The 2-blocks of defect 4

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Abstract

We show that the major counting conjectures of modular representation theory are satisfied for 2-blocks of defect at most 4 except one possible case. In particular we determine the invariants of such blocks.

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1 Introduction

Let B be a 2-block of a finite group G with defect group D. Then there are several open conjectures regarding the number $k(B)$ of irreducible ordinary characters of B and the number $l(B)$ of irreducible Brauer characters of B. The aim of this paper is to show that most of these conjectures are fulfilled if D is small. More precisely we will assume that B has defect at most 4, i. e. D has order at most 16. We denote the number of irreducible ordinary characters of height i in B by $k_i(B)$ for $i \ge 0$.

An essential tool is the following recent theorem by Kessar and Malle [17].

Theorem 1.1 (Kessar, Malle, 2011)**.** *For every p-block B of a finite group with abelian defect group we have $k(B) = k_0(B)$.*

For $|D| \le 8$ the block invariants and conjectures for B are known by the work of Brauer [5], Olsson [23] and Kessar-Koshitani-Linckelmann [16]. So we assume that D has order 16.

2 The elementary abelian case

Let $I(B)$ be the inertial quotient of B and set $e(B) := |I(B)|$.

Proposition 2.1. *Let B be a block of a finite group G with elementary abelian defect group D of order 16. Then one of the following holds:*

(i) B *is nilpotent. Then* $e(B) = l(B) = 1$ *and* $k(B) = k_0(B) = 16$.

(ii) $e(B) = l(B) = 3$, $\mathrm{C}_D(I(B)) = 1$ *and* $k(B) = k_0(B) = 8$.

(iii) $e(B) = l(B) = 3$, $|\mathrm{C}_D(I(B))| = 4$ *and* $k(B) = k_0(B) = 16$.

(iv) $e(B) = l(B) = 5$ *and* $k(B) = k_0(B) = 8$.

(v) $e(B) = l(B) = 7$ *and* $k(B) = k_0(B) = 16$.

(vi) $e(B) = l(B) = 9$ *and* $k(B) = k_0(B) = 16$.

(vii) $e(B) = 9$, $l(B) = 1$ *and* $k(B) = k_0(B) = 8$.

(viii) $e(B) = l(B) = 15$ *and* $k(B) = k_0(B) = 16$.

(ix) $e(B) = 15$, $l(B) = 7$ *and* $k(B) = k_0(B) = 8$.

(x) $e(B) = 21$, $l(B) = 5$ *and* $k(B) = k_0(B) = 16$.

Moreover, all cases except possibly case (ix) *actually occur.*

Proof. First of all by Theorem 1.1 we have $k(B) = k_0(B)$. The inertial quotient $I(B)$ is a subgroup of $\mathrm{Aut}(D) \cong \mathrm{GL}(4, 2)$ of odd order. It follows that $e(B) \in \{1, 3, 5, 7, 9, 15, 21\}$ (this can be shown with GAP [13]). If $e(B) \neq 21$, the inertial quotient is necessarily abelian. Then by Corollary 1.2(ii) in [29] there is a nontrivial subsection (u, b) such that $l(b) = 1$. Hence, Corollary 2 in [6] implies that $|D| = 16$ is a sum of $k(B)$ odd squares. This shows $k(B) \in \{8, 16\}$ for these cases. In order to determine $l(B)$ we calculate the numbers $l(b)$ for all nontrivial subsections (u, b). Here it suffices to consider a set of representatives of the orbits of D under $I(B)$, since B is a controlled block. If $e(B) = 1$, the block is nilpotent and the result is clear. We discuss the remaining cases separately:

Case 1: $e(B) = 3$
Here by results of Usami and Puig (see [40, 28]) there is a perfect isometry between B and its Brauer correspondent in $\mathrm{N}_G(D)$. According to two different actions of $I(B)$ on D, we get $k(B) = 8$ if $\mathrm{C}_D(I(B)) = 1$ or $k(B) = 16$ if $|\mathrm{C}_D(I(B))| = 4$. In both cases we have $l(B) = 3$.

Case 2: $e(B) = 5$
Then there are four subsections $(1, B)$, (u_1, b_1), (u_2, b_2) and (u_3, b_3) with $l(b_1) = l(b_2) = l(b_3) = 1$ up to conjugation. In [37] it was shown that $k(B) = 16$ is impossible. Hence, $k(B) = 8$ and $l(B) = 5$.

