On Loewy lengths of blocks

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Abstract

We give a lower bound on the Loewy length of a p-block of a finite group in terms of its defect. Afterwards we discuss blocks with small Loewy length. Since blocks with Loewy length at most 3 are known, we focus on blocks of Loewy length 4 and provide a relatively short list of possible defect groups. It turns out that p-solvable groups can only admit blocks of Loewy length 4 if $p = 2$. However, we find (principal) blocks of simple groups with Loewy length 4 and defect 1 for all $p \equiv 1 \pmod 3$. We also consider sporadic, symmetric and simple groups of Lie type in defining characteristic. Finally, we give stronger conditions on the Loewy length of a block with cyclic defect group in terms of its Brauer tree.

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1 Introduction

Let F be an algebraically closed field of characteristic $p > 0$, and let B be a block of the group algebra FG of a finite group G over F. Moreover, let D be a defect group of B. We denote the inertial index of B by $e(B)$.

For a finite-dimensional F-algebra A, we denote by $J(A)$ the Jacobson radical and by $LL(A)$ the Loewy length of A. Similarly, we denote by $LL(M)$ the Loewy length of a finitely generated A-module M. For a finite p-group P, we denote by $r(P)$ its rank and by $\exp(P)$ its exponent.

One aim of this paper is to give a general lower bound on $LL(B)$ in terms of the defect of B. This is established in the next section by making use of work by Oppermann [40] and Külshammer [32]. Since this inequality is usually very crude, we provide a different approach in terms of a certain fixed point algebra on $Z(D)$. Here our result on the Loewy length of a fixed point algebra might be of general interest. Finally, for blocks with cyclic defect groups we express the Loewy length via Brauer trees.

The third section deals with blocks of small Loewy length. After stating the known result about Loewy length at most 3, we determine the possible defect groups for blocks with Loewy length 4. For fixed $p \geq 5$ we get at most 12 isomorphism types of these groups. Since blocks of small Loewy length in solvable groups are well understood, we turn to blocks of (almost) (quasi)simple groups. Symmetric (and thus also alternating) groups can be completely handled, while for sporadic groups and simple groups of Lie type in defining characteristic we restrict to principal blocks. Here we develop general methods and reductions.

2 Defect and Loewy length of blocks

Lemma 2.1. *Let P be a finite p-group of order p^δ, exponent p^ϵ and (normal) rank ρ. Then*

$$\delta \leq \binom{\rho + 1}{2}(2\epsilon - 1).$$

Proof. Let A be a maximal abelian normal subgroup of P. Then $|A| = p^\alpha$ where $\alpha \leq \epsilon\rho$. Moreover, we have $A = C_P(A)$, and $P/A = P/C_P(A)$ is isomorphic to a subgroup of $\mathrm{Aut}(A)$. By III.3.19 in [22], $|\mathrm{Aut}(A)|$ divides

$$p^{\rho(\alpha-\rho)}(p^\rho - 1)(p^\rho - p)\cdots(p^\rho - p^{\rho-1}).$$

Thus $|P/A| = p^\sigma$ where

$$\sigma \leq \rho(\alpha - \rho) + 1 + 2 + \cdots + (\rho - 1) \leq \rho^2(\epsilon - 1) + \binom{\rho}{2}.$$

Hence an elementary calculation shows that

$$\delta = \alpha + \sigma \leq \binom{\rho+1}{2}(2\epsilon - 1).$$

□

Lemma 2.2. *Let B be a p-block with defect group D and Loewy length $\lambda > 1$. Then $\rho \leq \lambda - 1$ and $\epsilon \leq 1 + \lfloor \log_p(\lambda - 1) \rfloor$ where $p^\epsilon = \exp(D)$ and $\rho = r(D)$.*

Proof. The first inequality follows from Oppermann's proof [40, Corollary 3] of a conjecture by Benson. Moreover, a result by Külshammer (see [32, K. Korollar] or [34, Equation (21)]) implies that $\exp(D)/p < \lambda$. This proves the second inequality. □

Theorem 2.3. *Let B be a p-block of defect δ and Loewy length $\lambda > 1$. Then*

$$\delta \leq \binom{\lambda}{2}(2\lfloor \log_p(\lambda - 1) \rfloor + 1).$$

Proof. Let D be a defect group of B. Moreover, let $\exp(D) = p^\epsilon$ and $r(D) = \rho$. Then the results above imply:

$$\delta \leq \binom{\rho+1}{2}(2\epsilon - 1) \leq \binom{\lambda}{2}(2\lfloor \log_p(\lambda - 1) \rfloor + 1).$$

□

If G is p-solvable then, by a result of Koshitani [28, Theorem], we always have $\lambda > \delta(p - 1)$. However, this bound is not valid for arbitrary finite groups as one can see from our examples in the last section of this paper. We are wondering whether, in general, we have $\lambda > \delta$.

Our next result gives a different bound for the Loewy length of a block.

Proposition 2.4. *Let B be a p-block with defect group D and Loewy length λ. Moreover, let b be a p-block of $DC_G(D)$ such that $b^G = B$, and denote by $T := N_G(D, b)$ the inertial subgroup of b. Then T acts on $Z(D)$, and we have $LL(FZ(D)^T) \leq LL(Z(B)) \leq \lambda$ where*

$$FZ(D)^T := \{x \in FZ(D) : txt^{-1} = x \text{ for } t \in T\}$$

denotes the algebra of fixed points. In particular, if T acts trivially on $Z(D)$ then $LL(FZ(D)) \leq \lambda$.

