Fusion invariant characters of p-groups

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Abstract

We consider complex characters of a p-group P, which are invariant under a fusion system $\mathcal{F}$ on P. Extending a theorem of Bárcenas–Cantarero to non-saturated fusion systems, we show that the number of indecomposable $\mathcal{F}$-invariant characters of P is greater or equal than the number of $\mathcal{F}$-conjugacy classes of P. We further prove that these two quantities coincide whenever $\mathcal{F}$ is realized by a p-solvable group. On the other hand, we observe that this is false for constrained fusion systems in general. Finally, we construct a saturated fusion system with an indecomposable $\mathcal{F}$-invariant character, which is not a summand of the regular character of P. This disproves a recent conjecture of Cantarero–Combariza.

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1 Introduction

Let $\mathcal{F}$ be a fusion system (not necessarily saturated) on a finite p-group P (we refer the reader to [1] for terminology). Elements $x, y \in P$ are called $\mathcal{F}$-*conjugate* if there exists a morphism $f : \langle x \rangle \to P$ in $\mathcal{F}$ such that $f(x) = y$. We denote the number of $\mathcal{F}$-conjugacy classes of P by $k(\mathcal{F})$. A complex class function χ of P is called $\mathcal{F}$-*invariant* if χ is constant on the $\mathcal{F}$-conjugacy classes of P. These characters can often be used to construct new characters of finite groups via the Broué–Puig $*$-construction introduced in [3]. Further motivation and background can be found in the recent paper of Cantarero–Combariza [4].

We call an $\mathcal{F}$-invariant character of P *indecomposable* if it is not the sum of two (non-zero) $\mathcal{F}$-invariant characters (this is unrelated to the characters of indecomposable modules). Let $\mathrm{Ind}_{\mathcal{F}}(P)$ be the set of indecomposable $\mathcal{F}$-invariant characters of P. The following lemma is well-known among experts in lattice theory (it follows from *Gordan's lemma*), but perhaps less known among representation theorists.

Lemma 1. *There are only finitely many indecomposable $\mathcal{F}$-invariant characters of P.*

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Proof. Let $\mathrm{Irr}(P) = \{\chi_1, \ldots, \chi_k\}$. For $\psi \in \mathrm{Ind}_{\mathcal{F}}(P)$ let $c(\psi) = ([\psi, \chi_i] : i = 1, \ldots, k) \in \mathbb{N}_0^k$. We define a partial order on $\mathbb{N}_0^k$ by $a \le b :\Longleftrightarrow b - a \in \mathbb{N}_0^k$. It is easy to see that the set $\{c(\psi) : \psi \in \mathrm{Ind}_{\mathcal{F}}(P)\}$ is an antichain in $\mathbb{N}_0^k$ with respect to $\le$, i.e. no two distinct elements are comparable. Therefore, it is enough to show that every antichain in $\mathbb{N}_0^k$ is finite.

By way of contradiction, suppose that $c^{(1)}, c^{(2)}, \ldots$ is an infinite antichain in $\mathbb{N}_0^k$. We may replace this sequence by an infinite subsequence such that $c_1^{(1)} \le c_1^{(2)} \le \ldots$. This sequence can in turn be replaced by a subsequence such that $c_2^{(1)} \le c_2^{(2)} \le \ldots$. Repeating this process k times yields an infinite sequence $c^{(1)} \le c^{(2)} \le \ldots$. But this is impossible since the original sequence was an antichain. □

In the theory of lattices, the set $\mathrm{Ind}_{\mathcal{F}}(P)$ is sometimes called the *Hilbert basis* of the semigroup of $\mathcal{F}$-invariant characters. Since for every $k \ge 2$, the poset $\mathbb{N}_0^k$ contains antichains of arbitrary finite lengths (e.g. $(n, 1, *, \ldots, *)$, $(n-1, 2, *, \ldots, *), \ldots$ for any $n \in \mathbb{N}$), it is not easy to give an upper bound on $|\mathrm{Ind}_{\mathcal{F}}(P)|$. In the last section of this paper we construct examples with $|\mathrm{Ind}_{\mathcal{F}}(P)| > |P|$. However, since there are only finitely many fusion systems on a given p-group P, it is clear that $|\mathrm{Ind}_{\mathcal{F}}(P)|$ can be bounded by a function in $|P|$. A related question for quasi-projective characters has been raised by Willems–Zalesski [13, Question 4.2].

