Isotypies for the quasisimple groups with exceptional Schur multiplier

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Abstract

Let B be a block with abelian defect group D of a quasisimple group G such that $G/\operatorname{Z}(G)$ has exceptional Schur multiplier. We show that B is isotypic to its Brauer correspondent in $\operatorname{N}_G(D)$ in the sense of Broué. The proof uses methods from a previous paper [Sambale, 2015] and relies ultimately on computer calculations. Moreover, we verify the Alperin-McKay Conjecture for all blocks of G.

Keywords: Broué's Conjecture, isotypies, exceptional Schur multiplier, Alperin-McKay
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1 Introduction

This paper is concerned with the character theoretic version of Broué's Abelian Defect Group Conjecture [2]. It asks if there is a perfect isometry between a block of a finite group with abelian defect group and its Brauer correspondent provided by Brauer's First Main Theorem. In the past years this question has been answered affirmatively for many series of finite groups. For example the conjecture has been verified for the symmetric groups, the alternating groups and the double covers of those two (see [25, 6, 9, 4, 18]). Linear, unitary and some symplectic groups are handled in [7, 32, 13, 23]. Moreover, the conjecture is true for all principal 2-blocks with abelian defect groups by work of Fong and Harris [8].

In [27] we were able to show the existence of a distinguished perfect isometry, called *isotypy*, for the sporadic simple groups completing work by Rouquier [25]. We used several theoretical methods from block theory, but ultimately relied on computer calculations. The aim of the present paper is to use similar techniques in order to prove the existence of isotypies for another finite class of groups.

According to the Atlas [5, Table 5], a finite simple group has *exceptional Schur multiplier* if and only if it belongs to one of the following families:

(i) sporadic groups (by convention),

(ii) A_6, A_7,

(iii) $\operatorname{PSL}(3,2)$, $\operatorname{PSL}(3,4)$, $\operatorname{PSU}(4,2)$, $\operatorname{PSU}(4,3)$, $\operatorname{PSU}(6,2)$, $\operatorname{Sz}(8)$, $\operatorname{P\Omega}(7,3)$, $\operatorname{PSp}(6,2)$, $\operatorname{P\Omega}^+(8,2)$, $G_2(3)$, $G_2(4)$, $F_4(2)$, ${}^2E_6(2)$.

We note that this list is a bit arbitrary in the following sense. The group $\operatorname{PSL}(2,4)$ has an exceptional Schur multiplier when considered as a group of Lie type A_1, but $\operatorname{PSL}(2,4) \cong A_5$ and A_5 has a non-exceptional Schur multiplier among the alternating groups. The same applies to $\operatorname{PSL}(4,2) \cong A_8$. In view of the results mentioned above we will not consider these groups in the following. Also, $\operatorname{PSL}(3,2) \cong \operatorname{PSL}(2,7)$ and $\operatorname{PSU}(4,2) \cong \operatorname{PSp}(4,3)$ while $\operatorname{PSL}(2,7)$ and $\operatorname{PSp}(4,3)$ have non-exceptional Schur multipliers, but it will not hurt to include these groups. The main theorem of the paper is given as follows.

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Theorem 1. *Let B be a p-block with abelian defect group D of a quasisimple group G such that $G/\mathrm{Z}(G)$ has exceptional Schur multiplier. Then B is isotypic to its Brauer correspondent b in $\mathrm{N}_G(D)$.*

Note that we do not assume that G itself has exceptional center. In particular, we also cover all the simple groups with exceptional Schur multiplier. We will use the notation from Theorem 1 throughout the whole paper.

Since the quasisimple sporadic groups have been handled in [27], it remains to deal with the two remaining classes. In contrast to [27], we are not able to deal with the automorphism groups of quasisimple groups here, because the character tables of some of these extensions are unknown. For example, there are over one hundred extensions of $\mathrm{PSL}(3,4)$. We do not recall the technical definitions of a perfect isometry and an isotypy, because this has been outsourced to [11] and is not needed in the present paper. We will assume that the reader is familiar with the notation and methods used in [27], but for convenience we give a very brief outline:

(i) Use the character table of G in GAP [10] to obtain the number of irreducible (Brauer) characters $k(B)$ (resp. $l(B)$), the decomposition matrix Q_1 (up to basic sets) and the elementary divisors of the Cartan matrix of B.

(ii) Determine the inertial quotient $I(B)$ of B and conclude that Alperin's Weight Conjecture (AWC) $l(B) = l(b)$ and the Alperin-McKay Conjecture (AMC) $k(B) = k(b)$ hold for B (cf. [15]).

(iii) Check that the decomposition matrices of B and b coincide up to basic sets.

(iv) Construct the generalized decomposition matrix by using Q_1, the Broué-Puig $*$-construction and contributions of subsections.

(v) Apply [11, Theorem 2] to obtain a perfect isometry and show that it is in fact an isotypy.

By Kessar-Malle [15], the blocks with abelian defect groups of G are precisely those with height 0 characters only. Therefore, step (i) is routine. Step (ii) needs some work, because neither AWC nor AMC are verified for some of our groups yet (although this is claimed in Späth [30, p. 3], it is still work in progress by Breuer [1]). Step (iii) can be achieved by an algorithm we have developed in [27]. We will not show the details here. Step (iv) has been done in [27] for all but one new case we encounter in the present paper. Finally, the methods for step (v) in [27] work for blocks of defect 2 and we have to review them in order to handle some blocks of defect 3.

After Theorem 1 has been established for G, we use the opportunity to prove AMC for all blocks of G in the last section (see Theorem 3 below). We also illustrate with the group $F_4(2)$ how the methods help to verify the so-called *inductive AM-condition* introduced by Späth [30].

For the definition of the quasisimple groups we use a mixture of the standard notation for the simple groups and the Atlas [5] notation for the centers. However, we often describe subgroups by a more precise notation. In particular, cyclic groups of order n are denoted by Z_n (the letter C is reserved for Cartan matrices). Now we make some remarks about the proof of Theorem 1. As in [27], we can and will restrict ourselves to faithful blocks of defect at least 2. If $p = 2$, we may assume that B has defect at least 4 (see [14]). Among these, it turns out that we only encounter one 2-block B with defect 4 of $G = 4^2.\,\mathrm{PSL}(3,4)$. Obviously, $D \subseteq \mathrm{Z}(G)$ and B is nilpotent. Hence, we are done by [3]. For the odd primes, the only interesting blocks occur for $p \in \{3,5,7\}$. We discuss these individually.

