group of order at least 16 (see [33, Theorem 8.1]).

In order to say something about odd primes, we need to invoke a stronger conjecture known as Robinson's *ordinary weight conjecture* (see [33, Conj. 2.7]). Robinson gave the following consequence of his conjecture which is relevant to our work.

Lemma 3.4 ([31, Lemma 4.7])**.** *Let B be a p-block of G with defect group D satisfying the ordinary weight conjecture. Assume that there exists $\chi \in \mathrm{Irr}(B)$ such that $\chi(1)_p = |G|_p/p^2$. Then $|D| = p^2$ or D has maximal class. Let $\mathcal{F}$ be the fusion system of B. If $|D| \ge p^4$, then D contains an $\mathcal{F}$-radical, $\mathcal{F}$-centric subgroup Q of order p^3 such that $\mathrm{SL}_2(p) \le \mathrm{Out}_{\mathcal{F}}(Q) \le \mathrm{GL}_2(p)$. In particular, $Qd(p)$ is involved in G. If $p = 2$, then $Q \cong Q_8$ and if $p > 2$, then Q is the extraspecial group p_+^{1+2} with exponent p.*

Conversely, if B is any block (satisfying the ordinary weight conjecture) with an $\mathcal{F}$-radical, $\mathcal{F}$-centric subgroup Q as above, then $\mathrm{Irr}(B)$ contains a character χ with $\chi(1)_p = |G|_p/p^2$.

Observe that a p-group P of order $|P| \ge p^3$ has maximal class if and only if there exists $x \in P$ with $|C_P(x)| = p^2$ (see [16, Satz III.14.23]). We will verify that latter condition in a special case in Proposition 5.7.

Lemma 3.4 implies for instance that the group $p_+^{1+2} \rtimes \mathrm{SL}_2(p)$ contains irreducible characters χ with $\chi(1)_p = p^2$. Hence, for every prime p there are p-blocks of defect 4 with characters of defect 2. As another consequence we conditionally extend Landrock's result mentioned in the proof of Theorem 3.2 to odd primes.

Proposition 3.5. *Let B be a p-block of G satisfying the ordinary weight conjecture. If there exists $\chi \in \mathrm{Irr}(B)$ such that $\chi(1)_p = |G|_p/p^2$, then $k_0(B) \le p^2$.*

Proof. By Lemma 3.4, a defect group D of B has maximal class or $|D| = p^2$. In any case $|D/D'| = p^2$. The ordinary weight conjecture implies the Alperin–McKay conjecture (blockwise) and by [23], the Alperin–McKay conjecture implies Olsson's conjecture $k_0(B) \leq |D/D'| = p^2$. □

If we also assume the Eaton-Moretó conjecture [8] for B, it follows that $k_1(B) > 0$ in the situation of Proposition 3.5. This is because a p-group P of maximal class has a (unique) normal subgroup N such that P/N is non-abelian of order p^3. Hence, P has an irreducible character of degree p.

Proposition 3.6. *Let B be a p-block of a p-solvable group G with $\chi \in \mathrm{Irr}(B)$ such that $\chi(1)_p = |G|_p/p^2$. Then one of the following holds:*
(1) *B has defect 2 or 3.*
(2) *$p = 2 = l(B)$ and B has defect group Q_{16} or SD_{16}. Both cases occur.*
(3) *$p = 3$ and B has defect group $\texttt{SmallGroup}(3^4, a)$ with $a \in \{7, 8, 9\}$. All three cases occur.*

Proof. By Haggarty [12], B has defect at most 4. Moreover, if B has defect 4, then $p \leq 3$. Let D be a defect group of B. Since the ordinary weight conjecture holds for p-solvable groups, Lemma 3.4 implies that D has maximal class. If $p = 2$, then the fusion system $\mathcal{F}$ of B contains an $\mathcal{F}$-radical, $\mathcal{F}$-centric subgroup isomorphic to Q_8. Hence, $D \in \{Q_{16}, SD_{16}\}$. On the other hand, every fusion system of a block of a p-solvable group is constrained. This implies that there is only one $\mathcal{F}$-radical, $\mathcal{F}$-centric subgroup in D. It follows from [33, Thm. 8.1] that $l(B) = 2$. Examples are given by the two double covers of $\mathfrak{S}_4$.

Now suppose that $p = 3$. According to GAP [37], there are four possibilities for D: $\texttt{SmallGroup}(3^4, a)$ with $a \in \{7, 8, 9, 10\}$. In case $a = 10$, D has no extraspecial subgroup of order 27 and exponent 3. Hence, Lemma 3.4 excludes this case. Conversely, examples for the remaining three cases are given by the (solvable) groups $\texttt{SmallGroup}(3^4 \cdot 8, b)$ with $b \in \{531, 532, 533\}$. □

In the following we (conditionally) classify the possible defect groups in case $p = 3$. This relies ultimately on Blackburn's classification of the 3-groups of maximal class. Unfortunately, there is no such classification for $p > 3$.

Proposition 3.7. *Let B be a 3-block of G satisfying the ordinary weight conjecture. Suppose that there exists $\chi \in \mathrm{Irr}(B)$ such that $\chi(1)_3 = |G|_3/9$. If B has defect $d \geq 4$, then there are at most three possible defect groups of order 3^d up to isomorphism. If d is even, they all occur, and if d is odd, only one of them occurs. In particular, we have examples for every defect $d \geq 2$.*

Proof. In case $d = 4$ we can argue as in Proposition 3.6. Thus, suppose that $d \geq 5$. By Lemma 3.4, a defect group D of B has maximal class and contains a radical, centric subgroup $Q \cong 3^{1+2}_+$. In particular, B is not a controlled block. Since $d \geq 5$, it is known that D has 3-rank 2 (see [7, Thm. A.1]). Hence, the possible fusion systems $\mathcal{F}$ of B are described in [7, Thm. 5.10][1]. It turns out that D is one of the groups $B(3, d; 0, \gamma, 0)$

[1]Chris Parker has informed the authors that some fusion systems have been overlooked in [7]. However these fusion systems do not have a radical, centric subgroup isomorphic to 3^{1+2}_+, see `https://arxiv.org/abs/1809.01957v1`

with $\gamma \in \{0, 1, 2\}$. If d is odd, then $\gamma = 0$. In all these cases examples are given such that $\mathrm{Out}_{\mathcal{F}}(Q) \cong \mathrm{SL}_2(3)$. We can pick for instance the principal 3-blocks of $3.\mathrm{PGL}_3(q)$ and ${}^2F_4(q)$ for a suitable prime power q. An inspection of the character tables in [36, 24] shows the existence of χ. □

Now we are in a position to cover the second part of Theorem C (recall that the ordinary weight conjecture for all blocks of all finite groups implies Alperin's weight conjecture).