Case 3: $e(B) = 7$
There are again four subsections $(1, B)$, (u_1, b_1), (u_2, b_2) and (u_3, b_3) up to conjugation. But in this case $l(b_1) = l(b_2) = 1$ and $l(b_3) = 7$ by [16]. Thus, $k(B) = 16$ and $l(B) = 7$.

Case 4: $e(B) = 9$
There are four subsections $(1, B)$, (u_1, b_1), (u_2, b_2) and (u_3, b_3) such that $l(b_1) = 1$ and $l(b_2) = l(b_3) = 3$ up to conjugation. This gives the possibilities (vi) and (vii).

Case 5: $e(B) = 15$
Here $I(B)$ acts regularly on $D \setminus \{1\}$. Thus, there are only two subsections $(1, B)$ and (u, b) such that $l(b) = 1$. This gives the possibilities (viii) and (ix).

Case 6: $e(B) = 21$
Here $I(B)$ is nonabelian. Hence, we get four subsections $(1, B)$, (u_1, b_1), (u_2, b_2) and (u_3, b_3) up to conjugation. We have $l(b_1) = l(b_2) = 3$ and $l(b_3) = 5$ by [16]. Since $I(B)$ has a fixed point on D, it follows that $l(B) = 5$ and $k(B) = 16$ by Theorem 1 in [45].

For all cases except (vii) and (ix) examples are given by the principal block of $D \rtimes I(B)$. In case (vii) we can take a nonprincipal block of the group `SmallGroup(432,526)` $\cong D \rtimes E$ where E is the extraspecial group of order 27 and exponent 3 (see "small groups library"). □

We will see later that case (ix) would contradict Alperin's Weight Conjecture. Now we investigate the differences between the cases (vi) and (vii).

Lemma 2.2. *Let B be a block of a finite group G with elementary abelian defect group D of order* 16. *If $e(B) = l(B) = 9$, then the elementary divisors of the Cartan matrix of B are* $1, 1, 1, 1, 4, 4, 4, 4, 16$. *Moreover, the two $I(B)$-stable subgroups of D of order* 4 *are lower defect groups of B. Both occur with* 1*-multiplicity* 2.

Proof. Let C be the Cartan matrix of B. As in the proof of Proposition 2.1 there are four subsections $(1, B)$, (u_1, b_1), (u_2, b_2) and (u_3, b_3) such that $l(b_1) = 1$ and $l(b_2) = l(b_3) = 3$ up to conjugation. In order to determine C up to basic sets, we need to investigate the generalized decomposition numbers $d^{u_i}_{rs}$ for $i = 1, 2, 3$. The block b_2 dominates a block $\overline{b_2}$ of $\mathrm{C}_G(u_2)/\langle u_2 \rangle$ with defect group $D/\langle u_2 \rangle$ and inertial index 3. Thus, as in the proof of Theorem 3 in [36] the Cartan matrix of b_2 has the form

$$\begin{pmatrix} 8 & 4 & 4 \\ 4 & 8 & 4 \\ 4 & 4 & 8 \end{pmatrix}$$

up to basic sets. Since $k(B) = 16$, we may assume that the numbers $d^{u_2}_{rs}$ take the form

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & . & . & . & . & . & . & . & . \\ 1 & 1 & 1 & 1 & . & . & . & . & 1 & 1 & 1 & 1 & . & . & . & . \\ 1 & 1 & 1 & 1 & . & . & . & . & . & . & . & . & 1 & 1 & 1 & 1 \end{pmatrix}^{\mathrm{T}}.$$

For the column of decomposition numbers $d^{u_1}_{rs}$ we have essentially the following possibilities:

$$(i) : (1, 1, 1, 1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1)^{\mathrm{T}},$$
$$(ii) : (1, 1, 1, -1, 1, -1, -1, -1, 1, -1, -1, -1, 1, -1, -1, -1)^{\mathrm{T}},$$
$$(iii) : (1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1)^{\mathrm{T}}.$$