2 The case $p = 3$

2.1 The case $|D| = 9$

We first assume that B has defect 2. By [27, Theorem 3], it suffices to show that AMC and AWC hold for B. In most cases the numbers $k(B)$ and $l(B)$ determine $I(B)$ (see [16, 33]).

The second column in Table 1 lists the positions of the blocks according to the character table library in GAP. The blocks marked with an asterisk are conjugate under an outer automorphism. In particular, these blocks are isomorphic as algebras. If $|I(B)| \leq 4$, then the existence of isotypies follows from [3, 31, 22]. The remaining cases are handled separately.

G	labeling of block(s)	$k(B)$	$l(B)$	$I(B)$	$D \rtimes I(B)$
A_6	1	6	4	Z_4	$\texttt{SmallGroup}(36,9)$
$2.A_6$	3				
A_7	1				
$2.A_7$	3				
$2.{}^2E_6(2)$	18				
$3.A_7$	2	9	2	Z_2	$S_3 \times Z_3$
$3.\operatorname{PSU}(6,2)$	2				
$6.\operatorname{PSU}(6,2)$	7				
$3.{}^2E_6(2)$	5, 8*, 9*				
$\operatorname{PSL}(3,4)$	1	6	5	see 2.1.1	see 2.1.1
$2.\operatorname{PSL}(3,4)$	6				
$4_1.\operatorname{PSL}(3,4)$	9*, 10*				
$4_2.\operatorname{PSL}(3,4)$	9*, 10*				
${}^2E_6(2)$	2, 3				
$F_4(2)$	2, 6	9	5	D_8	$S_3 \wr Z_2$
$2.F_4(2)$	17*, 18*				
${}^2E_6(2)$	4				
$2.{}^2E_6(2)$	17				
$3^2.\operatorname{PSU}(4,3)$	2	9	1	1	Z_3^2

Table 1: faithful 3-blocks with defect group Z_3^2

2.1.1 The case $k(B) = 6$ and $l(B) = 5$

Let B be the principal block of $G = \operatorname{PSL}(3,4)$ with defect group D. Then it is easy to show with GAP that $I(B) \cong \operatorname{N}_G(D)/D \cong Q_8$. Clearly, AMC and AWC follow. If B is a block of a cover of G, then we also have $I(B) \cong Q_8$, since $\operatorname{N}_G(D)/\operatorname{C}_G(D)$ cannot get any bigger. Since Q_8 has a trivial Schur multiplier, AMC and AWC are also satisfied in these cases.

Next let $G = {}^2E_6(2)$. This group is too large to compute the inertial quotients directly. However, the Atlas shows that G contains a subgroup $D \cong Z_3^2$ such that $\operatorname{N}_G(D) \cong (D \rtimes Q_8) \times (\operatorname{PSU}(3,3) \rtimes Z_2)$ is a maximal subgroup. The group $\operatorname{PSU}(3,3)$ has exactly one 3-block of defect 0 which must necessarily be stable under $\operatorname{PSU}(3,3) \rtimes Z_2$. Hence, $\operatorname{PSU}(3,3) \rtimes Z_2$ has two blocks of defect 0 and $\operatorname{N}_G(D)$ has two blocks b_1 and b_2 with defect group D. By Brauer's First Main Theorem, b_i^G have defect group D and $I(b_i^G) \cong Q_8$. This gives AMC and AWC for the second and third block of G. Altogether we have shown that the unknowns in Table 1 represent Q_8 and $\texttt{SmallGroup}(72,41)$ respectively.

2.1.2 The case $I(B) \cong D_8$

We introduce a general lemma which was apparently not known before.

Lemma 2. *Let B be a block of a finite group G with defect group $D \cong Z_3^2$ and $I(B) \cong D_8$. Then AMC and AWC hold for B.*

Proof. Let b_D be a Brauer correspondent of B in $\operatorname{C}_G(D)$, and let $b := b_D^{\operatorname{N}_G(D)}$. By [16], it suffices to show that $l(B) = 5$ if and only if $l(b) = 5$. In order to do so, we follow the proof of [26, Proposition 13.4]. Suppose first that $l(B) = 5$. Then by [27, Theorem 3], B is isotypic to the principal block of $D \rtimes D_8$. It follows that the multiplicity of 3 as an elementary divisor of the Cartan matrix of B is 2. Hence, there exists a lower defect group $P \le D$ for B of order 3. Let $b_P := b_D^{\operatorname{C}_G(P)}$. With the notation of [26, Section 1.8] we have $m_B^{(1)}(P, b_P) > 0$. As usual, b_P dominates a block $\overline{b_P}$ with defect 1 and $I(\overline{b_P}) \cong \operatorname{C}_{I(B)}(P) \cong Z_2$. Hence, $l(b_P) = l(\overline{b_P}) = 2$. Now [26, Proposition 1.43] implies $m_B^{(1)}(P, b_P) = 1$. We get further from [26, Proposition 1.42] that $m_{B_P}^{(1)}(P) = 1$ where $B_P := b_P^{\operatorname{N}_G(P, b_P)}$. The block B_P has defect group D and inertial quotient Z_2^2. By [16], $l(B_P) \in \{1, 4\}$. Since the Cartan matrix of B_P has 3 as an elementary divisor, we conclude that $l(B_P) = 4$. By the Fong-Reynolds Theorem, we may identify B_P with $B_P^{\mathrm{N}_G(P)}$. By [27, Theorem 3], the multiplicity of 3 as an elementary divisor of the Cartan matrix of B_P is 2. Hence, there must be another lower defect group $Q \le D$ of B_P such that $D = P \times Q$. With $\beta_Q := b_D^{\mathrm{C}_G(Q)\cap \mathrm{N}_G(P)}$ and $\beta := b_D^{\mathrm{N}_G(Q,\beta_Q)\cap \mathrm{N}_G(P)}$ we obtain