Proof. Since $J(Z(B)) \subseteq J(B)$ we have $LL(Z(B)) \leq LL(B) = \lambda$. By a result of Broué [6, Proposition (III)1.1], $Z(B)$ has an ideal I (the kernel of the Brauer homomorphism) such that $Z(B)/I$ is isomorphic to $FZ(D)^T$; in particular, we have

$$LL(FZ(D)^T) = LL(Z(B)/I) \leq LL(Z(B)) \leq \lambda.$$

□

We are going to compare the Loewy lengths of $FZ(D)$ and of $FZ(D)^T$, in the situation of the proposition above. In order to do this, we quote Lemma 2.3.1 in [38] which the authors attribute to D. J. Benson.

Lemma 2.5 (Benson)**.** *Let T be a finite group whose order is not divisible by $p = \mathrm{char}(F)$, let A be a commutative T-algebra over F, and let I be a T-stable ideal of A. Then $I^{|T|} \subseteq I^T \cdot A$.*

We are going to apply this in the proof of the following result.

Proposition 2.6. *Let P be a finite abelian p-group, and let T be a p'-subgroup of $\mathrm{Aut}(P)$. Then*

$$LL(FP^T) \geq \frac{LL(FP)-1}{|T|} + 1.$$

Proof. We apply Benson's Lemma with $A := FP$ and $I := J(FP)$. Then we obtain $J(FP)^{|T|} \subseteq J(FP)^T \cdot A = J(FP^T) \cdot FP$. Thus

$$J(FP)^{|T|(\lambda-1)} \subseteq J(FP^T)^{(\lambda-1)} \cdot FP \subseteq \mathrm{Soc}(FP^T) \cdot FP$$

where $\lambda := LL(FP^T)$. But FP^T is a symmetric F-algebra (cf. Section 2 in [26], for example). Since FP^T is a local F-algebra, we have $\dim \mathrm{Soc}(FP^T) = 1$. Thus $\mathrm{Soc}(FP^T) = FP^+$ where $P^+ := \sum_{g \in P} g$. We conclude that $\mathrm{Soc}(FP^T) \cdot FP = P^+ \cdot FP = \mathrm{Soc}(FP)$, so that $J(FP)^{|T|(\lambda-1)+1} = 0$. Thus $LL(FP) \leq |T|(\lambda - 1) + 1$, and

$$\frac{LL(FP)-1}{|T|} \leq \lambda - 1 = LL(FP^T) - 1.$$

The result follows. □

The results above lead to the following consequence.

Corollary 2.7. *Let B be a block with maximal Brauer pair (D, b) and inertial group $T = N_G(D, b)$. Moreover, let $\overline{T} := T/C_G(Z(D))$. Then*

$$LL(B) \geq LL(Z(B)) \geq LL(FZ(D)^{\overline{T}}) \geq \frac{LL(FZ(D))-1}{|\overline{T}|} + 1.$$

We recall that the Loewy length of the group algebra $FZ(D)$ of the abelian group $Z(D)$ can be computed easily:

(i) If A_1 and A_2 are finite-dimensional F-algebras then, as is well known, we have

$$LL(A_1 \otimes A_2) = LL(A_1) + LL(A_2) - 1.$$

(ii) Now let $P = C_{p^{a_1}} \times \cdots \times C_{p^{a_r}}$ where $a_1, \ldots, a_r \in \mathbb{N}$. Then (i) implies that

$$LL(FP) = p^{a_1} + \cdots + p^{a_r} - r + 1.$$

We also observe the following consequence of our results.

Corollary 2.8. *Let B be a p-block with a cyclic defect group D. Then*

$$LL(B) \geq LL(Z(B)) = \frac{|D|-1}{e(B)} + 1.$$

Proof. It remains to prove that

$$LL(Z(B)) \leq \frac{|D|-1}{e(B)} + 1.$$

Let b be the Brauer correspondent of B in $N_G(D)$. Then, as is well known, the blocks B and b are perfectly isometric; in particular, their centers are isomorphic. By a result of Külshammer [33, A. Theorem], b is Morita equivalent to FH where H denotes the semidirect product of D and the inertial factor group $\overline{T}$. Thus $Z(B) \cong Z(b) \cong Z(FH)$. Recall that $\dim Z(FH) = \frac{|D|-1}{e(B)} + e(B)$. It follows easily that

$$Z(FH) = FD^T \oplus I_1(FH)$$

where $I_1(FH)$ is the subspace of $Z(FH)$ spanned by all class sums of defect 0, an ideal in $Z(FH)$ contained in $\mathrm{Soc}(FH)$. We conclude that

$$LL(FD^T) = LL(Z(FH)) = LL(Z(b)) = LL(Z(B)) \geq \frac{|D|-1}{e(B)} + 1 = \dim FD^T,$$

and the result follows. □

If in the situation of Corollary 2.8 the Brauer tree Γ of B is known, one can compute $LL(B)$ explicitly as follows. We attach to each vertex v of Γ a multiplicity m_v which is $\frac{|D|-1}{e(B)}$ if v is exceptional and 1 otherwise. Also, each vertex v of Γ has a degree d_v which equals the number of edges with endpoint v. If S is a simple B-module and v, w are the endpoints of the edge in Γ corresponding to S, then the Loewy length of the projective cover P_S of S satisfies

$$LL(P_S) = \max\{d_v m_v + 1, d_w m_w + 1\}.$$

Thus the Loewy length of B equals

$$LL(B) = \max\{d_v m_v + 1 : v \text{ vertex of } \Gamma\}. \tag{1}$$

For later purpose we consider the Loewy length with respect to normal subgroups and quotients.