2 The number of indecomposable $\mathcal{F}$-invariant characters

The following result was shown for saturated fusion systems by Bárcenas–Cantarero [2, Lemma 2.1] using some advanced category theory. Our proof applies to arbitrary fusion systems.

Theorem 2. *The space of $\mathcal{F}$-invariant class functions of P is spanned by $\mathrm{Ind}_{\mathcal{F}}(P)$. In particular, $|\mathrm{Ind}_{\mathcal{F}}(P)| \ge k(\mathcal{F})$.*

Proof. By a theorem of Park [9], there exists a finite group G such that $P \le G$ and the morphisms of $\mathcal{F}$ are induced by conjugation in G. In particular, $k(\mathcal{F})$ is the number of G-conjugacy classes, which intersect P. Let T be the part of the character table of G, whose columns belong to elements in P. Since the character table is invertible, T has full rank. Hence, the (G-invariant) restrictions χ_P for $\chi \in \mathrm{Irr}(G)$ span the space of G-invariant class functions on P. Since each χ_P can be decomposed into G-invariant indecomposable characters, the claim follows. □

Since Park's result, which we used in the proof, relies on computations in the Burnside ring, we like to offer a conceptually simpler proof for saturated fusion systems:

Proof of Theorem 2 for saturated fusion systems. Let

$$\zeta = \sum_{\chi \in \mathrm{Irr}(P)} a_\chi \chi$$

be $\mathcal{F}$-invariant where $a_\chi \in \mathbb{C}$ for $\chi \in \mathrm{Irr}(P)$. We define an equivalence relation on $\mathrm{Irr}(P)$ by $\chi \sim \psi$ if and only if there exist positive integers s, t such that $sa_\chi = ta_\psi$. For an equivalence class $T \subseteq \mathrm{Irr}(P)$ let $\zeta^{(T)} := \sum_{\chi \in T} a_\chi \chi$. There exists a some $z \in \mathbb{C}$ such that $z\zeta^{(T)}$ is a character of P. Since $\zeta = \sum_T \zeta^{(T)}$, it suffices to show that $\zeta^{(T)}$ is $\mathcal{F}$-invariant.

Recall that by Alperin's fusion theorem, every morphism in $\mathcal{F}$ is a composition of automorphisms of some subgroups of P (see [1, Theorem I.3.5]). For every $Q \le P$, the restricted class function ζ_Q is invariant under $\mathrm{Aut}_{\mathcal{F}}(Q)$. Let $\chi,\psi\in\mathrm{Irr}(P)$ such that $\chi\not\sim\psi$. Then, by the definition of $\sim$, we have $[a_\chi\chi_Q,\tau]\ne[a_\psi\psi_Q,\tau]$ for every $\tau\in\mathrm{Irr}(Q)$. It follows that each $(\zeta^{(T)})_Q$ is $\mathrm{Aut}_{\mathcal{F}}(Q)$-invariant. Again by Alperin's fusion theorem, $\zeta^{(T)}$ is $\mathcal{F}$-invariant. □

The argument (Alperin's fusion theorem) in our second proof does not work for arbitrary fusion systems. For instance, $P\cong C_4\rtimes C_4$ can be embedded (regularly) into the symmetric group S_{16} such that all elements of order 4 in P are conjugate. However, if we choose $x,y\in P$ of order 4 such that $P=\langle x,y\rangle$, then the conjugation of x to y cannot be realized by a composition of automorphisms of subgroups of P. As a matter of fact, the only saturated fusion system on P is the trivial system (see [11, Theorem 1]).