$$m_\beta^{(1)}(Q) = m_{B_P}^{(1)}(Q, \beta_Q) = 1.$$

Therefore, β is a block with defect group D, $I(\beta) = Z_2^2$ and $l(\beta) = 4$. Since $\mathrm{N}_G(P) \cap \mathrm{N}_G(Q, \beta_Q) \le \mathrm{N}_G(D)$, we are able to go up to b. Let $\widetilde{b}_P := b_D^{\mathrm{C}_G(P)\cap \mathrm{N}_G(D)}$, and let $\widetilde{B}_P := b_D^{\mathrm{N}_G(P,\widetilde{b}_P)\cap \mathrm{N}_G(D)}$. If we identify $\widetilde{B}_P$ with $\widetilde{B}_P^{\mathrm{N}_G(P)\cap \mathrm{N}_G(D)}$ as before, then (Q, β_Q) is also a $\widetilde{B}_P$-subpair. Hence,

$$1 = m_\beta^{(1)}(Q) = m_{\widetilde{B}_P}(Q, \beta_Q)$$

and $l(\widetilde{B}_P) = 4$. Since B_P and $\widetilde{B}_P$ have exactly the same fusion system, P must also be a lower defect group of $\widetilde{B}_P$. This shows that

$$1 = m_{\widetilde{B}_P}^{(1)}(P) = m_b(P, \widetilde{b}_P).$$

It follows that 3 is an elementary divisor of the Cartan matrix of b. Consequently, we must have $l(b) = 5$ (in case $l(b) = 2$ the elementary divisors are 1 and 9 by [27, Theorem 3]). It is easy to see that the converse implication $l(b) = 5 \Rightarrow l(B) = 5$ can be shown similarly. □

2.2 The case $|D| > 9$

Now we turn to 3-blocks with defect at least 3. There are only two instances: the fourth block of $3.{}^2E_6(2)$ and the 17-th block of $6.{}^2E_6(2)$. Both have defect 3 and dominate the fourth block of ${}^2E_6(2)$ and the 17-th block of $2.{}^2E_6(2)$ respectively (see Table 1). It follows that these blocks have inertial quotient D_8 and AMC and AWC hold (see [28, Lemma 2] and [21, Theorem 7.6]). Moreover, the defect group must be elementary abelian of order 27.

It remains to lift the isotypies from the proper quotient groups to the whole groups. We can handle both blocks above simultaneously. Let $\overline{B}$ be the block of $G/\mathrm{Z}(G)_3$ dominated by B. Let $\mathrm{Z}(G)_3 = \langle z \rangle$. If $\overline{\mathcal{R}}$ is a set of representatives for the $I(\overline{B})$-conjugacy classes of $D/\mathrm{Z}(G)_3$, then $\mathcal{R} := \{rz^i : r \in \overline{\mathcal{R}},\ i = 0, 1, 2\}$ represents the $I(B)$-classes of D. Let $\overline{\mathcal{R}} = \{1, u, v\}$. From the proof of [27, Theorem 3] we know the generalized decomposition matrices $\overline{Q_u}$ and $\overline{Q_v}$ of $\overline{B}$. Then also the ordinary decomposition matrix $\overline{Q_1}$ of $\overline{B}$ can be computed. By [24],

$$\mathrm{Irr}(B) = \{\chi * \lambda : \chi \in \mathrm{Irr}(\overline{B}),\ \lambda \in \mathrm{Irr}(\langle z \rangle) \subseteq \mathrm{Irr}(D)\}.$$

Hence, it is easy to compute the (generalized) decomposition matrices Q_1, Q_u and Q_v of B by [28, Lemma 10]. Since (uz^i, b_u) is a B-subsection for $i = 0, 1, 2$, we also obtain the matrices Q_{z^i}, Q_{uz^i} and Q_{vz^i}. The same arguments apply to the Brauer correspondent b. Hence by [11, Theorem 2], there exists a perfect isometry $I : \mathrm{CF}(G, B) \to \mathrm{CF}(\mathrm{N}_G(D), b)$ where $\mathrm{CF}(G, B)$ denotes the space of class functions of G with basis $\mathrm{Irr}(B)$.

Now we go into the details of [27, Section 2.7] in order to show that I is an isotypy. Let β_u be a Brauer correspondent of b in $\mathrm{N}_G(D) \cap \mathrm{C}_G(u)$ where u now stands for any element of $\mathcal{R} \setminus \{1\}$. We need to show that the isometry

$$I^u : \mathrm{CF}(\mathrm{C}_G(u)_{3'}, b_u) \to \mathrm{CF}(\mathrm{C}_{\mathrm{N}_G(D)}(u)_{3'}, \beta_u)$$

extends to a perfect isometry

$$\widehat{I^u} : \mathrm{CF}(\mathrm{C}_G(u), b_u) \to \mathrm{CF}(\mathrm{C}_{\mathrm{N}_G(D)}(u), \beta_u)$$

which does not depend on the generator u of $\langle u \rangle$. This is clear whenever $u \in \mathrm{Z}(G)$. Thus, let $u \in \mathcal{R} \setminus \mathrm{Z}(G)$. Then b_u and β_u have defect group D, inertial index 2 and focal subgroup $[D, I(b_u)]$ (resp. $[D, I(\beta_u)]$) of order 3. Hence, the Cartan matrix of b_u (resp. β_u) is given by

$$9 \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

up to basic sets (see [31]). Since $\mathrm{C}_D(I(b_u))$ acts semiregularly on $\mathrm{Irr}(B)$ via the $*$-construction, the decomposition matrix of b_u (resp. β_u) is given by

$$(\underbrace{X^{\mathrm{T}},\ldots,X^{\mathrm{T}}}_{9\text{ times}})^{\mathrm{T}} \qquad\qquad X:=\begin{pmatrix}1&.\\.&1\\1&1\end{pmatrix}.$$

At this point we can follow the arguments of [27, Section 2.7] word by word.