Proposition 2.9. *Suppose that B dominates a block b of G/N where N is a normal subgroup of G. Then $LL(b) \le LL(B)$.*

Proof. Let $f : FG \longrightarrow F[G/N]$ denote the canonical epimorphism. Then $f(B) = b_1 \oplus \cdots \oplus b_r$ where $b_1 = b, b_2, \ldots, b_r$ are the blocks of G/N dominated by B. Thus

$$f(J(B)) = J(f(B)) = J(b_1 \oplus \cdots \oplus b_r) = J(b_1) \oplus \cdots \oplus J(b_r).$$

Let $\lambda := LL(B)$. Then $J(B)^\lambda = 0$, and

$$0 = f(J(B)^\lambda) = f(J(B))^\lambda = J(b_1)^\lambda \oplus \cdots \oplus J(b_r)^\lambda.$$

Thus $LL(b_i) \le \lambda$ for $i = 1, \ldots, r$. $\square$

Proposition 2.10. *Suppose that B covers a block b of a normal subgroup H of G. Then $LL(b) \le LL(B)$. Similarly, we have $LL(P_{F_H}) \le LL(P_{F_G})$ where P_{F_G} is the projective cover of the trivial FG-module F_G. Moreover, if p does not divide $|G : H|$, then $LL(B) = LL(b)$.*

Proof. Let V be an indecomposable projective b-module. Then V is a direct summand of $\operatorname{Res}^G_H(U)$ for an indecomposable projective B-module U. Recall that $J(FH) \subseteq J(FG)$. Thus, $J(B)^t = 0$ implies that $0 = J(FG)^t U \supseteq J(FH)^t U \supseteq J(b)^t V$. Thus $J(b)^t = 0$.

It is clear that P_{F_H} is a direct summand of P_{F_G}. Let $t := LL(P_{F_G})$. Then $0 = J(FG)^t P_{F_G} \supseteq J(FH)^t P_{F_G} \supseteq J(FH)^t P_{F_H}$, so that $LL(P_{F_H}) \le t$.

The last statement is a result of Koshitani and Miyachi [30, (4.1) Lemma(i)]. $\square$

3 Blocks with small Loewy length

Theorem 2.3 gives a crude bound on the defect of a block if its Loewy length is given. If the Loewy length is small, we have more precise results.

Proposition 3.1 (Okuyama [39, Theorem 1])**.** *Let B be a p-block with defect δ and Loewy length λ. Then the following holds:*

(i) $\lambda = 1$ *if and only if* $\delta = 0$.

(ii) $\lambda = 2$ *if and only if* $\delta = 1$ *and* $p = 2$.

(iii) $\lambda = 3$ *if and only if one of the following holds*

  (a) $p = \delta = 2$ *and B is Morita equivalent to* $F[C_2 \times C_2]$ *or to* FA_4,

  (b) $p > 2$, $\delta = 1$, $e(B) \in \{p-1, (p-1)/2\}$ *and the Brauer tree of B is a straight line with the exceptional vertex at the end (if it exists).*

Hence, we turn to blocks of Loewy length 4 in the following.

Lemma 3.2. *Let P be a finite p-group of exponent p and rank $\rho \le 3$ where $p \ge 5$. Then P is isomorphic to one of the following groups (p-powers of generators and not mentioned commutator relations between generators are defined to be trivial):*

(i) C_p, $C_p \times C_p$, $C_p \times C_p \times C_p$ *or* p_+^{1+2} *(extraspecial of order p^3),*

(ii) $C_p \times p_+^{1+2}$,

(iii) $\langle a, b, c, d \mid [b, d] = a,\ [c, d] = b\rangle$,

(iv) p_+^{1+4} *(extraspecial of order p^5),*

(v) $\langle a, b, c, d, e \mid [d, e] = a,\ [a, d] = b = [e, c]\rangle$,

(vi) $\langle a, b, c, d, e \mid [d, e] = a,\ [a, d] = b,\ [a, e] = c\rangle$,

(vii) $\langle a, b, c, d, e \mid [d, e] = a,\ [a, d] = b,\ [a, e] = c = [b, d]\rangle$,

(viii) $\langle a, b, c, d, e, f \mid [b, a] = c,\ [c, a] = d,\ [d, a] = e,\ [d, b] = [e, b] = [c, d] = f\rangle$ *where* $p \ge 7$,

(ix) $\langle a, b, c, d, e, f \mid [b, a] = c,\ [c, a] = d,\ [d, a] = e = [c, b],\ [d, b] = [e, b] = [c, d] = f\rangle$ *where* $p \ge 7$.