Now we restrict ourselves further to non-exotic saturated fusion systems. Here we can prove a stronger theorem, which resembles the fact that Brauer characters are restrictions of generalized characters (see [8, Corollary 2.16]).

Theorem 3. *Let G be a finite group with Sylow p-subgroup P. Then every G-invariant character ζ of P is the restriction of a generalized character of G.*

Proof. We extend ζ to a class function $\hat{\zeta}$ of G in the following way: Every $g\in G$ is conjugate to an element of the form $xy=yx$ where $x\in P$ and y is a p'-element. We define $\hat{\zeta}(g):=\zeta(x)$ (this is well-defined since ζ is G-invariant). Now we use Brauer's induction theorem to show that $\hat{\zeta}$ is a generalized character of G (along the lines of [8, proof of Lemma 2.15]). To this end, let $N\le G$ be a nilpotent subgroup with Sylow p-subgroup $Q\trianglelefteq N$. After conjugation, we may assume that $Q\le P$. Then $\hat{\zeta}_Q=\zeta_Q$ is a character of $Q\cong N/\mathrm{O}_{p'}(N)$ and $\hat{\zeta}_N$ is the inflation of ζ_Q to N. In particular, $\hat{\zeta}_N$ is a (generalized) character of N. Hence, $\hat{\zeta}$ is a generalized character of G, which restricts to ζ. □

Obviously, every G-invariant character of P is a summand of a restriction of a character of G. However, an indecomposable character is not necessarily a summand of a restriction of an irreducible character of G. A counterexample will be given in the last section of the paper.

The following lemma of Cantarero–Combariza [4, Corollary 2.9] characterizes equality in Theorem 2.

Lemma 4. *For every fusion system $\mathcal{F}$ on P we have $|\mathrm{Ind}_{\mathcal{F}}(P)|=k(\mathcal{F})$ if and only if every $\mathcal{F}$-invariant character of P can be decomposed* uniquely *into indecomposable characters.*

Proof. If $|\mathrm{Ind}_{\mathcal{F}}(P)|=k(\mathcal{F})$, then $\mathrm{Ind}_{\mathcal{F}}(P)$ is a basis of the space of $\mathcal{F}$-invariant class functions and the result follows. Now assume that $|\mathrm{Ind}_{\mathcal{F}}(P)|>k(\mathcal{F})$. Since the dimension of the $\mathbb{Q}$-vectorspace spanned by $\mathrm{Ind}_{\mathcal{F}}(P)$ is bounded by $k(\mathcal{F})$, the set $\mathrm{Ind}_{\mathcal{F}}(P)$ is linearly dependent over $\mathbb{Q}$. Hence, there exist integers $c_\psi\in\mathbb{Z}$ (not all zero) such that

$$\sum_{\psi\in\mathrm{Ind}_{\mathcal{F}}(P)}c_\psi\psi=0.$$

Since the degree of each character is positive, not all c_ψ can have the same sign. If we bring the negative coefficients to the right hand side, we end up with two distinct decompositions of an $\mathcal{F}$-invariant character. □

Cantarero and Combariza [4, Lemma 2.17] have proven that $|\mathrm{Ind}_{\mathcal{F}}(P)|=k(\mathcal{F})$ holds for controlled fusion systems (among other cases). A controlled fusion system is realized by a group of the form $P\rtimes H$ for some p'-group H. Our main theorem generalizes this result to the larger class of p-solvable groups.

Theorem 5. *Let $\mathcal{F}$ be the (saturated) fusion system on a Sylow p-subgroup P of a p-solvable group G. Then $|\mathrm{Ind}_{\mathcal{F}}(P)| = k(\mathcal{F})$.*

Proof. We apply Isaacs' theory of π-partial characters, where $\pi = \{p\}$ (see [7, p. 71]). Every indecomposable $\mathcal{F}$-invariant character χ of P extends uniquely to a class function $\hat{\chi}$ on the set of p-elements of G. By [7, Corollary 3.5], $\hat{\chi}$ is an irreducible p-partial character of G. The number of those characters is exactly $k(\mathcal{F})$ by [7, Theorem 3.3]. □

We remark that every fusion system of a p-solvable group is constrained by the Hall–Higman lemma. Conversely, by the model theorem [1, Theorem I.4.9], every constrained fusion system is realized by a p-constrained group. However, Theorem 5 does not hold for constrained fusion systems in general as we are about to see.