3 The case $p = 5$

Here only blocks of defect 2 occur and we encounter the following pairs of invariants

$$(k(B),l(B))\in\{(14,6),(16,10),(16,14),(20,14),(20,16)\}.$$

All but the second of these pairs did already occur in [27]. However, we need to be careful, since AMC and AWC are not known a priori. Nevertheless, the methods from [27] imply

$$I(B)\cong\begin{cases}\mathrm{SL}(2,3)\rtimes Z_4 & \text{if } (k(B),l(B))=(20,16),\\ Z_4\wr Z_2 & \text{if } (k(B),l(B))=(20,14),\\ D_{12} & \text{if } (k(B),l(B))=(14,6).\end{cases}$$

Now suppose that $k(B) = 16$ and $l(B) = 10$. Here we obtain $I(B) \cong D_8 * Z_4$ (central product) or $I(B) \cong Z_4 \wr Z_2$. By way of contradiction, suppose that the second case occurs. Then $I(B)$ acts imprimitively on D (as a linear group). It follows that there are two non-trivial subsections (x, b_x) and (y, b_y) such that $l(b_x) = 4$ and $l(b_y) = 2$. From the character table of G we can show that the smallest ordinary contribution $25m^1_{ii}$ is 14 (see [27] for notation). Since $M_1 + M_x + M_y = 1_{16}$, we conclude that $m^x_{ii} \leq 9$. The Cartan matrix C_x of b_x is given by $C_x = 5(1 + \delta_{ij})^4_{i,j=1}$. Now Plesken's algorithm shows that there is no possibility for M_x. This contradiction shows that $I(B) \cong D_8 * Z_4$.

Finally, let $k(B) = 16$ and $l(B) = 14$. Then $I(B) \cong \mathrm{SL}(2, 3) \rtimes Z_2$ or $I(B) \cong Z_{24} \rtimes Z_2$. These cases are harder to distinguish, because they lead to the same generalized decomposition matrix. However, these blocks only occur if $G/\mathrm{Z}(G)$ is $\mathrm{P}\Omega^+(8, 2)$, $F_4(2)$ or ${}^2E_6(2)$. In any case D is a Sylow 5-subgroup of G. Let B_0 be the principal block of G. Then $k(B_0) = 20$ and $l(B_0) = 16$. Hence, by what we have already shown, $\mathrm{N}_G(D)/\mathrm{C}_G(D) \cong \mathrm{SL}(2, 3) \rtimes Z_4$. Since this quotient does not contain $Z_{24} \rtimes Z_2$, we derive that $I(B) \cong \mathrm{SL}(2, 3) \rtimes Z_2$.

G	labeling of block(s)	$k(B)$	$l(B)$	$I(B)$	$D \rtimes I(B)$
$\mathrm{P}\Omega^+(8,2)$ $2.\mathrm{P}\Omega^+(8,2)$	1 27	16	10	$D_8 * Z_4$	`SmallGroup`(400, 207)
$F_4(2)$ $2.F_4(2)$ ${}^2E_6(2)$ $2.{}^2E_6(2)$	1 53 1 79	20	16	$\mathrm{SL}(2,3) \rtimes Z_4$	`PrimitiveGroup`(25, 19)
$G_2(4)$ $2.G_2(4)$	1 14	14	6	D_{12}	`SmallGroup`(300, 25)
${}^2E_6(2)$ $2.{}^2E_6(2)$	5 78	16	14	$\mathrm{SL}(2,3) \rtimes Z_2$	`SmallGroup`(1200, 947)
$3.{}^2E_6(2)$ $6.{}^2E_6(2)$	78*, 79* 257*, 258*	20	14	$Z_4 \wr Z_2$	`SmallGroup`(800, 1191)

Table 2: faithful 5-blocks with non-cyclic abelian defect groups

In the next step we verify AWC (and therefore also AMC, since $k(B)-l(B)$ is locally determined) for the blocks listed in Table 2. The groups $\mathrm{SL}(2, 3) \rtimes Z_2$ and $\mathrm{SL}(2, 3) \rtimes Z_4$ have trivial Schur multiplier. Hence, if $I(B)$ is one of these, then $l(b) = k(I(B)) = l(B)$. This verifies the conjectures for these cases. Then also Theorem 1 follows for these cases, because the generalized decomposition matrix was already constructed in [27].

3.1 The case $I(B) \cong Z_4 \wr Z_2$

First we deal with the blocks of $3.{}^2E_6(2)$. In order to do so, we need to investigate the group $G = {}^2E_6(2)$ more closely. Here we have two blocks with defect group $D \in \mathrm{Syl}_5(G)$ (listed in Table 2). By Brauer's First Main Theorem and the remark above, we have $k(\mathrm{N}_G(D)/D) = l(\mathrm{N}_G(D)) = 16 + 14 = 30$. Moreover, $\mathrm{C}_G(D)/D$ has exactly three 5-blocks, i. e. $k(\mathrm{C}_G(D)/D) = 3$. Hence, $\mathrm{C}_G(D)/D$ is isomorphic to Z_3 or to S_3. Since the action of $\mathrm{N}_G(D)/D$ on $\mathrm{Irr}(\mathrm{C}_G(D)/D)$ has only two orbits, we must have $\mathrm{C}_G(D)/D \cong Z_3$. Now a GAP calculations shows that $\mathrm{N}_G(D)/D \cong \texttt{SmallGroup}(288, 70)$ or $\texttt{SmallGroup}(288, 404)$. Next we turn to $3.G$. This group has two non-faithful blocks (not listed in Table 2) coming from G and a pair B_1, B_2 of conjugate faithful blocks (of defect 2). Suppose that AWC fails for B_1. Then it must also fail for B_2. Let b_i be the Brauer correspondent of B_i in $\mathrm{N}_{3.G}(D)$. The Schur multiplier of $I(B_i) \cong Z_4 \wr Z_2$ is Z_2. It is easy to show that this implies $l(b_i) = 5$. Hence, $k(\mathrm{N}_{3.G}(D)/D) = 30 + 5 + 5 = 40$. Now another GAP calculation shows that there is no such group $\mathrm{N}_{3.G}(D)/D$. This contradiction gives $l(b_i) = 14$. Moreover, we obtain $\mathrm{N}_{3.G}(D)/D \cong \texttt{SmallGroup}(864, 707)$ and a posteriori $\mathrm{N}_G(D)/D \cong \texttt{SmallGroup}(288, 404)$.