Now we use a GAP program to enumerate the possible decomposition numbers $d^{u_3}_{rs}$. After that the ordinary decomposition matrix M can be calculated as the orthogonal space. Then $C = M^{\mathrm{T}}M$ up to basic sets. It turns out that in some cases C has 2 as an elementary divisor. Using the notion of lower defect groups as described in [24] we show that these cases cannot occur. If 2 is an elementary divisor of C, then there exists a lower defect group $Q \le D$ of order 2. With the notation of [24] we have $m^{(1)}_B(Q) > 0$. By Theorem 7.2 in [24] there is a block b_Q of $\mathrm{N}_G(Q) = \mathrm{C}_G(Q)$ such that $b_Q^G = B$ and $m^{(1)}_{b_Q}(Q) > 0$. In particular the Cartan matrix of b_Q has 2 as elementary divisor. Hence, b_Q is conjugate to b_2 or b_3. But we have seen above that all elementary divisors of the Cartan matrix of b_2 (and also b_3) must be divisible by 4. This contradiction shows that 2 does not occur as elementary divisor of C. After excluding these cases the GAP program reveals the following two possibilities for the elementary divisors of C: $1, 1, 1, 1, 4, 4, 4, 4, 16$ or $1, 1, 4, 4, 4, 4, 4, 4, 16$.

Now we have to look at the lower defect group multiplicities more carefully. The calculation above and (7G) in [4] imply

$$4 \le \sum_{R \in \mathcal{R}} m^{(1)}_B(R)$$

where $\mathcal{R}$ is a set of representatives for the G-conjugacy classes of subgroups of G of order 4. After combining this with the formula (2S) of [7] we get

$$4 \le \sum_{(R,b_R) \in \mathcal{R}'} m^{(1)}_B(R, b_R)$$

where $\mathcal{R}'$ is a set of representatives for the G-conjugacy classes of B-subpairs (R, b_R) such that R has order 4. Let b_D be a Brauer correspondent of B in $\mathrm{C}_G(D)$. Then, after changing the representatives if necessary we may assume $(R, b_R) \le (D, b_D)$ for $(R, b_R) \in \mathcal{R}'$. Then it is well known that $b_R = b_D^{\mathrm{C}_G(R)}$ is uniquely determined by R. Since the fusion of these subpairs is controlled by $\mathrm{N}_G(D, b_D)$, we get

$$4 \le \sum_{R \in \mathcal{R}''} m^{(1)}_B(R, b_R)$$

where $\mathcal{R}''$ is a set of representatives for the $I(B)$-conjugacy classes of subgroups of D of order 4.

Now let $Q \le D$ of order 4 such that $m^{(1)}_B(Q, b_Q) > 0$. Then by (2Q) in [7] we have $m^{(1)}_{B_Q}(Q) > 0$ where $B_Q := b_Q^{\mathrm{N}_G(Q,b_Q)}$. If Q is not fixed under $I(B)$, then we would have the contradiction $e(B_Q) = l(B_Q) = 1$. Thus, we have shown that Q is stable under $I(B)$. Hence,

$$4 \le m^{(1)}_B(Q, b_Q) + m^{(1)}_B(P, b_P) \tag{1}$$

where $P \ne Q$ is the other $I(B)$-stable subgroup of D of order 4. Since 16 is always an elementary divisor of C, we have $m^{(1)}_{B_Q}(D) = 1$. Observe that b_Q has defect group D and inertial index 3, so that $l(b_Q) = 3$ by Proposition 2.1. Now Theorem 5.11 in [24] and the remark following it imply

$$3 = l(b_Q) \ge m^{(1)}_{B_Q}(Q) + m^{(1)}_{B_Q}(D).$$

(Notice that in Theorem 5.11 it should read $B \in \mathrm{Bl}(G)$ instead of $B \in \mathrm{Bl}(Q)$.) Thus, $m^{(1)}_{B_Q}(Q) \le 2$ and similarly $m^{(1)}_{B_P}(P) \le 2$. Now Equation (1) yields $m^{(1)}_B(Q, b_Q) = m^{(1)}_B(P, b_P) = 2$. In particular, 4 occurs as elementary divisor of C with multiplicity 4. It is easy to see that we also have $m^{(1)}_B(Q) = m^{(1)}_B(P) = 2$ which proves the last claim. □

Proposition 2.3. *Let B be a block of a finite group G with elementary abelian defect group D of order 16. If $e(B) = 9$, then Alperin's Weight Conjecture holds for B.*

Proof. Let b_D be a Brauer correspondent of B in $\mathrm{C}_G(D)$, and let B_D be the Brauer correspondent of B in $\mathrm{N}_G(D, b_D)$. Then it suffices to show that $l(B) = l(B_D)$. By Proposition 2.1 we have to consider two cases $l(B) \in \{1, 9\}$. As in Lemma 2.2 we set $b_R := b_D^{\mathrm{C}_G(R)}$ for $R \le D$.