3 Counterexamples

In [4, table on p. 5206] and [5], the authors list some fusion systems $\mathcal{F}$ where $|\mathrm{Ind}_{\mathcal{F}}(P)| > k(\mathcal{F})$, including the system on $P \cong D_{16}$ of the group $\mathrm{PSL}(2, 17)$. This fusion system has two conjugacy classes of essential subgroups. The authors seem to have overlooked the "smaller" fusion system of $\mathrm{PGL}(2, 7)$ with only one class of essential subgroups (still on D_{16}). With the notation

$$P = \langle x, y \mid x^8 = y^2 = 1,\ x^y = x^{-1}\rangle$$

the character table of P is:

	1	x	x^3	x^2	x^4	y	xy
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	-1	1	1	1	-1
χ_3	1	-1	-1	1	1	-1	1
χ_4	1	1	1	1	1	-1	-1
χ_5	2	0	0	-2	2	0	0
χ_6	2	$\sqrt{2}$	$-\sqrt{2}$	0	-2	0	0
χ_7	2	$-\sqrt{2}$	$\sqrt{2}$	0	-2	0	0

We may assume that x^4 and y are $\mathcal{F}$-conjugate, but the other classes of P are not fused. The $\mathcal{F}$-invariant characters of P must agree on the fifth and sixth column of the character table. Hence, we are looking for non-negative integral vectors orthogonal to $(0, 0, 1, 1, 1, -1, -1)$. Now it is easy to see that

$$\mathrm{Ind}_{\mathcal{F}}(P) = \{\chi_1,\ \chi_2,\ \chi_3 + \chi_6,\ \chi_3 + \chi_7,\ \chi_4 + \chi_6,\ \chi_4 + \chi_7,\ \chi_5 + \chi_6,\ \chi_5 + \chi_7\}.$$

In particular, $|\mathrm{Ind}_{\mathcal{F}}(P)| = 8 > 6 = k(\mathcal{F})$.

To turn this into a constrained fusion system, we set $G := \mathrm{PGL}(2, 7)$ and choose an irreducible faithful $\mathbb{F}_2G$-module V of dimension 6. Then

$$\hat{G} := V \rtimes G = \mathrm{PrimitiveGroup}(64, 64) = \mathrm{TransitiveGroup}(16, 1802)$$

(notation from GAP [6]) is a 2-constrained group with Sylow 2-subgroup $\hat{P} := V \rtimes P$. Let $\hat{\mathcal{F}}$ be the corresponding constrained fusion system. The inflations of the eight G-invariant indecomposable characters of P are $\hat{\mathcal{F}}$-indecomposable. According to the proof of Theorem 2, there must be at least $k(\mathcal{F})-6$ other indecomposable character arsing as summands of $\chi_{\hat{P}}$, where $\chi \in \mathrm{Irr}(\hat{G})$ with $V \nsubseteq \mathrm{Ker}(\chi)$. In particular, $|\mathrm{Ind}_{\mathcal{F}}(\hat{P})| > k(\hat{\mathcal{F}})$.

In [4, Conjecture 2.19], the authors have conjectured that for saturated fusion systems $\mathcal{F}$, every indecomposable $\mathcal{F}$-invariant character of P is a summand of the regular character. As a consequence of Theorem 5, we obtain this for p-solvable groups.