With this information in hand, we can go to $6.G$. This group has four non-faithful blocks coming from $3.G$ and two more non-faithful blocks coming from $2.G$. Moreover, it has two conjugate faithful blocks B_1 and B_2. If we suppose as above that AWC fails for B_i, then it follows that $k(\mathrm{N}_{6.G}(D)/D) = 16+16+14+14+14+14+5+5 = 98$. On the other hand, $\mathrm{N}_{6.G}(D)/D$ is a central extension of $\texttt{SmallGroup}(864, 707)$ with Z_2. Using GAP we see that no such group exists. Hence, AWC holds for B_i.

3.2 The remaining cases

The blocks of $G_2(4)$, $2.G_2(4)$ and $\mathrm{P}\Omega^+(8, 2)$ are very easy to handle directly, since one can construct $\mathrm{N}_G(D)$ in GAP. Now let $G = 2.\,\mathrm{P}\Omega^+(8, 2)$. This group is not stored in GAP, so we need some trick. It is well-known that the principal block of G has a trivial 2-cocycle. Hence, it follows that AWC holds for this block and it remains to consider the non-principal block B of G. We know that $\mathrm{N}_G(D)/D$ is a central extension of $\mathrm{N}_G(D)/\,\mathrm{C}_G(D) \cong D_8 * Z_4$ with Z_2. We have to show that this is not a Schur extension (otherwise $l(b) < l(B)$). With the Atlas notation, $\mathrm{N}_G(D)$ is contained in the 15-th maximal subgroup M of G. Fortunately, the character table of M is stored in GAP. It turns out that M has the form $A_5^2.(Z_4 \times Z_2)$ and has 50 conjugacy classes. Using the `grpconst` package in GAP, we are able to show that M is uniquely determined as $M = \texttt{TransitiveGroup}(20, 543)$. This leads to $\mathrm{N}_G(D)/D \cong \texttt{SmallGroup}(32, 25)$ and $k(\mathrm{N}_G(D)/D) = 20$. Hence, AWC holds for B.

To complete the proof of Theorem 1 for all 5-blocks, we still need to construct the generalized decomposition matrix for the blocks B with $k(B) = 16$ and $l(B) = 10$, since this case is not covered by [27]. Here all irreducible characters of B and b are p-rational. In particular, the generalized decomposition matrix is integral. The ordinary decomposition matrix Q_1 and the corresponding contribution matrix M_1 of B (and of b) are given by

$$Q_1 = \begin{pmatrix}
1 & . & . & . & . & . & . & . & . & . \\
. & 1 & . & . & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & 1 & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . & . \\
. & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & 1 & . & . \\
. & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & . & 1 \\
. & 1 & . & . & 1 & . & 1 & 1 & 1 & 1 \\
1 & . & 1 & 1 & . & 1 & . & . & 1 & 1 \\
. & 1 & 1 & . & . & 1 & 1 & . & 1 & 1 \\
. & 1 & . & 1 & . & 1 & . & 1 & 1 & 1 \\
1 & . & 1 & . & 1 & . & 1 & . & 1 & 1 \\
1 & . & . & 1 & 1 & . & . & 1 & 1 & 1
\end{pmatrix}$$

$$25M_1 = \begin{pmatrix} 16 & 6 & -4 & -4 & -4 & 1 & 1 & 1 & -3 & -3 & -2 & 3 & -2 & -2 & 3 & 3 \\ 6 & 16 & 1 & 1 & 1 & -4 & -4 & -4 & -3 & -3 & 3 & -2 & 3 & 3 & -2 & -2 \\ -4 & 1 & 16 & 1 & 1 & -4 & -4 & 6 & -3 & -3 & -2 & 3 & 3 & -2 & 3 & -2 \\ -4 & 1 & 1 & 16 & 1 & -4 & 6 & -4 & -3 & -3 & -2 & 3 & -2 & 3 & -2 & 3 \\ -4 & 1 & 1 & 1 & 16 & 6 & -4 & -4 & -3 & -3 & 3 & -2 & -2 & -2 & 3 & 3 \\ 1 & -4 & -4 & -4 & 6 & 16 & 1 & 1 & -3 & -3 & -2 & 3 & 3 & 3 & -2 & -2 \\ 1 & -4 & -4 & 6 & -4 & 1 & 16 & 1 & -3 & -3 & 3 & -2 & 3 & -2 & 3 & -2 \\ 1 & -4 & 6 & -4 & -4 & 1 & 1 & 16 & -3 & -3 & 3 & -2 & -2 & 3 & -2 & 3 \\ -3 & -3 & -3 & -3 & -3 & -3 & -3 & -3 & 19 & -6 & 1 & 1 & 1 & 1 & 1 & 1 \\ -3 & -3 & -3 & -3 & -3 & -3 & -3 & -3 & -6 & 19 & 1 & 1 & 1 & 1 & 1 & 1 \\ -2 & 3 & -2 & -2 & 3 & -2 & 3 & 3 & 1 & 1 & 14 & -6 & 4 & 4 & 4 & 4 \\ 3 & -2 & 3 & 3 & -2 & 3 & -2 & -2 & 1 & 1 & -6 & 14 & 4 & 4 & 4 & 4 \\ -2 & 3 & 3 & -2 & -2 & 3 & 3 & -2 & 1 & 1 & 4 & 4 & 14 & 4 & 4 & -6 \\ -2 & 3 & -2 & 3 & -2 & 3 & -2 & 3 & 1 & 1 & 4 & 4 & 4 & 14 & -6 & 4 \\ 3 & -2 & 3 & -2 & 3 & -2 & 3 & -2 & 1 & 1 & 4 & 4 & 4 & -6 & 14 & 4 \\ 3 & -2 & -2 & 3 & 3 & -2 & -2 & 3 & 1 & 1 & 4 & 4 & -6 & 4 & 4 & 14 \end{pmatrix}$$

up to basic sets. There are three non-trivial B-subsections (x, b_x), (y, b_y) and (z, b_z). We have $l(b_x) = l(b_y) = l(b_z) = 2$ and the Cartan matrix of these blocks is $C_x = C_y = C_z = 5\left(\begin{smallmatrix} 3 & 2 \\ 2 & 3 \end{smallmatrix}\right)$ up to basic sets. By the shape of C_x, we have $25m_{ii}^x \geq 2$ and similarly for y and z. Since $M_1 + M_x + M_y + M_z = 1_{16}$, it follows that $25m_{ii}^x \leq 7$. Now Plesken's algorithm gives two essentially different solutions for Q_x and we need to adjust these to Q_1:

$$\begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 & . & . & . & . & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & . & . & . & . & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^{\mathrm{T}}, \tag{3.1}$$

$$\begin{pmatrix} 2 & 2 & 1 & . & . & . & . & . & . & . & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & . & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^{\mathrm{T}}. \tag{3.2}$$