We start with the assumption $l(B) = 9$. Then by Lemma 2.2 there is an $I(B)$-stable subgroup $Q \le D$ of order 4 such that $m_{B_Q}^{(1)}(Q) = m_B^{(1)}(Q, b_Q) > 0$ where $B_Q := b_Q^{\mathrm{N}_G(Q,b_Q)}$. In particular $l(B_Q) = 9$. Let $P \le D$ be the other $I(B)$-stable subgroup of order 4. Moreover, let $b'_P := b_D^{\mathrm{N}_G(Q,b_Q)\cap \mathrm{C}_G(P)}$ such that (P, b'_P) is a B_Q-subpair. Then by the same argument we get

$$m_\beta^{(1)}(P) = m_{B_Q}^{(1)}(P, b'_P) > 0$$

where $\beta := (b'_P)^{\mathrm{N}_G(Q,b_Q)\cap \mathrm{N}_G(P,b'_P)}$ is a block with defect group D and $l(\beta) = 9$. Now $D = QP$ implies

$$\mathrm{N}_G(D, b_D) \le \mathrm{N}_G(Q, b_Q) \cap \mathrm{N}_G(P, b'_P) \le \mathrm{N}_G(D).$$

Since $B_D^{\mathrm{N}_G(Q,b_Q)\cap \mathrm{N}_G(P,b'_P)} = \beta$, it follows that $l(B_D) = 9$ as desired.

Now let us consider the case $l(B) = 1$. Here we can just follow the same lines except that we have $m_{B_Q}^{(1)}(Q) = 0$ and $m_\beta^{(1)}(P) = 0$. □

We want to point out that Usami showed in [42] that in case $2 \ne p \ne 7$ there is a perfect isometry between a p-block with abelian defect group D and inertial index 9 and its Brauer correspondent in $\mathrm{N}_G(D)$.

3 The Ordinary Weight Conjecture

For most 2-blocks of defect 4 Robinson's Ordinary Weight Conjecture (OWC) [30] is known to hold. In this section we handle the remaining cases.

Proposition 3.1. *Let B be a block of a finite group G with minimal nonabelian defect group*

$$D := \langle x, y \mid x^{2^r} = y^2 = [x, y]^2 = [x, x, y] = [y, x, y] = 1\rangle$$

of order $2^{r+2} \ge 16$. Then the Ordinary Weight Conjecture holds for B.

Proof. The block invariants of B were determined and several conjectures were verified in [35]. In order to prove the OWC we use the version in Conjecture 6.5 in [15]. Let $\mathcal{F}$ be the fusion system of B. We may assume that $\mathcal{F}$ is nonnilpotent. Let $z := [x, y]$. Then it was shown in [35] that $Q := \langle x^2, y, z\rangle \cong C_{2^{r-1}} \times C_2^2$ and D are the only $\mathcal{F}$-centric and $\mathcal{F}$-radical subgroups of D. Moreover, $\mathrm{Out}_{\mathcal{F}}(Q) = \mathrm{Aut}_{\mathcal{F}}(Q) \cong S_3$ and $\mathrm{Out}_{\mathcal{F}}(D) = 1$. Hence, it follows easily that $\mathbf{w}(D, d) = k^d(D) = k^d(B)$ for all $d \in \mathbb{N}$ where $k^d(D)$ is the number of characters of defect d in D. Thus, it suffices to show $\mathbf{w}(Q, d) = 0$ for all $d \in \mathbb{N}$ by Theorem 3.6 in [35]. Since Q is abelian, we have $\mathbf{w}(Q, d) = 0$ unless $d = r + 1$. Thus, let $d = r + 1$. Up to conjugation $\mathcal{N}_Q$ consists of the trivial chain $\sigma : 1$ and the chain $\tau : 1 < C$, where $C \le \mathrm{Out}_{\mathcal{F}}(Q)$ has order 2. We consider the chain σ first. Here $I(\sigma) = \mathrm{Out}_{\mathcal{F}}(Q) \cong S_3$ acts faithfully on $\Omega(Q) \cong C_2^3$ and thus fixes a four-group. Hence, the characters in $\mathrm{Irr}(Q)$ split in 2^{r-1} orbits of length 3 and 2^{r-1} orbits of length 1 under $I(\sigma)$. For a character $\chi \in \mathrm{Irr}(D)$ lying in an orbit of length 3 we have $I(\sigma, \chi) \cong C_2$ and thus $w(Q, \sigma, \chi) = 0$. For the 2^{r-1} stable characters $\chi \in \mathrm{Irr}(D)$ we get $w(Q, \sigma, \chi) = 1$, since $I(\sigma, \chi) = \mathrm{Out}_{\mathcal{F}}(Q)$ has precisely one block of defect 0.