Corollary 3.8. *Let B be a 3-block of G with defect d satisfying the ordinary weight conjecture and Alperin's weight conjecture. If there exists $\chi \in \mathrm{Irr}(B)$ such that $\chi(1)_3 = |G|_3/9$, then*

$$l(B) \le \begin{cases} 8 & \text{if } 4 \ne d \equiv 0 \pmod 2, \\ 9 & \text{if } d \equiv 1 \pmod 2, \\ 10 & \text{if } d = 4. \end{cases}$$

Proof. Let D be a defect group of B. We may assume that $|D| \ge 27$. Then D has maximal class. In case $|D| = 27$ and $\exp(D) = 9$, Watanabe has shown that $l(B) \le 2$ without invoking any conjecture (see [33, Thm. 1.33 and 8.8]). Now assume that $D \cong 3^{1+2}_+$. Then the possible fusion systems $\mathcal{F}$ of B are given in [32]. To compute $l(B)$ we use Alperin's weight conjecture in the form [33, Conj. 2.6]. Let $Q \le D$ be $\mathcal{F}$-radical and $\mathcal{F}$-centric. For $Q = D$ we have $\mathrm{Out}_{\mathcal{F}}(D) \le SD_{16}$. Hence, regardless of the Külshammer–Puig class, D contributes at most 7 to $l(B)$ (for a definition of the Külshammer–Puig class see [33, Theorem 7.3]). For $Q < D$ we have $\mathrm{Out}_{\mathcal{F}}(Q) \in \{\mathrm{SL}_2(3), \mathrm{GL}_2(3)\}$. The groups $\mathrm{SL}_2(3)$ and $\mathrm{GL}_2(3)$ have trivial Schur multiplier and exactly one respectively two irreducible characters of 3-defect 0. Hence, each $\mathcal{F}$-conjugacy class of such a subgroup Q contributes at most 2 to $l(B)$. There are at most two such subgroups up to conjugation. Now an examination of the tables in [32] yields the claim for $d = 3$. Note that $l(B) = 9$ only occurs for the exceptional fusion systems on ${}^2F_4(2)'$ and ${}^2F_4(2)$.

Now let $d \ge 5$. As in Proposition 3.7 there are at most three possibilities for D and the possible fusion systems $\mathcal{F}$ are listed in [7, Thm. 5.10]. We are only interested in those cases where there exists an $\mathcal{F}$-radical, $\mathcal{F}$-centric, extraspecial subgroup of order 27. We have $\mathrm{Out}_{\mathcal{F}}(D) \le C_2 \times C_2$. Hence, D contributes at most 4 to $l(B)$. Now assume that $Q < D$ is $\mathcal{F}$-radical and $\mathcal{F}$-centric (i. e., $\mathcal{F}$-Alperin in the notation of [7]). Then as above, $\mathrm{Out}_{\mathcal{F}}(Q) \in \{\mathrm{SL}_2(3), \mathrm{GL}_2(3)\}$. There are at most three such subgroups up to conjugation. The claim $l(B) \le 9$ follows easily. In case $l(B) = 9$, $\mathcal{F}$ is the fusion system of ${}^2F_4(q^2)$ for some 2-power q^2 or $\mathcal{F}$ is exotic. In both cases d is odd. The principal block of ${}^2F_4(q^2)$ shows that $l(B) = 9$ really occurs (see [24]).

It remains to deal with the case $d = 4$. By the results of [7], we may assume that $D \cong C_3 \wr C_3$. The fusion systems on this group seem to be unknown. Therefore, we have to analyse the structure of D by hand. Up to conjugation, D has the following candidates of $\mathcal{F}$-radical, $\mathcal{F}$-centric subgroups: $Q_1 \cong C_3 \times C_3$, $Q_2 \cong 3^{1+2}_+$, $Q_3 \cong C_3 \times C_3 \times C_3$ and $Q_4 = D$. We may assume that $Q_1 \le Q_2$. As before, Q_2 must be $\mathcal{F}$-radical and $\mathrm{Out}_{\mathcal{F}}(Q_2) \in \{\mathrm{SL}_2(3), \mathrm{GL}_2(3)\}$. Hence, Q_1 is conjugate to D' under $\mathrm{Aut}_{\mathcal{F}}(Q_2)$. Since $C_D(D') = Q_3$, we conclude that Q_1 is not $\mathcal{F}$-centric. Now let $\alpha \in \mathrm{Aut}_{\mathcal{F}}(Q_2)$ the automorphism inverting the elements of $Q_2/Z(Q_2)$. Then α acts trivially on $Z(Q_2)$. By the saturation property of fusion systems, α extends to D. Since Q_3 is the only abelian maximal subgroup of D, the extension of α restricts to Q_3. Since $Z(Q_2) \le Q_3$, it follows from a GAP computation that $\mathrm{Out}_{\mathcal{F}}(Q_3) \in \{\mathfrak{S}_4, \mathfrak{S}_4 \times C_2\}$. Using similar arguments we end up with two configurations:

(i) $\mathrm{Out}_{\mathcal{F}}(Q_2) \cong \mathrm{SL}_2(3)$, $\mathrm{Out}_{\mathcal{F}}(D) \cong C_2$ and $\mathrm{Out}_{\mathcal{F}}(Q_3) \cong \mathfrak{S}_4$.