(This is very similar to the case $(k(B), l(B)) = (16, 12)$ in [27]). The $*$-construction shows that $25M_x \equiv 25M_y \equiv 25M_z \pmod 5$ and $3 \cdot 25M_1 \equiv 25M_x \pmod 5$. This implies

$$\begin{aligned} 25m_{ii}^1 = 16 &\iff \{25m_{ii}^x, 25m_{ii}^y, 25m_{ii}^z\} = \{3, 3, 3\}, \\ 25m_{ii}^1 = 19 &\iff \{25m_{ii}^x, 25m_{ii}^y, 25m_{ii}^z\} = \{2, 2, 2\}, \\ 25m_{ii}^1 = 14 &\iff \{25m_{ii}^x, 25m_{ii}^y, 25m_{ii}^z\} = \{7, 2, 2\}. \end{aligned}$$

Suppose that (3.2) occurs for Q_x. Then there are distinct characters $\chi, \psi \in \mathrm{Irr}(B)$ such that $25m_{\chi\psi}^x = \pm 7$ and $25m_{\chi\psi}^y = 25m_{\chi\psi}^z = \pm 2$. Then $25m_{\chi\psi}^1 = \mp(7 + 2 + 2) = \mp 11$, but this is a contradiction. Hence, Q_x, Q_y and Q_z have shape (3.1). Since $25m_{1,1}^1 = 16$, we may assume that the first row of Q_x, Q_y and Q_z is $(1, 0)$ (after interchanging the irreducible Brauer characters of b_x, b_y and b_z if necessary). Since $25m_{1,2}^1 = 6$, the second row of all three matrices must be $(0, 1)$. Now in the third row we have some choices. Since x, y and z are still interchangeable, we may assume that the third row of Q_x is $(0, 1)$ and the third row of Q_y and Q_z is $(1, 0)$. After interchanging y and z if necessary, we may assume that the fourth rows of these matrices are $(1, 0)$, $(0, 1)$ and $(1, 0)$ respectively. Then all the remaining entries are uniquely determined and can be computed in a straight-forward manner.

$$\begin{aligned} Q_x &= \begin{pmatrix} 1 & . & . & 1 & 1 & . & . & 1 & 1 & 1 & -1 & -1 & 2 & -1 & -1 & 1 \\ . & 1 & 1 & . & . & 1 & 1 & . & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 2 \end{pmatrix}^{\mathrm{T}}, \\ Q_y &= \begin{pmatrix} 1 & . & 1 & . & 1 & . & 1 & . & 1 & 1 & -1 & -1 & -1 & 2 & 1 & -1 \\ . & 1 & . & 1 & . & 1 & . & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 2 & -1 \end{pmatrix}^{\mathrm{T}}, \\ Q_z &= \begin{pmatrix} 1 & . & 1 & 1 & . & 1 & . & . & 1 & 1 & 2 & 1 & -1 & -1 & -1 & -1 \\ . & 1 & . & . & 1 & . & 1 & 1 & 1 & 1 & 1 & 2 & -1 & -1 & -1 & -1 \end{pmatrix}^{\mathrm{T}}. \end{aligned}$$

This completes the proof in case $p = 5$.

4 The case $p = 7$

Here again, only blocks of defect 2 are of interest. In all cases we have $k(B) = 27$ and $l(B) = 21$. The arguments in [27] show that the inertial quotient is always isomorphic to $\mathrm{SL}(2,3) \times Z_3$.

G	labeling of block(s)	$k(B)$	$l(B)$	$I(B)$	$D \rtimes I(B)$
$F_4(2)$	1	27	21	$\mathrm{SL}(2,3) \times Z_3$	`PrimitiveGroup`$(49, 27)$
$2.F_4(2)$	58				
${}^2E_6(2)$	1				
$2.{}^2E_6(2)$	73				
$3.{}^2E_6(2)$	77^*, 78^*				
$6.{}^2E_6(2)$	231^*, 232^*				

Table 3: faithful 7-blocks with non-cyclic abelian defect groups

The Schur multiplier of $\mathrm{SL}(2,3) \times Z_3$ is Z_3. The AMC and AWC hold for the principal blocks of $F_4(2)$ and ${}^2E_6(2)$, since here the Külshammer-Puig 2-cocycle is trivial. In both cases the principal block is the only block with maximal defect. Hence, we have $\mathrm{N}_G(D)/D \cong \mathrm{SL}(2,3) \times Z_3$ for both simple groups G. This implies that $\mathrm{N}_{2.G}(D)/D \cong \mathrm{SL}(2,3) \times Z_6$ and the conjectures also hold for the blocks of the double covers. Now let $G = 3.{}^2E_6(2)$. According to the Atlas, $F_4(2)$ is a maximal subgroup of ${}^2E_6(2)$. Therefore, $\mathrm{N}_G(D) \subseteq F_4(2) \times Z_3$ and again the conjectures follow. The blocks of $6.{}^2E_6(2)$ are completely similar. The existence of the isotypy follows from [27]. This completes the proof of Theorem 1.

5 The Alperin-McKay Conjecture

Recall that $k_0(B)$ denotes the number of irreducible characters of height 0 in the block B.

Theorem 3. *The Alperin-McKay Conjecture holds for all blocks of quasisimple groups G such that $G/\mathrm{Z}(G)$ has exceptional Schur multiplier.*

Proof. Most cases have already been handled by work of several mathematicians. On Breuer's website [1] many simple groups are listed for which the so-called inductive AM-condition from [30] is fulfilled. These include the sporadic groups, A_6 and A_7 as well as some groups of Lie type. It follows from [30, Theorem C] that AMC holds for all the corresponding quasisimple groups.