Theorem 6. *Let $\mathcal{F}$ be the fusion system on a Sylow p-subgroup P of a p-solvable group. Then every indecomposable $\mathcal{F}$-invariant character of P is a summand of the regular character of P.*

Proof. This follows from Theorem 5 and [4, Remark 2.18]. For the convenience of the reader we repeat the short proof of the latter result: Let ψ be an indecomposable $\mathcal{F}$-invariant character of P. Let

$$m := \max\big\{[\psi, \chi] : \chi \in \mathrm{Irr}(P)\big\}.$$

Then ψ is a summand of $m\rho$, where ρ is the regular character of P. By the hypothesis and Lemma 4, $m\rho$ has a unique decomposition into indecomposable $\mathcal{F}$-invariant characters. Since ρ itself is $\mathcal{F}$-invariant (remember that $\rho(x) = 0$ for all $x \in P \setminus \{1\}$), ψ must appear as a summand of ρ. □

On the other hand, we provide a counterexample to [4, Conjecture 2.19]. Let $\mathcal{F}$ be the fusion system on a Sylow 2-subgroup P of the automorphism group of the Mathieu group $G = \mathrm{Aut}(M_{22}) \cong M_{22} \rtimes C_2$. Then $|P| = 2^8$ and $k(\mathcal{F}) = 10$. Let $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_{21}\}$ and $\mathrm{Irr}(P) = \{\lambda_1, \ldots, \lambda_{34}\}$. Let

$$A := \big([(\chi_i)_P, \lambda_j]\big)_{i,j} \in \mathbb{Z}^{34\times 21}.$$

By Theorem 3, $\mathrm{Ind}_{\mathcal{F}}(P)$ is in one-to-one correspondence to the Hilbert basis of the semigroup

$$\{x \in \mathbb{Z}^{21} : Ax \geq 0\}.$$

Using the `nconvex`-package [10] in GAP, we compute $|\mathrm{Ind}_{\mathcal{F}}(P)| = 25$. Moreover, 14 indecomposable $\mathcal{F}$-invariant characters are not summands of the regular character of P and six are not summands of restrictions of irreducible characters of G. The source code is available at [12]. The symmetric group $G = S_{12}$ is a counterexample for $p = 2, 3$. As promised in the introduction, $G = S_{10}$ for $p = 2$ provides an example where $|\mathrm{Ind}_{\mathcal{F}}(P)| = 266 > 256 = |P|$.

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References

[1] M. Aschbacher, R. Kessar and B. Oliver, *Fusion systems in algebra and topology*, London Mathematical Society Lecture Note Series, Vol. 391, Cambridge University Press, Cambridge, 2011.

[2] N. Bárcenas and J. Cantarero, *A completion theorem for fusion systems*, Israel J. Math. **236** (2020), 501–531.

[3] M. Broué and L. Puig, *Characters and local structure in G-algebras*, J. Algebra **63** (1980), 306–317.

[4] J. Cantarero and G. Combariza, *Uniqueness of factorization for fusion-invariant representations*, Comm. Algebra **51** (2023), 5187–5208.

[5] J. Cantarero and J. Gaspar, *Fusion-invariant representations for symmetric groups*, to appear in Bull. Iran. Math. Soc., arXiv: 2305.17587v1.

[6] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.12.2*; 2022, (`http://www.gap-system.org`).

[7] I. M. Isaacs, *Characters of solvable groups*, Graduate Studies in Mathematics, Vol. 189, American Mathematical Society, Providence, RI, 2018.

[8] G. Navarro, *Characters and blocks of finite groups*, London Mathematical Society Lecture Note Series, Vol. 250, Cambridge University Press, Cambridge, 1998.

[9] S. Park, *Realizing fusion systems inside finite groups*, Proc. Amer. Math. Soc. **144** (2016), 3291–3294.

[10] K. Saleh and S. Gutsche, *NConvex, A Gap package to perform polyhedral computations, Version 2022.09-01*, `https://homalg-project.github.io/pkg/NConvex`.

[11] B. Sambale, *Fusion systems on metacyclic* 2*-groups*, Osaka J. Math. **49** (2012), 325–329.

[12] B. Sambale, *GAP code to compute indecomposable F-invariant characters*, `https://github.com/BrauerSuzuki/GAP-codes/`.

[13] W. Willems and A. E. Zalesski, *Quasi-projective and quasi-liftable characters*, J. Algebra **442** (2015), 548–559.