(ii) $\mathrm{Out}_{\mathcal{F}}(Q_2) \cong \mathrm{GL}_2(3)$, $\mathrm{Out}_{\mathcal{F}}(D) \cong C_2 \times C_2$ and $\mathrm{Out}_{\mathcal{F}}(Q_3) \cong \mathfrak{S}_4 \times C_2$.

In the first case we have $l(B) \le 5$ (occurs for the principal block of $\mathrm{L}_4(4)$) and in the second case $l(B) \le 10$ (occurs for the principal block of $\mathrm{L}_6(2)$). □

Concerning the primes $p > 3$ we note that for example the principal 5-block of $\mathrm{U}_6(4)$ has defect 6 and an irreducible character of defect 2. However, we do not know if for any prime $p \ge 5$ and any $d \ge 5$ there are p-blocks of defect d with irreducible characters of defect 2.

4. Symmetric, alternating and sporadic groups

In this section we discuss the validity of our conjectures for alternating, symmetric and sporadic groups.

4.1. The irreducible characters of the symmetric group $\mathfrak{S}_n$ are parametrised by partitions λ of n, and we shall write $\chi_\lambda \in \mathrm{Irr}(\mathfrak{S}_n)$ for the character labelled by λ. Its degree is given by the well-known hook formula. We first address Conjecture B.

Proposition 4.1. *Conjecture B holds for the alternating and symmetric groups at any prime.*

Proof. First consider $G = \mathfrak{S}_n$. Let p be a prime. For $\chi = \chi_\lambda \in \mathrm{Irr}(\mathfrak{S}_n)$ let $w \ge 0$ be its defect, i.e., such that $p^w \chi(1)_p = |\mathfrak{S}_n|_p$. It is immediate from the hook formula that this can only happen if there are at most w ways to move a bead upwards on its respective ruler in the p-abacus diagram of λ. But then clearly the p-core of λ can be reached by removing at most w p-hooks, so λ lies in a p-block of weight at most w. But it is well-known that any such block B has less than p^w modular irreducible characters (see e.g. [25, Prop. 5.2]).

We now consider the alternating groups where we first assume that p is odd. Clearly $\chi \in \mathrm{Irr}(\mathfrak{A}_n)$ satisfies our hypothesis if and only if it lies below a character χ_λ of $\mathfrak{S}_n$ which does. But then χ_λ lies in a p-block B of $\mathfrak{S}_n$ of weight at most w, as shown before. If B is not self-associate, that is, if the parametrising p-core is not self-dual, then B and its conjugate B' both lie over a block B_0 of $\mathfrak{A}_n$, namely the one containing χ, with the same invariants. So we are done by the case of $\mathfrak{S}_n$. If B is self-associate then an easy estimate shows that again $l(B) < p^w$ (see the proof of [25, Prop. 5.2]).

Finally consider $p = 2$ for alternating groups. Here the number of modular irreducibles in a 2-block B of weight w is $\pi(w)$ if w is odd, respectively $\pi(w) + \pi(w/2)$ if w is even, with $\pi(w)$ denoting the number of partitions of w. Now $\pi(w) \le 2^{w-1}$, and moreover $\pi(w) + \pi(w/2) \le 2^{w-1}$ for even $w \ge 4$, so for $w \ge 3$ we have $l(B) \le 2^{w-1}$. On the other hand the difference $d(B) - \mathrm{ht}(\chi)$ is at most one smaller for $\chi \in \mathrm{Irr}(\mathfrak{A}_n)$ than for a character $\tilde{\chi}$ of $\mathfrak{S}_n$ lying above χ. Thus for all $w \ge 3$ we have $l(B) \le 2^{w-1}$ is at most $2^{d(B)-\mathrm{ht}(\chi)}$ for all $\chi \in \mathrm{Irr}(B)$, as required. For $w = 2$ the defect groups of B are abelian and the claim is easily verified. □

The next statement follows essentially from a result of Scopes:

Theorem 4.2. *Conjecture A holds for the alternating and symmetric groups at any prime.*

Proof. We first consider symmetric groups. Let $\chi = \chi_\lambda \in \mathrm{Irr}(\mathfrak{S}_n)$ be such that $p^2\chi(1)_p = |\mathfrak{S}_n|_p$ for some prime p. By the hook formula this happens only if exactly two beads can be moved in exactly one way on their respective ruler in the p-abacus diagram of λ, or if one bead can be moved in two ways. In either case the p-core of λ can be reached by removing two p-hooks, so λ lies in a p-block of weight 2.

If B is a p-block of weight 2 of $\mathfrak{S}_n$, with p odd, then all decomposition numbers are either 0 or 1 by Scopes [34, Thm. I], and furthermore any row in the decomposition matrix has at most 5 non-zero entries. This proves our claim for $\mathfrak{S}_n$ and odd primes. For $p = 2$ the defect groups of B of weight 2 are dihedral of order 8, and our claim also follows.

We now consider the alternating groups. First assume that p is odd. Then $\chi \in \mathrm{Irr}(\mathfrak{A}_n)$ satisfies our hypothesis if and only if it lies below a character χ_λ of $\mathfrak{S}_n$ which does. But then χ_λ lies in a p-block of B of $\mathfrak{S}_n$ of weight 2, as shown before. If B is not self-associate then B and its conjugate B' both lie over a block B_0 of $\mathfrak{A}_n$ with the same invariants, in particular, with the same decomposition numbers. So we are done by the case of $\mathfrak{S}_n$. If B is self-associate then it is easy to count that B contains $(p-1)/2$ self-associate characters and $(p+1)^2/2$ that are not. That is, $(p+1)^2/2$ characters in B restrict irreducibly, while $(p-1)/2$ of them split. It is clear from the $\mathfrak{S}_n$-result that the decomposition numbers are at most two, and $l(B) \le (p^2+6p-3)/4 < p^2$, so the conjecture holds.