The covers of $\mathrm{PSL}(3,4)$ and $\mathrm{PSU}(6,2)$ can also be handled by computer, since one can construct the universal covers by `PerfectGroup`$(967680, 4)$ and `AtlasGroup("(2^2x3).U6(2)")` efficiently in GAP. Now let G be the universal cover $(3^2 \times 4).\mathrm{PSU}(4,3)$, and let $P \in \mathrm{Syl}_2(G)$. Then P is also a Sylow 2-subgroup of $\mathrm{SU}(4,3)$. This makes it possible to show $\mathrm{Z}(G)_2 \le P'$. Moreover, $\mathrm{N}_G(P)/\mathrm{Z}(G) = \mathrm{N}_{G/\mathrm{Z}(G)}(P\,\mathrm{Z}(G)/\mathrm{Z}(G)) = P\,\mathrm{Z}(G)/\mathrm{Z}(G)$ and $\mathrm{N}_G(P) \cong P \times Z_3^2$. Therefore, AMC predicts that any 2-block of maximal defect of a cover of $\mathrm{PSU}(4,3)$ has exactly $|P : P'| = 8$ irreducible characters of height 0. This can be confirmed easily in GAP.

Now let G be any quasisimple group such that $G/\mathrm{Z}(G)$ has exceptional Schur multiplier. Then blocks with abelian defect groups are dealt with by Theorem 1. Moreover, it suffices to consider the faithful p-blocks where $p \in \{2, 3\}$. Finally, if $p = 2$, we may restrict ourselves to blocks of defect at least 5 by using [26, Theorem 13.6]. Table 4 lists the remaining cases.

We see that all blocks have maximal defect except the ones in the last two rows. We first deal with these blocks of maximal defect. If p is the defining characteristic, things should be a bit easier, but apparently the result is only known if in addition $p \ge 5$ (see [30, Theorem 8.4]). So we need to handle these cases as well. Assume that $\mathrm{Z}(G) = 1$. Then the principal block B_0 is the only block of maximal defect. Hence, AMC for B_0 follows from the ordinary McKay Conjecture which has been established by Malle [19]. Actually, Malle only considered groups with exceptional centers, but in order to prove the claim he uses [19, Corollary 2.2] which covers the simple groups as well.

G	p	labeling of block(s)	defect
$\mathrm{PSU}(4,3)$ $2.\,\mathrm{PSU}(4,3)$ $4.\,\mathrm{PSU}(4,3)$	3	1 3 4^*, 5^*	6
$3_1.\,\mathrm{PSU}(4,3)$ $3_2.\,\mathrm{PSU}(4,3)$ $6_1.\,\mathrm{PSU}(4,3)$ $6_2.\,\mathrm{PSU}(4,3)$ $12_1.\,\mathrm{PSU}(4,3)$ $12_2.\,\mathrm{PSU}(4,3)$	3	1 1 3 3 4^*, 5^* 4^*, 5^*	7
$3^2.\,\mathrm{PSU}(4,3)$ $(3^2\times 2).\,\mathrm{PSU}(4,3)$ $(3^2\times 4).\,\mathrm{PSU}(4,3)$	3	1 3 4^*, 5^*	8
$\mathrm{P}\Omega(7,3)$ $3.\,\mathrm{P}\Omega(7,3)$	2	1 6^*, 7^*	9
$2.\,\mathrm{P}\Omega(7,3)$ $6.\,\mathrm{P}\Omega(7,3)$	2	1 6, 7	10
$\mathrm{P}\Omega(7,3)$ $2.\,\mathrm{P}\Omega(7,3)$	3	1 3	9
$3.\,\mathrm{P}\Omega(7,3)$ $6.\,\mathrm{P}\Omega(7,3)$	3	1 3	10
$\mathrm{P}\Omega^+(8,2)$	2	1	12

G	p	labeling of block(s)	defect
$2.\,\mathrm{P}\Omega^+(8,2)$	2	1	13
$2^2.\,\mathrm{P}\Omega^+(8,2)$	2	1	14
$\mathrm{P}\Omega^+(8,2)$ $2.\,\mathrm{P}\Omega^+(8,2)$	3	1 7	5
$F_4(2)$	2	1	24
$2.F_4(2)$	2	1	25
$F_4(2)$ $2.F_4(2)$	3	1 16	6
${}^2E_6(2)$ $3.{}^2E_6(2)$	2	1 3^*, 4^*	36
$2.{}^2E_6(2)$ $6.{}^2E_6(2)$	2	1 3^*, 4^*	37
$2^2.{}^2E_6(2)$ $(2^2\times 3).{}^2E_6(2)$	2	1 3^*, 4^*	38
${}^2E_6(2)$ $2.{}^2E_6(2)$	3	1 16	9
$3.{}^2E_6(2)$ $6.{}^2E_6(2)$	3	1 16	10
$3.{}^2E_6(2)$ $6.{}^2E_6(2)$	3	2, 3 18	3

Table 4: some faithful blocks with non-abelian defect groups

If $|\mathrm{Z}(G)|$ is divisible by p, then the McKay Conjecture can be reduced to a strong form of the McKay Conjecture for the group $G/\mathrm{Z}(G)_p$ (see proof of [12, Theorem 2.1]). Now assume that $\mathrm{Z}(G)$ is exceptional or p is the defining characteristic. Then the (strong form of the) McKay Conjecture is true by [19, 29] (for $p = 2$ one can also argue by [20]). In this way we see that the ordinary McKay Conjecture holds for all groups in Table 4.

Suppose that $|\mathrm{Z}(G)| = 2$. Then AMC holds for the principal block, because it is non-faithful. Hence, the McKay Conjecture implies AMC for the remaining block of maximal defect (if it exists). Now assume that $|\mathrm{Z}(G)| = 3$. Then there are at most two faithful blocks B_1 and B_2 of maximal defect. It turns out that these are conjugate under an outer automorphism. In particular, $k_0(B_1) = k_0(B_2)$ and similarly for the Brauer correspondents in $\mathrm{N}_G(D)$. Since AMC is already satisfied for the non-faithful block(s), the claim follows again from the McKay Conjecture. The other blocks of maximal defect where $|\mathrm{Z}(G)| > 3$ can be handled in the same manner.