Proof. The result is obvious when $|P| \le p^3$. The groups of order p^4 can be found in III.12.6 of [22], for example. The groups of order p^5 can be found in a paper by Schreier [43, §6E], for example. The groups of order p^6 and exponent p can be found in a paper by Wilkinson [52, Table 1], for example. We slightly adjust the generators and relations for sake of simplicity and uniformity. □

Although the proof shows that the groups in Lemma 3.2 are pairwise non-isomorphic, it is often useful to know some invariants which distinguish these groups:

- The groups in (ii) and (iii) have order p^4. The one in (iii) has a cyclic center whereas the one in (ii) does not.
- The groups in (iv) – (vii) have order p^5. The one in (iv) has a derived subgroup of order p, the one in (v) a derived subgroup of order p^2, and the ones in (vi) and (vii) have derived subgroups of order p^3. Moreover, the group in (vii) has a cyclic center while the one in (vi) does not.
- The groups in (viii) and (ix) have both order p^6. They have maximal nilpotency class and are exceptional in the sense of Definition III.14.5 of [22]. Let P be one of these two groups, and set $P_1 := C_P(K_2(P)/K_4(P))$. Then $P_1 = \langle b, c, d, e, f\rangle$ is a characteristic maximal subgroup of P in both cases, and $r(P_1) = 3$. If P is of type (viii) then P_1 is of type (iv), and if P is of type (ix) then P_1 is of type (v).

We add some examples. A Sylow 5-subgroup of the sporadic simple group Co_1 has exponent 5 and rank 3 (see [9] and Table 5.6.1 on p. 303 in [15]); it is the group appearing in Lemma 3.2(iii), for $p = 5$ (this follows for example from the inclusion $5_+^{1+2} \rtimes \mathrm{GL}(2, 5) \le Co_1$, see Table 5.3 on p. 211 in [53]). Similarly, a Sylow 7-subgroup of the sporadic simple group M called the Monster has exponent 7 and rank 3; it is the group appearing in Lemma 3.2(viii), for $p = 7$ (see Table 5.6 on p. 258 in [53]). Also, for $p \ge 7$ the Sylow p-subgroups of the exceptional groups of Lie type $G_2(p)$ are isomorphic to the groups in Lemma 3.2(viii) (follows from $G_2(p) \le \mathrm{GL}(7, p)$, Table 3.3.1 on p. 108 in [15] and Table 4.1 on p. 127 in [53]).

Proposition 3.3. *Let B be a p-block of Loewy length 4 where $p \ge 5$. Then the defect groups of B are isomorphic to one of the p-groups appearing in Lemma 3.2.*

Proof. Let D be a defect group of B. Then Theorem 2.3 implies that $|D| \le p^6$. Moreover, Lemma 2.2 shows that $r(D) \le 3$ and $\exp(D) \le p$. □

Combining Theorem 2.3 and Proposition 3.3 gives the following bound on the defect δ of a block with Loewy length 4:

$$\delta \leq \begin{cases} 18 & \text{if } p \leq 3, \\ 5 & \text{if } p = 5, \\ 6 & \text{if } p \geq 7. \end{cases}$$

A result by Koshitani [27] implies that even $\delta \leq 3$ whenever $p = 2$ and B is the principal block (see Theorem 4.5 below). It is perhaps of interest that there are at most 12 possibilities for the isomorphism type of D when $p \geq 7$, and at most 10 possibilities for $p = 5$. We do not expect, however, that all these p-groups really occur as defect groups of p-blocks of Loewy length 4. In fact, we only have examples for C_p whenever $p > 2$.

Suppose that B is a p-block with defect group D and Loewy length 4. Moreover, suppose that $|Z(D)| = p$. Then $4 \geq (p-1)/e(B) + 1$ by Corollary 2.7, i.e. $e(B) \geq (p-1)/3$. Thus the inertial index has to be "large". Now suppose that $Z(D) \cong C_p \times C_p$. Then $4 \geq 2(p-1)/e(B) + 1$, i.e. $e(B) \geq 2(p-1)/3$, and again $e(B)$ has to be "large". We use this to impose further conditions on the list in Lemma 3.2.

Proposition 3.4. *Let P be a p-group and $\varphi : \mathrm{Aut}(P) \to \mathrm{Aut}(Z(P))$ the restriction map. Then the following holds:*

(i) If P is one of the groups from Lemma 3.2(i),(iii),(iv),(v),(viii)*, then φ is surjective.*

(ii) If P is the group from Lemma 3.2(ii)*, then $\varphi(\mathrm{Aut}(P)) \cong (C_p \rtimes C_{p-1}) \times C_{p-1}$.*

(iii) If P is the group from Lemma 3.2(vi)*, then*

$$\varphi(\mathrm{Aut}(P)) = \begin{cases} \mathrm{SL}(2,p) \rtimes C_{(p-1)/3} & \text{if } p \equiv 1 \pmod 3, \\ \mathrm{Aut}(Z(P)) \cong \mathrm{GL}(2,p) & \text{otherwise.} \end{cases}$$

(iv) If P is the group from Lemma 3.2(vii)*, then*

$$\varphi(\mathrm{Aut}(P)) = \begin{cases} C_{(p-1)/5} & \text{if } p \equiv 1 \pmod 5, \\ \mathrm{Aut}(Z(P)) \cong C_{p-1} & \text{otherwise.} \end{cases}$$

(v) If P is the group from Lemma 3.2(ix)*, then*

$$\varphi(\mathrm{Aut}(P)) = \begin{cases} C_{(p-1)/7} & \text{if } p \equiv 1 \pmod 7, \\ \mathrm{Aut}(Z(P)) \cong C_{p-1} & \text{otherwise.} \end{cases}$$

Proof. If P is abelian, there is nothing to prove. If P is extraspecial, then the claim follows from [54, Theorem 1]. Now, let us assume that P is the group from Lemma 3.2(iii). Choose a primitive root ω modulo p. Then it is easy to see that the map $c \mapsto c^\omega$, $d \mapsto d$ is an automorphism of P whose restriction generates $\mathrm{Aut}(Z(P)) = \mathrm{Aut}(\langle a \rangle)$. In case of Lemma 3.2(v) we can use the automorphism $c \mapsto c$, $d \mapsto d$ and $e \mapsto e^\omega$ for the same conclusion. The case (viii) will be handled later.