Finally the case $p = 2$ for $\mathfrak{A}_n$ follows by Theorem 3.2, as the Alperin–McKay conjecture is known to hold for all blocks of $\mathfrak{A}_n$, see [27]. □

The case of faithful blocks for the double covering groups of $\mathfrak{A}_n$ and $\mathfrak{S}_n$ seems considerably harder to investigate, at least in as far as decomposition numbers are concerned, due to the missing analogue of the theorem of Scopes for this situation.

Proposition 4.3. *Conjecture B holds for the 2-fold covering groups of alternating and symmetric groups at any odd prime, and Conjecture A holds for these groups at $p = 2$.*

Proof. As the Alperin–McKay conjecture has been verified for all blocks of the covering groups of alternating and symmetric groups [27], our claim for the prime $p = 2$ follows from Theorem 3.2.

So now assume that p is an odd prime, and first consider $G = 2.\mathfrak{S}_n$, a 2-fold covering group of $\mathfrak{S}_n$, with $n \ge 5$. By the hook formula for spin characters [29, (7.2)], the p-defect of any spin character $\chi \in \mathrm{Irr}(G)$ is at least the weight of the corresponding p-block B. On the other hand, by [25, Prop. 5.2] the blocks of G satisfy the $l(B)$-conjecture, so Conjecture B holds. Now for any p-block of $2.\mathfrak{A}_n$ there exists a height and defect group preserving bijection to a p-block of a suitable $2.\mathfrak{S}_m$, so the claim for $2.\mathfrak{A}_n$ also follows. □

4.2. We now turn to verifying our conjectures for the sporadic quasi-simple groups.

Proposition 4.4. *Let B be a p-block of a covering group of a sporadic simple group or of ${}^2F_4(2)'$. Then:*

(a) *B satisfies Conjecture B, unless possibly when B is as in the first four lines of Table 1.*
(b) *B satisfies Conjecture A unless possibly when B is as in the last four lines of Table 1.*

Table 1. Blocks in sporadic groups

G	p	$d(B)$	max ht	$l(B)$	$\chi(1)$	Conj. A(1)
Ly	2	8	5	9	4 997 664	ok
Co_1	3	9	6	29	469 945 476	
Co_1	5	4	2	29	210 974 400	ok
$2.Co_1$	5	4	2	29	1 021 620 600	
J_4	3	3	1	9	5 chars in 2 blks	

Proof. For most blocks of sporadic groups, the inequality (1) can be checked using the known character tables and Brauer tables; the only remaining cases are listed in Table 1, where the first four lines contain cases in which Conjecture B might fail, while the last four lines are those cases where Conjecture A might fail.

In two of these remaining cases we can show that at least Conjecture A(1) holds. For Ly, the tensor product of the 2-defect 0 characters of degree 120064 with the irreducible character of degree 2480 is projective, has non-trivial restriction to the principal block, but does not contain the (unique) character χ of defect 2 of degree 4 997 664. Thus $|\mathrm{IBr}(\chi^0)| \leq 8$.

For Co_1 the tensor products of irreducible 5-defect zero characters with irreducible characters, restricted to the principal block, span a 7-dimensional space of projective characters not containing the unique defect 2 character χ of degree 210 974 400, so $|\mathrm{IBr}(\chi^0)| \leq 22$. □

5. Groups of Lie type

In this section we consider our conjectures for quasi-simple groups of Lie type G. We prove both Conjectures A and B when p is the defining characteristic of G. On the other hand, we only treat one series of examples in the case of non-defining characteristic.

5.1. Defining characteristic. We need an auxiliary result about root systems:

Lemma 5.1. *Let Φ be an indecomposable root system and denote by $N(\Phi)$ its number of positive roots. If $\Psi \subset \Phi$ is any proper subsystem of Φ then $N(\Phi) - N(\Psi) \geq n$, where n is the rank of Φ.*

In fact, our result is more precise in that we determine the minimum of $N(\Phi) - N(\Psi)$ for each type, see Table 5.1.

Proof. The values of $N(\Phi)$ for indecomposable root systems Φ are given as in Table 5.1 (see e.g. [26, Tab. 24.1]). The possible proper subsystems can be determined by the algorithm of Borel–de Siebenthal (see [26, §13.2]). For Φ of type A_n the largest proper subsystem Ψ has type A_{n-1}, and then $N(\Phi) - N(\Psi) = \binom{n+1}{2} - \binom{n}{2}$. For Φ of type B_n, we need to consider subsystems of types D_n and $B_{n-1}B_1$, of which the second always has the larger number of positive roots. Next, for type D_n with $n \geq 4$, the largest subsystems are those of types A_{n-1} and D_{n-1}, which lead to the entries in our table. Finally, it is straightforward to handle the possible subsystems for Φ of exceptional type. □

Table 2. Maximal subsystems

Φ	A_n	B_n, C_n	D_n $(n \geq 4)$	G_2	F_4	E_6	E_7	E_8
$N(\Phi)$	$\binom{n+1}{2}$	n^2	$n^2 - n$	6	24	36	63	120
max. $N(\Psi)$	$\binom{n}{2}$	$n^2 - n$	$n^2 - 3n + 2$	2	16	20	36	64
$N(\Phi) - N(\Psi)$	n	n	$2n - 2$	4	8	16	27	56

Corollary 5.2. *Let G be a finite quasi-simple group of Lie type in characteristic p. Let $\chi \in \mathrm{Irr}(G)$ lie in the p-block B. Then $l(B) < |G_p|/\chi(1)_p$. In particular Conjecture B holds for G at the prime p.*

Proof. The faithful p-blocks of the finitely many exceptional covering groups can be seen to satisfy Conjecture B by inspection using the Atlas [5]. Note that the non-exceptional Schur multiplier of a simple group of Lie type has order prime to the characteristic (see e.g. [26, Tab. 24.2]). Thus we may assume that G is the universal non-exceptional covering group of its simple quotient S. Hence, G can be obtained as the group of fixed points $\mathbf{G}^F$ of a simple simply connected linear algebraic group $\mathbf{G}$ over an algebraic closure of $\mathbb{F}_p$ under a Steinberg endomorphism $F : \mathbf{G} \to \mathbf{G}$.