It remains to deal with the blocks not of maximal defect. Thus, let B be a 3-block of $3.{}^2E_6(2)$ or $6.{}^2E_6(2)$ with non-abelian defect group D of order 27. Then $k(B) = 16$, $k_0(B) = 6$ and $l(B) = 5$. Moreover, B dominates a block $\overline{B}$ of $G/\mathrm{Z}(G)_3$ with defect 2 and inertial quotient Q_8 (see Table 1). Consequently, $I(B) \cong Q_8$ and D has exponent 3. Let b be the Brauer correspondent of B in $\mathrm{N}_G(D)$ as usual. Let (u, β_u) be a b-subsection. Then β_u dominates a block of $\mathrm{N}_G(D)/\mathrm{Z}(G)_3 = \mathrm{N}_{G/\mathrm{Z}(G)_3}(D/\mathrm{Z}(G)_3)$ with defect 2. Since AMC holds for $\overline{B}$, we conclude that $k(b) = k(B) = 16$ and $l(b) = 5$ (see [26, Theorem 1.35]). Moreover, by inflation we have $k_0(b) \ge k(\overline{B}) = 6$. Now assume that (u, β_u) is non-major. Then $l(\beta_u) = 1$ and the Cartan matrix of β_u is (9). Hence, the non-vanishing generalized decomposition numbers d^u_{ij} are $(2, 1, \ldots, 1)$ or $(1, \ldots, 1)$. This shows that $k_0(b) \in \{6, 9\}$ (see [26, Proposition 1.36]). On the other hand, the major subsections show that the six inflated characters are the only p-rational irreducible character of B. Since the non-p-rational characters come in pairs, the number of non-zero numbers d^u_{ij} is even. Therefore, $k_0(b) = 6$ and AMC is satisfied. □

In the situation of Theorem 3 it is desirable to prove the inductive AM-condition [30, Definition 7.2] as well. However, this often requires an embedding of $\mathrm{N}_G(D)$ in G which is usually not constructed in the framework of our methods. Moreover, one needs good knowledge about $\mathrm{Aut}(G)$. For example, the group $\mathrm{PSL}(3, 4)$ is relatively small, but still does not appear on Breuer's list [1]. Nevertheless, we can verify a special case where the situation is particularly simple.

Proposition 4. *The inductive AM-condition holds for $F_4(2)$.*

Proof. We use the inductive AM-condition from [17, Definition 6.2]. The Schur multiplier and the outer automorphism group of $F_4(2)$ both have order 2. First let $p > 2$. In view of [30, Lemma 8.1] and Theorem 3, it suffices to show that AMC holds for all p-blocks of $G = 2.F_4(2).2$. This is true for all blocks with cyclic defect groups. Now assume that $p = 3$. Then there are blocks B of defect 2 with $k(B) = 9$ and $l(B) = 7$. By [16, 33], $I(B) \cong SD_{16}$ and AMC holds. Now let B be a 3-block with defect group $D \in \mathrm{Syl}_3(G)$. By the proof of [19, Theorem 4.1], $\mathrm{N}_G(D) \cong \mathrm{PSL}(4,3).2^2 \times Z_2$ and AMC follows easily.

Next suppose that $p = 5$. Here we have blocks of defect 2 such that $(k(B), l(B)) \in \{(20,16),(16,14)\}$. These can be handled as in Section 3. Finally, let $p = 7$. There are blocks of defect 2 with $k(B) = 27$ and $l(B) = 24$. By [27], we have $I(B) \cong 2.S_4^- \times Z_3$ where $2.S_4^-$ is the binary octahedral group. Since $I(B)$ has trivial Schur multiplier and $k(I(B)) = l(B)$, AMC is satisfied for these blocks.

It remains to prove the inductive AM-condition for $F_4(2)$ with respect to the prime $p = 2$. The universal cover $G = 2.F_4(2)$ has two 2-blocks. The non-principal block has defect 1 so that the inductive AM-condition holds by [17]. Thus, it suffices to consider the principal block B of G. Since the McKay Conjecture holds for $p = 2$, also the AMC holds for B. In particular, we have a bijection $\Lambda : \mathrm{Irr}_0(G) = \mathrm{Irr}_0(B) \to \mathrm{Irr}_0(\mathrm{N}_G(D)) = \mathrm{Irr}_0(b)$. Every orbit of $\mathrm{Aut}(G)$ on $\mathrm{Irr}_0(G)$ has length 1 or 2. Since $k_0(G) = 32$ and $k_0(G.2) = 16$, we obtain that exactly 8 characters in $\mathrm{Irr}_0(G)$ are stable under $\mathrm{Aut}(G)$. Since the McKay Conjecture also holds for $G.2$, there are also exactly 8 stable characters in $\mathrm{Irr}_0(\mathrm{N}_G(D))$. Hence, we can make Λ equivariant. From the character tables it follows that all characters in $\mathrm{Irr}_0(G)$ are inflations from $G/\mathrm{Z}(G)$. This shows that

$$\mathrm{Irr}(\mathrm{Z}(G)|\chi) = \{1_{\mathrm{Z}(G)}\} = \mathrm{Irr}(\mathrm{Z}(G)|\Lambda(\chi)) \qquad (\chi \in \mathrm{Irr}_0(G)).$$

We still need to verify part (iii) of [17, Definition 6.2]. With the notation of this paper we have $\chi \in \mathrm{Irr}_0(G)$, $M = \mathrm{N}_G(D)$, $Z = \mathrm{Z}(G)$, $\overline{G} = F_4(2)$ and $A \in \{F_4(2), F_4(2).2\}$. In particular, the equation

$$\mathrm{Irr}(\mathrm{C}_A(\overline{G})|\widetilde{\chi}) = \mathrm{Irr}(\mathrm{C}_A(\overline{G})|\widetilde{\chi}')$$

becomes trivially satisfied, since $\mathrm{C}_A(\overline{G}) = 1$. Also the last equality $\mathrm{bl}(\widetilde{\chi}_J) = \mathrm{bl}(\ldots)^J$ is trivial, because all groups J with $\overline{G} \le J \le A$ have only one 2-block of maximal defect. This completes the proof. □

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