Now consider the chain τ. Here $I(\tau) = C$ and the characters in $\mathrm{Irr}(Q)$ split in 2^{r-1} orbits of length 2 and 2^r orbits of length 1 under $I(\tau)$. For a character $\chi \in \mathrm{Irr}(D)$ in an orbit of length 2 we have $I(\tau, \chi) = 1$ and thus $w(Q, \tau, \chi) = 1$. For the 2^r stable characters $\chi \in \mathrm{Irr}(D)$ we get $I(\tau, \chi) = I(\tau) = C$ and $w(Q, \tau, \chi) = 0$.

Taking both chains together, we derive

$$\mathbf{w}(Q, d) = (-1)^{|\sigma|+1}2^{r-1} + (-1)^{|\tau|+1}2^{r-1} = 2^{r-1} - 2^{r-1} = 0.$$

This proves the OWC. □

Now we consider the OWC for blocks with abelian defect groups D of order 2^d. Here of course D is the only $\mathcal{F}$-centric and $\mathcal{F}$-radical subgroup of D and $I(B) = \mathrm{Out}_{\mathcal{F}}(D)$ has odd order. In particular $\mathcal{N}_D$ consists only of the trivial chain. Moreover, $\mathbf{w}(D, d') = 0$ unless $d' = d$. If we assume in addition that $I(B)$ is cyclic, then

$$\mathbf{w}(D, d) = \sum_{\chi \in \mathrm{Irr}(D)/I(B)} |I(B) \cap I(\chi)| \tag{2}$$

where $I(B) \cap I(\chi) := \{\alpha \in I(B) : {}^{\alpha}\chi = \chi\}$. In connection with Theorem 1.1, the OWC predicts $k(B) = \mathbf{w}(D, d)$.

Now let us consider the case where D is elementary abelian of order 16. Then if $21 \neq e(B) \neq 9$, the OWC follows easily from Proposition 2.1 and Equation (2) except if case (ix) occurs (where OWC does not hold). Now assume $e(B) = 21$. Here the number of 2-blocks of defect 0 in $I(B)$ (which is denoted by $z(kI(B))$ in [15] where k is an algebraically closed field of characteristic 2) is 5. We have to insert this number for $|I(B) \cap I(\chi)|$ in Equation (2) if χ is invariant under $I(B)$. Now the OWC also follows in this case. We will deal with the remaining case $e(B) = 9$ in the next section.

4 The general case

Theorem 4.1. *Let B be a 2-block of a finite group G with defect group D of order at most 16. Then one of the following holds:*

(i) The following conjectures are satisfied for B:

- *Alperin's Weight Conjecture [2]*
- *Brauer's $k(B)$-Conjecture [3]*
- *Brauer's Height-Zero Conjecture [3]*
- *Olsson's Conjecture [25]*
- *Alperin-McKay Conjecture [1]*
- *Robinson's Ordinary Weight Conjecture [30]*
- *Gluck's Conjecture [14]*
- *Eaton's Conjecture [9]*
- *Eaton-Moretó Conjecture [11]*
- *Malle-Navarro Conjecture [22]*

Moreover, the Gluing Problem [21] for B has a unique solution.

(ii) $D \cong C_2^4$, $e(B) = 15$, $k(B) = k_0(B) = 8$, $l(B) = 7$ and $D \notin \mathrm{Syl}_2(G)$. The Cartan matrix of B is given by

$$\begin{pmatrix} 6 & 5 & 5 & 5 & 5 & 5 & 7 \\ 5 & 6 & 5 & 5 & 5 & 5 & 7 \\ 5 & 5 & 6 & 5 & 5 & 5 & 7 \\ 5 & 5 & 5 & 6 & 5 & 5 & 7 \\ 5 & 5 & 5 & 5 & 6 & 5 & 7 \\ 5 & 5 & 5 & 5 & 5 & 6 & 7 \\ 7 & 7 & 7 & 7 & 7 & 7 & 10 \end{pmatrix}$$

up to basic sets. Alperin's Weight Conjecture and the Alperin-McKay Conjecture are not *satisfied for B.*