Now the order formula [26, Cor. 24.6] shows that $|G|_p = q^N$, where q is the underlying power of p defining G, and $N = N(\Phi) = |\Phi^+|$ denotes the number of positive roots of the root system Φ of $\mathbf{G}$. According to Lusztig's Jordan decomposition of ordinary irreducible characters of G, any $\chi \in \mathrm{Irr}(G)$ lies in some Lusztig series $\mathcal{E}(G, s)$, for s a semisimple element in the dual group G^*, and $\chi(1) = |G^* : C_{G^*}(s)|_{p'}\ \psi(1)$ for some unipotent character ψ of $C_{G^*}(s)$. In particular, $\chi(1)_p = \psi(1)_p$ is at most the p-part in $|C_{G^*}(s)|$, hence at most q^M for M the number of positive roots of the connected reductive group $C^\circ_{\mathbf{G}^*}(s)$. (Observe that $|C_{G^*}(s) : C^\circ_{G^*}(s)|$ is prime to p by [26, Prop. 14.20].) Now first assume that $s \neq 1$, so s is not central in G^* (which is of adjoint type by our assumption on $\mathbf{G}$). Then the character(s) in $\mathcal{E}(G, s)$ with maximal p-part correspond to the Steinberg character of $C^\circ_{\mathbf{G}^*}(s)$, of degree $q^{N(\Psi)}$, where Ψ is the root system of $C^\circ_{\mathbf{G}^*}(s)$, a proper subsystem of Φ. In particular, $|G|_p/\chi(1)_p \geq q^n$ by Lemma 5.1, with n denoting the Lie rank of $\mathbf{G}$.

We next deal with the Lusztig series of $s = 1$, that is, the unipotent characters of G. If $\chi \in \mathrm{Irr}(G)$ is unipotent then its degree is given by a polynomial in q, $\chi(1) = q^{a_\chi} f_\chi(q)$, where $f_\chi(X)$ has constant term ± 1, so that $\chi(1)_p = q^{a_\chi}$ (see [4, 13.8 and 13.9]). We let A_χ denote the degree of the polynomial $X^{a_\chi} f_\chi(X)$. Let $D(\chi)$ denote the Alvis–Curtis dual of χ. Then $D(\chi)(1) = q^{N - A_\chi} f'_\chi(q)$, for some polynomial f'_χ of the same degree as f_χ, so the degree polynomial of $D(\chi)$ is of degree $N - a_\chi$. We are interested in unipotent characters χ with large a_χ, that is, those for which the degree polynomial of the Alvis–Curtis dual has small degree $N - a_\chi$. The Alvis–Curtis dual of the trivial character is the Steinberg character, whose degree is just the full p-power q^N of $|G|$. The smallest possible degrees of degree polynomials of non-trivial unipotent characters for simple groups of Lie type are easily read off from the explicit formulas in [4, 13.8 and 13.9], they are given as follows:

Φ	A_n	B_n, C_n	D_n $(n \geq 4)$	G_2	F_4	E_6	E_7	E_8
$N - a_\chi$	n	$2n - 1$	$2n - 3$	5	11	11	17	29

It transpires that again $|G|_p/\chi(1)_p \geq q^n$ in all cases.

On the other hand, the p-modular irreducibles of G are parametrised by q-restricted weights of $\mathbf{G}$, so there are q^n of them, one of which is the Steinberg character, of defect zero. Thus $l(B) < q^n$ for all p-blocks B of G of positive defect which shows Conjecture B. $\square$

We now turn to Conjecture A. In view of Corollary 5.2 only its second assertion remains to be considered. The following result shows that the assumptions of Conjecture A are hardly ever satisfied:

Proposition 5.3. *Let G be a quasi-simple group of Lie type in characteristic p. Assume that G has an irreducible character $\chi \in \mathrm{Irr}(G)$ such that $|G|_p = p^2\,\chi(1)_p$. Then G is a central quotient of $\mathrm{SL}_2(p^2)$, $\mathrm{SL}_3(p)$, $\mathrm{SU}_3(p)$ or $\mathrm{Sp}_4(p)$.*

Proof. Again the finitely many exceptional covering groups can be handled by inspection using the Atlas [5] and so as in the proof of Corollary 5.2 we may assume that G is the universal non-exceptional covering group of its simple quotient S and so can be obtained as the group of fixed points of a simply connected simple algebraic group under a Steinberg map.

Using Lusztig's Jordan decomposition we see that our question boils down to

(1) determining the irreducible root systems Φ possessing a proper root subsystem Ψ such that $N(\Psi) \geq N(\Phi) - 2$, and
(2) finding the unipotent characters χ of G such that $p^2\chi(1)_p \geq |G|_p$.

The first issue can easily be answered using Lemma 5.1. According to Table 5.1 only $\mathbf{G}$ of types A_1, A_2 or B_2 are candidates, the first with $q = p^2$, that is, $G = \mathrm{SL}_2(p^2)$, and the other two only for $q = p$. For type A_2 this leads to $\mathrm{SL}_3(p)$ and $\mathrm{SU}_3(p)$, in the case of type B_2 we have $|\Phi(B_2)| - |\Phi(B_1^2)| = 2$, which only leads to $G = \mathrm{Sp}_4(p)$. Indeed, for the twisted Suzuki groups ${}^2B_2(q^2)$, where q^2 is even, there is no centraliser of root system A_1^2 in the dual group.