For the group $P \cong C_p \times p_+^{1+2}$ we have $C_p \cong P' \leq Z(P) \cong C_p \times C_p$. Thus, $\varphi(\mathrm{Aut}(P))$ consists of triangular matrices in $\mathrm{GL}(2,p)$. Obviously, $\varphi(\mathrm{Aut}(P))$ contains all diagonal automorphisms. If we write $P = \langle a, b, c, d \mid [b, c] = d \rangle$, then it is easy to see that the map $a \mapsto ad$, $b \mapsto b$ and $c \mapsto c$ is an automorphism. This shows that every triangular matrix lies in $\varphi(\mathrm{Aut}(P))$.

Next assume that P is the group from Lemma 3.2(vi). Here the map $\alpha : d \mapsto e$, $e \mapsto d^{-1}$ is an automorphism which acts on $Z(P) = \langle b, c \rangle$ as $\alpha(b) = c$ and $\alpha(c) = b^{-1}$. Similarly, the map β defined by $d \mapsto de$ and $e \mapsto e$ is an automorphism with $\beta(b) = bc$ and $\beta(c) = c$. It is well-known that $\langle \alpha, \beta \rangle \cong \mathrm{SL}(2,p) \leq \mathrm{Aut}(Z(P))$ (see Lemma 1.2.2 in [4] for example). Consider the automorphism $\gamma : d \mapsto d^\omega$, $e \mapsto e$. Then $\gamma(b) = b^{\omega^2}$ and $\gamma(c) = c^\omega$. Hence, γ corresponds to a matrix of determinant ω^3 in $\mathrm{GL}(2,p) \cong \mathrm{Aut}(Z(P))$. In particular, φ is surjective if $p \not\equiv 1 \pmod 3$. Finally, in case $p \equiv 1 \pmod 3$ it remains to show that $\varphi(\mathrm{Aut}(P))$ cannot be larger than $\langle \alpha, \beta, \gamma \rangle$. For this let $\tau \in \mathrm{Aut}(P)$ be arbitrary. Then $\tau(d) \equiv d^i e^j \pmod{P'}$ and $\tau(e) \equiv d^k e^l \pmod{P'}$ for some $i, j, k, l \in \mathbb{Z}$. It follows that $\tau(a) \equiv [d^i e^j, d^k e^l] \equiv a^{il-jk} \pmod{Z(P)}$ by III.1.2 and III.1.3 in [22]. This implies $\tau(b) = b^{i(il-jk)}c^{j(il-jk)}$ and $\tau(c) = b^{k(il-jk)}c^{l(il-jk)}$. Thus the corresponding element of $\mathrm{GL}(2,p)$ has determinant $(il-jk)^3$. This proves the claim in case of Lemma 3.2(vi).

Now let P be the group from Lemma 3.2(vii). Then $P' = \langle a,b,c\rangle$, $K_3(P) = [P',P] = \langle b,c\rangle$ and $Z(P) = \langle c\rangle$. In particular, P has maximal class and $P_1 := C_P(P'/Z(P)) = C_P(K_3(P)) = \langle a,b,c,e\rangle$ is characteristic in P. Let $\alpha(d) = d^{\omega}$ and $\alpha(e) = e^{\omega^2}$. As usual, one can prove that α is an automorphism such that $\alpha(c) = c^{\omega^5}$. Therefore, in case $p \not\equiv 1 \pmod 5$ we are done. For $p \equiv 1 \pmod 5$ we take an arbitrary automorphism $\beta \in \mathrm{Aut}(P)$. Since P_1 is characteristic, we may write $\beta(d) \equiv d^ie^j \pmod{P'}$ and $\beta(e) = e^k \pmod{P'}$ for some $i,j,k \in \mathbb{Z}$. By III.1.2 and III.1.3 in [22] we get $\beta(a) \equiv a^{ik} \pmod{K_3(P)}$. Moreover, $\beta(b) \equiv b^{i^2k} \pmod{Z(P)}$ and $\beta(c) = \beta([a,e]) = c^{ik^2}$. On the other hand, $\beta(c) = \beta([b,d]) = c^{i^3k}$. Thus, $k \equiv i^2 \pmod p$ and $\beta(c) = c^{i^5}$.