Proof. As explained earlier we may assume that $|D| = 16$. Then the situation splits in the following possibilities:

(a) D is metacyclic

(b) D is minimal nonabelian

(c) D is abelian, but nonmetacyclic

(d) $D \cong D_8 \times C_2$

(e) $D \cong Q_8 \times C_2$

(f) $D \cong D_8 * C_4$

We start with a remark about Gluck's Conjecture which only applies to rational defect groups of nilpotency class at most 2. By Corollary 3.2 and Lemma 2.1 in [14] we may assume that D is nonabelian of exponent 4 in order to prove Gluck's Conjecture. Moreover, Gluck's Conjecture is satisfied for nilpotent blocks.

In case (a) the block invariants are known by [5, 23, 39]. From this most of the conjectures follow trivially. Observe here that the nonabelian metacyclic groups of exponent 4 provide only nilpotent blocks. In particular Gluck's Conjecture follows. For the OWC we refer to [32] and for the Gluing Problem to [26].

In case (b), D has the form $D \cong \langle x, y \mid x^4 = y^2 = [x, y]^2 = [x, x, y] = [y, x, y] = 1\rangle$; in particular D is not rational. Then the result follows from [35] (for the OWC see Proposition 3.1). Again we skip the elementary details for the three (less-known) conjectures in (i). The last three cases (d), (e) and (f) were handled in [38, 32, 31] (for Gluck's Conjecture see [37]). It remains to consider case (c). Here it is known that the Gluing Problem has a unique solution (see [21]). We have two possibilities: $D \cong C_4 \times C_2 \times C_2$ or D is elementary abelian. We may assume that B is nonnilpotent.

In case $D \cong C_4 \times C_2 \times C_2$, 3 is the only odd prime divisor of $|\mathrm{Aut}(D)|$. Thus, by Usami and Puig (see [40, 28]) there is a perfect isometry between B and its Brauer correspondent in $\mathrm{N}_G(D)$. Then it is easy to see that the conjectures are true.

Now we consider the elementary abelian case. By Proposition 2.1, Brauer's $k(B)$-Conjecture, Brauer's Height-Zero Conjecture, Olsson's Conjecture, Eaton's Conjecture, the Eaton-Moretó Conjecture and the Malle-Navarro Conjecture are satisfied. For abelian defect groups, Alperin's Weight Conjecture is equivalent to $l(B) = l(b)$ where b is the Brauer correspondent of B in $\mathrm{N}_G(D)$. For $e(B) = 9$ this was shown in Proposition 2.3. Thus, assume $e(B) \neq 9$. By the main result in [20], b is Morita equivalent to a twisted group algebra of $D \rtimes I(B)$. Since $e(B) \neq 9$, the corresponding 2-cocycle must be trivial so that b is Morita equivalent to the group algebra of $D \rtimes I(B)$. This gives $l(b) = k(I(B))$. Now it can be seen that Alperin's Weight Conjecture holds unless case (ix) in Proposition 2.1 occurs.

Since $k(B) - l(B) = k_0(B) - l(B)$ is determined locally, the Alperin-McKay Conjecture follows from Alperin's Weight Conjecture. Now consider the Ordinary Weight Conjecture. By the remarks in the last section it suffices to look at the case $e(B) = 9$. Here again b is Morita equivalent to a twisted group algebra of $D \rtimes I(B)$. If the corresponding 2-cocycle α is trivial we have $l(B) = 9$ and $l(B) = 1$ otherwise. Then with the notation in [15] we have $z(k_\alpha I(B)) = 9$ or $z(k_\alpha I(B)) = 1$ respectively. Now the OWC follows as in the last section.

Now we consider the situation $e(B) = 15$, $k(B) = k_0(B) = 8$ and $l(B) = 7$ more closely. The arguments above imply that Alperin's Weight Conjecture and thus also the Alperin-McKay Conjecture are not fulfilled. In particular G is nonsolvable. The Cartan matrix C of B can be determined as in [37]. Here observe that $\det C = 16 = |D|$ a fact which is also predicted by Corollary 1 in [12].

Assume that $D \in \mathrm{Syl}_2(G)$. We spend the rest of the proof to derive a contradiction. By the first Fong reduction we may assume that B is quasiprimitive, i.e. that, for any normal subgroup N of G, B covers a unique block B_N of N. Note that $D \cap N$ is a Sylow 2-subgroup of N and a defect group of B_N.