Issue (2) has already been partly discussed in the proof of Corollary 5.2. Using the list of maximal q-powers occurring in unipotent character degrees given there it follows that examples can only possibly arise if the root system of $\mathbf{G}$ is of type A_1 or A_2. $\square$

Theorem 5.4. *Let G be a quasi-simple group of Lie type in characteristic p. Then Conjecture A holds for G and the prime p.*

Proof. By Proposition 5.3 we only need to consider the groups $\mathrm{SL}_2(p^2)$, $\mathrm{SL}_3(p)$, $\mathrm{SU}_3(p)$ and $\mathrm{Sp}_4(p)$. The ordinary character tables of all these groups are known and available for example in the Chevie system [10].

The p-modular irreducibles of $G = \mathrm{SL}_2(p^2)$ are indexed by p^2-restricted dominant weights, so there are exactly p^2 of them. One of them, the Steinberg representation, is of defect 0, so any p-block of G contains at most $p^2 - 1$ irreducible Brauer characters. All decomposition numbers are equal to 0 or 1 by Srinivasan [35], which deals with this case.

For $G = \mathrm{SL}_3(p)$ the only characters χ satisfying the assumptions of Conjecture A are the unipotent character χ_u of degree $p(p+1)$ and the regular characters χ_r of degree $p(p^2+p+1)$. The p-modular irreducibles of G are parametrised by p-restricted weights, so since G is of rank 2 there are p^2 of them, one of which is the Steinberg character, of defect zero. Thus $l(B) < p^2$ for all blocks of positive defect which already shows (1). Now assume that $p \geq 11$. By [13, Tab. 1 and §4] χ_u^0 has just two modular composition factors. Furthermore, projective indecomposables of G have degree at most $12p^3$ (see [13]), so any decomposition number occurring for χ_r is at most $12p^2/(p^2+p+1)$, which is smaller or equal to p for $p \geq 11$. The decomposition numbers for $p \leq 7$ are contained in [37].

The arguments for $G = \mathrm{SU}_3(p)$ are entirely similar, using the reference [14] for decomposition numbers when $p \geq 11$. The cases with $3 \leq p \leq 7$ are contained in [37].

Finally consider $G = \mathrm{Sp}_4(p)$. Here the only relevant characters χ are those of degree $\frac{1}{2}p^2(p^2 \pm 1)$ parametrised by involutions in the dual group with centraliser of type A_1^2; they only exist when p is odd, which we now assume. Then G has two p-blocks of positive defect both containing $(p^2-1)/2$ modular irreducibles. This already proves (1). The second claim follows from the explicit lists of decomposition numbers provided in Appendix 3 of [19]: indeed, all relevant decomposition numbers are bounded above by 2. □

5.2. Non-defining characteristic. The current knowledge about decomposition numbers for blocks of groups of Lie type in non-defining characteristic does not seem sufficient to prove even Conjecture A, not even for unipotent blocks in general. We hence just make some preliminary observations.

Example 5.5. This example shows that characters in blocks of groups of Lie type in non-defining characteristic can have rather large heights. Let $G = \mathrm{GL}_n(q)$, and ℓ a prime dividing $q-1$. Then all unipotent characters lie in the principal ℓ-block B_0 of G, and so do all characters in Lusztig series $\mathcal{E}(G,s)$ for any ℓ-element $t \in G^* \cong G$. Now assume that $n = \ell^a$ for some $a \geq 1$. Then $|G|_\ell = \ell^{cn+(n-1)/(\ell-1)}$ where ℓ^c is the precise power of ℓ dividing $q-1$. Let $T \leq \mathrm{GL}_n(q)$ be a Coxeter torus, of order q^n-1, and $t \in T$ an element of maximal ℓ-power order. It can easily be seen that then $o(t) = (q-1)_\ell \ell^a = \ell^{c+a}$ and t is regular, so $\mathcal{E}(G,t)$ consists of a single character χ, say. Now $\chi(1) = |G : C_G(t)|$ and hence

$$\chi(1)_\ell = |G|_\ell/|T|_\ell = \ell^{cn+(n-1)/(l-1)-c-a},$$

so χ has height $c(n-1)+(n-1)/(l-1)-a$ and defect $c+a$. So the height can become arbitrarily large by varying q for fixed n and in particular it is not bounded in terms of the (relative) Weyl group. Moreover, the unipotent characters form a basic set for B_0. As they are in bijection with $\mathrm{Irr}(\mathfrak{S}_n)$, $l(B_0)$ is the number of partitions of n and so grows exponentially in n.

Choosing $a = c = 1$, that is, $n = \ell$ and $\ell||(q-1)$ we obtain principal ℓ-blocks of defect $\ell+1$ containing a character of defect 2. These examples also give rise to similar blocks for the quasi-simple groups $\mathrm{SL}_n(q)$. We will deal with them in Proposition 5.9 below.

Now let G be the group of fixed points of a connected reductive linear algebraic group $\mathbf{G}$ under a Frobenius endomorphism F defining an $\mathbb{F}_q$-structure. Let ℓ be a prime different from the defining characteristic of G and set $e = e_\ell(q)$, the order of q modulo ℓ if $\ell > 2$, respectively the order of q modulo 4 if $\ell = 2$. We write $a_G(e)$ for the precise power of Φ_e dividing the order polynomial of the derived subgroup $[\mathbf{G},\mathbf{G}]$, that is, since $\mathbf{G} = [\mathbf{G},\mathbf{G}]Z(\mathbf{G})$, the precise power of Φ_e dividing the order polynomial of $\mathbf{G}/Z(\mathbf{G})$. If e is clear from the context, we will just denote it by a_G.

We start with an observation on defects of characters in unipotent e-Harish-Chandra series of G.