Finally, we turn to the groups of order p^6. First, let P be the group from Lemma 3.2(viii). For $i,j \in \mathbb{Z}$ such that $i \not\equiv 0 \not\equiv j \pmod p$ we define a map α by $\alpha(a) = a^i$ and $\alpha(b) = b^jc^{j(1-i)}d^k$ where $k := j(1-6i+5i^2)/12$. Since 12 is invertible modulo p, we can regard k as an integer. We have $\alpha(c) \equiv c^{ij} \pmod{K_3(P)}$, $\alpha(d) \equiv d^{i^2j}$ $\pmod{K_4(P)}$, $\alpha(e) = e^{i^3j} \pmod{Z(P)}$ and $\alpha(f) = \alpha([d,b]) = f^{i^2j^2-i^2j^2(1-i)} = f^{i^3j^2}$. Also, $\alpha(f) = \alpha([e,b]) = \alpha([c,d]) = f^{i^3j^2}$. Hence, the set of generators $\alpha(a),\ldots,\alpha(f)$ also satisfies the given relations. In order to prove that also the (not mentioned) trivial commutator relations in $\alpha(a),\ldots,\alpha(f)$ are fulfilled, we need to be more precise. The only difficult part is to show $[\alpha(b),\alpha(c)] = 1$. For this we need to determine $\alpha(c) \pmod{Z(P)}$. Since $P/Z(P) \cong (C_p \times C_p \times C_p \times C_p) \rtimes C_p$, we can regard $\alpha(c) \pmod{Z(P)}$ as an element of the vector space with basis b,c,d,e. The action of a is given by the matrix

$$\begin{pmatrix} 1 & . & . & . \\ 1 & 1 & . & . \\ . & 1 & 1 & . \\ . & . & 1 & 1 \end{pmatrix}.$$

Hence, $\alpha(c) = \alpha([b,a]) = [b^jc^{j(1-i)}d^k, a^i] \pmod{Z(P)}$ corresponds to the vector

$$\begin{pmatrix} 1 & . & . & . \\ i & 1 & . & . \\ \binom{i}{2} & i & 1 & . \\ \binom{i}{3} & \binom{i}{2} & i & 1 \end{pmatrix} \begin{pmatrix} j \\ j(1-i) \\ k \\ . \end{pmatrix} - \begin{pmatrix} j \\ j(1-i) \\ k \\ . \end{pmatrix} = \begin{pmatrix} . \\ ij \\ ij(1-i)/2 \\ ij(i^2-1)/12 \end{pmatrix}.$$

Since both $\alpha(b)$ and $\alpha(c)$ lie in the extraspecial group $\langle b,c,d,e,f\rangle$, the equation $[\alpha(b),\alpha(c)] = 1$ is just an elementary arithmetic expression in the involved exponents (this is the only place where we need the definition of k). Thus, $\alpha \in \mathrm{Aut}(P)$. Now by taking $(i,j) = (\omega,1)$ and $(i,j) = (1,\omega)$ we see that φ is surjective.

Now assume that P is the group from Lemma 3.2(ix). Let $\alpha \in \mathrm{Aut}(P)$ arbitrary. Since P is an exceptional group of maximal class, we have two characteristic maximal subgroups $P_1 := C_P(P'/K_4(P)) = \langle b,c,d,e,f\rangle$ and $P_1^* := C_P(K_4(P)) = \langle a,c,d,e,f\rangle$. Hence, we may write $\alpha(a) \equiv a^i \pmod{P'}$ and $\alpha(b) \equiv b^j \pmod{P'}$ for some $i \not\equiv 0 \not\equiv j \pmod p$. It follows that $\alpha(c) \equiv c^{ij} \pmod{K_3(P)}$, $\alpha(d) \equiv d^{i^2j} \pmod{K_4(P)}$ and $\alpha(e) = \alpha([c,b]) \equiv e^{ij^2} \pmod{Z(P)}$. On the other hand, $\alpha(e) = \alpha([d,a]) \equiv e^{i^3j} \pmod{Z(P)}$. Thus, $j \equiv i^2 \pmod p$. Finally, $\alpha(f) = \alpha([c,d]) = f^{i^3j^2} = f^{i^7}$. It remains to show that $\varphi(\mathrm{Aut}(P))$ cannot be smaller. We start by defining $\alpha(a) := a^{\omega}$ and $\alpha(b) := b^{\omega^2}d^k$ where $k := -7\omega^2(\omega^2-1)/12 \pmod p$. Then a tedious calculation gives $\alpha(c) = [\alpha(b),\alpha(a)] \equiv c^{i^3}d^{i^3(i-1)/2}e^{(7i^5-6i^4-i^3)/12} \pmod{Z(P)}$. Now one can show as before that the images of the generators under α satisfy the same relations. Hence, $\alpha \in \mathrm{Aut}(P)$. Moreover, $\alpha(f) = f^{\omega^7}$. This completes the proof. □

Corollary 3.5. *If* $p \equiv 1 \pmod 5$ *(respectively* $p \equiv 1 \pmod 7$*), then the group from Lemma 3.2*(vii) *(respectively* (ix)*) cannot occur as a defect group of a p-block with Loewy length* 4.

Proposition 3.6. *Let P be a finite p-group. Then the following holds:*

(i) $LL(FP) = 1$ *if and only if* $P = 1$.

(ii) $LL(FP) = 2$ *if and only if* $P = C_2$.

(iii) $LL(FP) = 3$ *if and only if* $P = C_3$ *or* $P = C_2 \times C_2$.

(iv) $LL(FP) = 4$ *if and only if* $P = C_4$ *or* $P = C_2 \times C_2 \times C_2$.