Suppose now that $N = \mathrm{O}(G)$. Then, by the second Fong reduction there exist a finite group G^* with a cyclic central subgroup N^* of odd order such that G^*/N^* is isomorphic to G/N, and a block B^* of G^* whose defect group D^* is isomorphic to D; moreover, B^* is Morita equivalent to B; in particular, we have $k(B^*) = k(B) = 8$ and $l(B^*) = l(B) = 7$.

Thus, Proposition 2.1 implies that $e(B^*) = 15$ as well, so that G^*, B^* is also a counterexample. So we may assume that $G = G^*$ and $B = B^*$. Then N is a central cyclic subgroup of odd order in G.

Let M/N be a minimal normal subgroup of G/N. Then $D \cap M$ is a Sylow 2-subgroup of M; in particular, $D \cap M \neq 1$. Then $D \cap M$ is stable under the inertial subgroup $\mathrm{N}_G(D, b)$ of B. Since $\mathrm{N}_G(D, b)$ acts transitively on $D \setminus \{1\}$, we must have $D = D \cap M \subseteq M$. Thus M/N is the only minimal normal subgroup of G/N, and $|G : M|$ is odd.

If M/N is abelian then $M = D \times N$; in particular, B has a normal defect group. But this is impossible since G, B is a counterexample.

Hence M/N is a direct product of isomorphic nonabelian finite simple groups which are transitively permuted under conjugation in G:

$$M/N = S_1/N \times \ldots \times S_t/N.$$

Thus $D = (D \cap S_1) \times \ldots \times (D \cap S_t)$ with isomorphic factors. Since $|D| = 2^4$, we must have $t = 1$, $t = 2$ or $t = 4$. Since $|G : M|$ is odd, this implies that $t = 1$. Hence M/N is a simple group with Sylow 2-subgroup D. By Walter's Theorem (see [44]), we must have $M/N = \mathrm{PSL}(2, 16)$. Note also that $M = \mathrm{F}^*(G)$. Since $\mathrm{PSL}(2, 16)$ has a trivial Schur multiplier and an outer automorphism group of order 4, we conclude that $G = M = \mathrm{PSL}(2, 16) \times N$. We may therefore clearly assume that $N = 1$. In this case B is the principal 2-block of $\mathrm{PSL}(2, 16)$, and $l(B) = 15$, a final contradiction. □

We remark that even more informations about 2-blocks of defect 4 can be extracted from [37]. For example Cartan matrices and the number of 2-rational and 2-conjugate characters of these blocks are known in most cases.

5 Invariants of blocks

In this section we give an overview in which cases the block invariants of p-blocks for arbitrary primes p are known. It should be pointed out that many p-groups provide only nilpotent fusion systems. For such defect groups all block invariants are known, and we will omit theses cases. The extraspecial group of order p^3 and exponent p^2 for an odd prime p is denoted by p_-^{1+2}. More generally, let M_{p^n} be the (unique) nonabelian group of order p^n with exponent p^{n-1}.

p	D	$I(B)$	classification used?	references
arbitrary	cyclic	arbitrary	no	[8]
arbitrary	abelian	$e(B) \leq 4$	no	[40, 28, 27]
arbitrary	abelian	S_3	no	[41]
≥ 7	abelian	$C_4 \times C_2$	no	[43]
$\notin \{2, 7\}$	abelian	C_3^2	no	[42]
2	metacyclic	arbitrary	no	[5, 23, 28, 39]
2	maximal class $*$ cyclic, incl. $* = \times$	arbitrary	only for $D \cong C_2^3$	[16, 38, 31, 32]
2	minimal nonabelian	arbitrary	only for one family where $\lvert D\rvert = 2^{2r+1}$	[35, 10]
2	minimal nonmetacyclic	arbitrary	only for $D \cong C_2^3$	[37]
2	$\lvert D\rvert \leq 16$	$\ncong C_{15}$	yes	this paper
2	$C_4 \wr C_2$	arbitrary	no	[19]
2	$D_8 * Q_8$	C_5	yes	[34]
2	$C_{2^n} \times C_2^3$, $n \geq 2$	arbitrary	yes	[34]
3	C_3^2	$\notin \{C_8, Q_8\}$	no	[18, 46]
$3, 5, 7, 11$	p_-^{1+2}	$e(B) \leq 2$	no	[33]
3	M_{81}	arbitrary	no	[33]

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