Lemma 5.6. *Let $\mathbf{G}, F, q$ be as above. Let χ be a unipotent character of $G = \mathbf{G}^F$ lying in the e-Harish-Chandra series of the e-cuspidal pair $(\mathbf{L}, \lambda)$. Let ℓ be a prime with $e = e_\ell(q)$ and set $\ell^c || \Phi_e(q)$. If χ has defect 2, then $c(a_G - a_L) \leq 2$. Moreover, if $c(a_G - a_L) = 2$ then χ is $e\ell^a$-cuspidal for all $a > 0$.*

Proof. As the unipotent character χ lies in the e-Harish-Chandra series of $(\mathbf{L}, \lambda)$, the Φ_e-part of its degree polynomial agrees with the one of λ (see [2]). As λ is e-cuspidal, the Φ_e-part in its degree polynomial equals a_L [2, Prop. 2.4]. So $|G|/\chi(1)$ is divisible by at least $\Phi_e^{a_G - a_L}$ and hence by $\ell^{c(a_G - a_L)}$. If χ has defect 2 this implies that $c(a_G - a_L) \leq 2$. If $c(a_G - a_L) = 2$ then, since $\ell | \Phi_{e\ell^a}$, $\chi(1)$ must be divisible by the same power of $\Phi_{e\ell^a}$ as $|G|$ for all $a > 0$, that is, χ must be $e\ell^a$-cuspidal. $\square$

Now assume that $\ell \geq 5$ is a prime that is good for $\mathbf{G}$. By the main result of Cabanes–Enguehard [3] the unipotent ℓ-blocks of G are then parametrised by e-cuspidal unipotent pairs $(\mathbf{L}, \lambda)$ up to conjugation. Let $B = B_G(\mathbf{L}, \lambda)$ be a unipotent ℓ-block of G. Then according to [3, Thm.] the irreducible characters in B are the constituents of $R_{G_t}^G(\hat{t}\chi_t)$ where t runs over ℓ-elements in G^*, $\mathbf{G}_t$ is a Levi subgroup of $\mathbf{G}$ dual to $C^\circ_{\mathbf{G}^*}(t)$, $\hat{t}$ is the linear character of G_t dual to t, and χ_t lies in the unipotent ℓ-block $B_{G_t}(\mathbf{L}_t, \lambda_t)$, where $(\mathbf{L}_t, \lambda_t)$ is a unipotent e-cuspidal pair of $\mathbf{G}_t$ such that $[\mathbf{L}, \mathbf{L}] = [\mathbf{L}_t, \mathbf{L}_t]$ and λ, λ_t have the same restriction to $[\mathbf{L}, \mathbf{L}]^F$. Now as $\mathbf{G}_t^* = C_{\mathbf{G}^*}(t)$, $R_{G_t}^G$ induces a bijection (with signs) $\mathcal{E}(G_t, \hat{t}) \longrightarrow \mathcal{E}(G, \hat{t})$ such that degrees are multiplied by $|G : G_t|_{p'}$. In particular this shows that χ and χ_t have the same ℓ-defect.

Proposition 5.7. *Let $\mathbf{G}, F, q, \ell$ be as above. Assume that the principal ℓ-block of G contains a character of ℓ-defect 2. Then a Sylow ℓ-subgroup S of G^* contains an element t with centraliser $|C_S(t)| = \ell^2$.*

Proof. The principal ℓ-block B_0 of G is the ℓ-block above the e-cuspidal pair $(\mathbf{L}, 1_L)$, where $\mathbf{L}$ is the centraliser of a Sylow e-torus of $\mathbf{G}$. In particular $a_L = 0$. Now let $\chi \in \mathrm{Irr}(B_0)$ of defect 2. By the result of Cabanes and Enguehard cited above, there is an ℓ-element $t \in G^*$ and a unipotent character $\chi_t \in \mathrm{Irr}(G_t)$ of defect 2 in the e-Harish-Chandra series of $(\mathbf{L}_t, 1_{L_t})$ with $[\mathbf{L}_t, \mathbf{L}_t] = [\mathbf{L}, \mathbf{L}]$. By Lemma 5.6 this implies that $ca_{G_t} \leq 2$. In particular Φ_e divides the order polynomial of $[\mathbf{G}_t, \mathbf{G}_t]$ at most twice, and so the same statement holds for the dual group $\mathbf{C} := C_{\mathbf{G}^*}(t)$. An inspection of the order formulas of the finite reductive groups shows that then the Sylow ℓ-subgroups of C are abelian (using that $\ell > 3$), and hence contained in a Sylow e-torus of $\mathbf{C}$. For χ_t to have defect 2 this forces $|C|_\ell = \ell^2$, so t is as claimed. $\square$

We will not attempt to classify the cases when Sylow ℓ-subgroups of finite reductive groups contain ℓ-elements with this property, even though that seems possible, since in most of these cases our knowledge on decomposition numbers would not suffice to settle Conjecture A anyway. We just discuss one particular case.

Lemma 5.8. *Let $G = \mathrm{PGL}_n(q)$ with $n \geq 4$ and ℓ a prime dividing $q - 1$. If G contains an ℓ-element t with $|C_G(t)|_\ell = \ell^2$ then one of*

(1) *$n = \ell$ and $C \cong \mathrm{GL}_1(q^\ell)$,*

(2) *$n = \ell + 1$ and $C \cong \mathrm{GL}_\ell(q) \times \mathrm{GL}_1(q)$, or*
(3) *$n = \ell^2$ and $C \cong \mathrm{GL}_1(q^{\ell^2})$;*
where C denotes the centraliser in $\mathrm{GL}_n(q)$ of a preimage of t under the natural map.