Proof. This follows from a result of Jennings [25, Theorems 3.7 and 5.5]. □

Proposition 3.7. *Let G be p-solvable, and let B be a block of FG with defect group D and Loewy length 4. Then $p = 2$, and one of the following holds:*

(i) $D \cong C_4$,

(ii) $D \cong C_2 \times C_2 \times C_2$,

(iii) $D \cong D_8$ *and* $l(B) = 2$.

Proof. Let δ be the defect of B. By a result of Koshitani [28, Theorem], we have $4 = LL(B) \geq \delta(p-1)+1$, i.e. $3 \geq \delta(p-1)$. Thus $p \leq 3$. Moreover, if $p = 3$ then $\delta = 1$. But then B is Morita equivalent to FC_3 or FS_3 which have both Loewy length 3, a contradiction. Thus we must have $p = 2$ and $\delta \leq 3$. If D is abelian or quaternion then we have $LL(B) = LL(FD)$, and the result follows from Proposition 3.6. So we can assume that D is dihedral of order 8. In this case B is Morita equivalent to FD or FS_4, and the result follows. □

For principal 2-blocks with Loewy length 4 the same defect groups occur as in Proposition 3.7 (see Theorem 4.5 below).

Corollary 3.8. *Let B be nilpotent with $LL(B) = 4$. Then $D \cong C_4$ or $D \cong C_2 \times C_2 \times C_2$.*

Proof. This follows from a result of Puig [41], in connection with Proposition 3.6. □

For cyclic defect groups we obtain the following consequence of Equation (1).

Corollary 3.9. *Let B be a p-block with cyclic defect group D and Brauer tree Γ. Then B has Loewy length 4 if and only if $|D| = p > 3$ and one of the following holds:*

(i) $e(B) = p - 1$, *and the valency of* Γ *is* 3.

(ii) $e(B) = \frac{p-1}{2}$, *the exceptional vertex is a leaf, and the valency of* Γ *is* 3.

(iii) $e(B) = \frac{p-1}{3}$, *the exceptional vertex is a leaf, and the valency of* Γ *is at most* 3.

Proof. By previous results, $|D| = p > 3$ and $e(B) \geq \frac{p-1}{3}$. Now, Equation (1) implies the result. □

Suppose that B is a tame block of defect δ and Loewy length 4. Then $p = 2$, and the defect groups of B have order 2^δ and exponent $2^{\delta-1}$. Thus Lemma 2.2 implies that $\delta \leq 3$. The symmetric group S_4 of degree 4 shows that this bound is sharp.

Now, let B be a 2-block with a metacyclic defect group D, and suppose that B has Loewy length 4. By a result of Sambale [42, Theorem 2], B is either nilpotent or tame, or D is homocyclic. If B is nilpotent, then $D \cong C_4$ by Corollary 3.8. If B is tame, then $|D| \leq 8$ as we just proved. And if D is homocyclic of order $2^{2\epsilon} > 4$, then, by a recent result of C. W. Eaton, R. Kessar, B. Külshammer and B. Sambale [11], B is Morita equivalent to its Brauer correspondent. In particular, we have $4 = LL(B) = LL(FD) = 2 \cdot 2^\epsilon - 1$, a contradiction. Thus altogether $|D| \leq 8$.

As another example we consider blocks of symmetric groups.

Theorem 3.10. *Let B be a p-block of the symmetric group S_n with Loewy length 4. Then $n = 4$ and B is the principal 2-block.*

Proof. Let w be the weight of B. Then the defect groups of B are Sylow p-subgroups of S_{pw}. Since their rank is at most 3, we must have $w \leq 3$. Assume first that $p \geq 5$. Then B has defect at most 3 and the defect groups are abelian. Now J. Scopes [44, Theorem 1] has proved that all blocks of symmetric groups of defect 2 have Loewy length 5, and K. M. Tan [47, Theorem 4.4] has proved that all blocks of symmetric groups of defect 3 and abelian defect groups have Loewy length 7. Thus we conclude that B has defect 1. Here by 6.3.9 in [24] the Brauer tree of B is a straight line and $e(B) = p - 1$. This contradicts Corollary 3.9.

Now let $p = 2$, and let D be a defect group of B. If $w = 1$, then $D \cong C_2$, and B is Morita equivalent to FD. This is a contradiction, since $LL(B) = 4$. Suppose next that $w = 2$. Then D is a dihedral group of order 8. The 2-core of B has the form $\kappa = (x, x-1, \ldots, 2, 1)$ for some $x \in \mathbb{N}_0$. If $x \geq w - 1 = 1$, then B is Scopes equivalent to the 2-block B_1 with 2-core (1) and weight 2, i.e. to the principal 2-block of S_5. Since $LL(B_1) > 4$, this is a contradiction. Thus we must have $x = 0$, and B is the only 2-block of S_4. It remains to deal with the case $w = 3$. Let again $\kappa = (x, x-1, \ldots, 2, 1)$ be the 2-core of B. If $x \geq 2$, then B is Scopes equivalent to the 2-block B_2 with 2-core $(2, 1)$ and 2-weight 3. This is a non-principal 2-block of S_9. But results by Benson [3, Theorem 2] in connection with Proposition 2.10 imply that $LL(B_2) > 4$. Suppose next that $x = 1$. Then B is the principal 2-block B_1 of S_7. However, as one can see from Benson's paper [2, Section 1.2], we have $LL(B_1) > 4$, a contradiction. It remains to consider the case $x = 0$. Then B is the principal 2-block of S_6. Again one can see from Benson's paper [2, Section 1.3] that this is a contradiction.