Proof. Let $\hat{t}$ be a preimage of t of ℓ-power order in $\hat{G} := \mathrm{GL}_n(q)$ under the natural surjection. Then $|C_{\hat{G}}(\hat{t})|_\ell \leq \ell^{2+c}$ where ℓ^c is the precise power of ℓ dividing $q-1$. Now

$$C := C_{\hat{G}}(\hat{t}) \cong \mathrm{GL}_{n_1}(q^{a_1}) \times \cdots \times \mathrm{GL}_{n_r}(q^{a_r})$$

for suitable $n_i, a_i \geq 1$ with $\sum n_i a_i = n$. Let s_i denote the projection of $\hat{s}$ into the ith factor of C. Then $s_i \in Z(\mathrm{GL}_{n_i}(q^{a_i})) \cong \mathbb{F}_{q^{a_i}}^\times$ is an ℓ-element generating the field $\mathbb{F}_{q^{a_i}}$. In particular, ℓ divides the cyclotomic polynomial $\Phi_{a_i}(q)$, and so $a_i = \ell^{f_i}$ for some $f_i \geq 0$ (see e.g. [26, Lemma 25.13]). Then $|C|_\ell \geq \sum_{i=1}^r (c + f_i) n_i$. Our assumption thus implies that $\sum n_i \leq 3$.

If $n_1 = 3$ then $c = 1$, $f_1 = 0$, so $a_1 = 1$ and $n = \sum n_i = 3$, which was excluded. Now assume that $\sum n_i = 2$. If $n_1 = 2$ then $C \cong \mathrm{GL}_2(q^{n/2})$ and $f_1 = 0$, whence $n = 2$. If $n_1 = n_2 = 1$ then $C \cong \mathrm{GL}_1(q^{a_1}) \times \mathrm{GL}_1(q^{a_2})$ and $2c + f_1 + f_2 \leq 2 + c$, so $c = 1$ and $f_1 + f_2 \leq 1$. This leads to $C \cong \mathrm{GL}_1(q)^2 \leq \mathrm{GL}_2(q)$ or to $C \cong \mathrm{GL}_\ell(q) \times \mathrm{GL}_1(q) \leq \mathrm{GL}_{\ell+1}(q)$, with $\ell || (q-1)$. Finally assume that $\sum n_i = 1$, so $n_1 = 1$ and $C \cong \mathrm{GL}_1(q^n)$ with $n = \ell^f$ and $c + f \leq 3$. If $c = 3$ then $f = 0$ and $n = 1$; if $c = 2$ then we find $C \cong \mathrm{GL}_1(q^\ell) \leq \mathrm{GL}_\ell(q)$, and if $c = 1$ then we also could have $C \cong \mathrm{GL}_1(q^{\ell^2}) \leq \mathrm{GL}_{\ell^2}(q)$. □

Proposition 5.9. *Conjecture A holds for all characters in the principal ℓ-block of $\mathrm{SL}_n(q)$ when $\ell | (q-1)$.*

Proof. Assume that the principal ℓ-block B_0 of $G := \mathrm{SL}_n(q)$ contains a character $\chi \in \mathrm{Irr}(B_0)$ of ℓ-defect 2. Then $\chi \in \mathcal{E}(G, t)$ for some ℓ-element $t \in G^* = \mathrm{PGL}_n(q)$ such that $|C_{G^*}(t)|_\ell = \ell^2$ by Proposition 5.7. According to Lemma 5.8 then either $n \leq 3$ or $n \in \{\ell, \ell+1, \ell^2\}$. For $n \leq 3$ the claim is easily checked from the known decomposition matrices. We consider the remaining cases in turn.

Assume that $n = \ell^f$ with $f \in \{1, 2\}$. Then $C_{\mathbf{G}^*}(t)$ is a Coxeter torus of PGL_n by Lemma 5.8(1) and (3), and t is a regular element. Its centraliser is disconnected, with $|C_{G^*}(t)| = \ell^f (q^{\ell^f} - 1)/(q-1)$, so the characters in these Lusztig series can only have defect 2 if $f = 1$, that is, $n = \ell$, which we assume from now on. The ℓ characters in $\mathcal{E}(G, t)$ are the constituents of the restriction to G of the unique character ψ in $\mathcal{E}(\tilde{G}, \hat{t})$, where $\tilde{G} = \mathrm{GL}_n(q)$. Let $\tilde{T}$ be a Coxeter torus of $\tilde{G}$. Then $\psi = \mathrm{R}_{\tilde{\mathbf{T}}}^{\tilde{\mathbf{G}}}(\theta)$ for an ℓ-element $\theta \in \mathrm{Irr}(\tilde{T})$ in duality with $\hat{t}$. The reduction of $\mathrm{R}_{\tilde{\mathbf{T}}}^{\tilde{\mathbf{G}}}(\theta)$ modulo ℓ thus coincides with the reduction of $\mathrm{R}_{\tilde{\mathbf{T}}}^{\tilde{\mathbf{G}}}(1)$. The latter decomposes as $\sum_{i=1}^{\ell} (-1)^{i-1} \chi_i$, where $\chi_i \in \mathrm{Irr}(B_0)$ is the unipotent character of $\tilde{G}$ parametrized by the hook partition $(i, 1^{\ell-i})$. Now the decomposition numbers of the unipotent characters in B_0 are known: the ℓ-decomposition matrix of the Iwahori–Hecke algebra of type $A_{\ell-1}$, that is, of the symmetric group $\mathfrak{S}_\ell$ embeds into that of G. Since ℓ divides $|\mathfrak{S}_\ell|$ just once, we obtain a Brauer tree with $\chi_i + \chi_{i+1}$ being projective for $i = 1, \ldots, \ell-1$. Adding up we see that ψ is irreducible modulo ℓ, hence the same is true for $\chi \in \mathcal{E}(G, t)$ and our claim is proved in this case.

Assume next that $n = \ell + 1$. Then by Lemma 5.8(2) the centraliser of t is a maximal torus with $|C_{G^*}(t)| = q^\ell - 1$. So again t is regular, but now with connected centraliser, hence $\mathcal{E}(G,t) = \{\chi\}$ consists of just one character. Again the principal block of $\mathfrak{S}_{\ell+1}$ has cyclic defect, and a computation as in the previous case shows that χ is in fact irreducible. The validity of Conjecture A follows. $\square$

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