Finally, assume $p = 3$. One can check with GAP [13] that there are no examples among principal blocks. In the non-principal cases calculations by Susanne Danz [10] show that the 3-blocks of weight 3 in symmetric groups do not have Loewy length 4. (There are twelve Scopes equivalence classes of such blocks which have weight 3.) Again, the 3-blocks of weight 2 can be excluded by [44, Theorem 1]. The case $w = 1$ is excluded by Corollary 3.9. Hence, there are no such blocks for $p = 3$. □

Note that Theorem 3.10 also handles the blocks of alternating groups by Proposition 2.10.

4 Principal blocks of Loewy length 4

We start with an easy consequence of Proposition 2.10.

Corollary 4.1. *Let B be the principal block of FG, and let b be the principal block of FH where H is a normal subgroup of G. Then $LL(b) \leq LL(B)$.*

Corollary 4.2. *Suppose that B is the principal block and that $LL(B) = 4$. Then $LL(FP) \leq 4$ where $P := O_p(G)$. In particular, we have $P = 1$ whenever $p > 3$.*

Proof. Corollary 4.1 implies that $LL(FP) \leq 4$. Thus Proposition 3.6 implies that $P = 1$ whenever $p > 3$. □

Next we state some results about projective covers.

Proposition 4.3 (Koshitani [29, Corollary]). *If the projective cover of the trivial FG-module has Loewy length 4, then $p = 2$.*

Proposition 4.4 (Okuyama [39, Theorem 2]). *If $p = 2$ and if the projective cover of the trivial FG-module has Loewy length 3, then the Sylow 2-subgroups of G are dihedral.*

The principal 2-blocks of Loewy length 4 are completely described by the following theorem.

Theorem 4.5 (Koshitani [27, Theorem 1.3]). *The principal 2-block of a finite group G has Loewy length 4 if and only if $O^{2'}(G/O_{2'}(G))$ is one of the following groups:*

(i) C_4,

(ii) $C_2 \times C_2 \times C_2$,

(iii) $C_2 \times \mathrm{PSL}(2, q)$ *for* $q \equiv 3 \pmod 8$,

(iv) $\mathrm{PGL}(2, q)$ *for* $q \equiv 3 \pmod 8$.

Therefore, we concentrate on odd primes p in the following.

Proposition 4.6. *Let B be the principal block, and suppose that $p > 2$ and $LL(B) = 4$. Then the projective cover U of the trivial FG-module F is uniserial of length 3, i.e.*

$$U = \begin{pmatrix} F \\ S \\ F \end{pmatrix}$$

where S is a non-trivial simple B-module. In particular, the Cartan matrix $\mathcal{C}$ of B has the following form:

$$\mathcal{C} = \begin{pmatrix} 2 & 1 & 0 & \cdots & 0 \\ 1 & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & * & * & \cdots & * \end{pmatrix}.$$

Proof. We have $LL(U) \leq LL(B) = 4$. However, Proposition 4.3 implies that $LL(U) \leq 3$. By a theorem of Webb [48, Theorem E], $\operatorname{Rad}(U)/\operatorname{Soc}(U)$ is indecomposable. Since $\operatorname{Rad}(U)/\operatorname{Soc}(U)$ is also semisimple, we conclude that $\operatorname{Rad}(U)/\operatorname{Soc}(U)$ is simple. Thus U has the desired form. This also gives the shape of $\mathcal{C}$. □

Corollary 4.7. *In the situation above, the decomposition matrix of B has the form*

$$\mathcal{D} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & * & * & \cdots & * \end{pmatrix}.$$

In particular, G has an irreducible character χ such that $\chi(x) = -1$ for every p-singular $x \in G$.

Proof. The first row corresponds to the trivial character of G, and the first column corresponds to the trivial FG-module. Thus, the first row of $\mathcal{D}$ has the desired form. Since the first Cartan invariant $c_{11} = 2$, we may assume that the first column of $\mathcal{D}$ has the desired form. Since the Cartan invariant $c_{12} = 1$, we then get the shape of the second column of $\mathcal{D}$. Since $c_{1i} = 0$ for $i > 2$ we get the zeroes in the second row of the decomposition matrix. The last statement follows by looking at the first column of $\mathcal{D}$. □

Proposition 4.8 (Brauer-Nesbitt [5, Theorem 12])**.** *The first Cartan invariant c_{11} of FG satisfies $c_{11} \geq \frac{|G|}{|G_{p'}|}$. Moreover, if $c_{11} = \frac{|G|}{|G_{p'}|}$, then G is p-nilpotent.*

Our next aim is a reduction to simple groups.

Lemma 4.9 (Koshitani-Miyachi [30, (4.2) Lemma(i)])**.** *Let X and Y be finite groups, and set $G := X \times Y$. We denote the principal blocks of G, X and Y by B, B_X and B_Y, respectively. Then $LL(B) = LL(B_X) + LL(B_Y) - 1$.*

Proposition 4.10. *Suppose that $p \geq 5$, that $O_{p'}(G) = 1$ and that the principal block of G has Loewy length 4. Then $E(G) = O^{p'}(G)$ is simple.*

Proof. We show first that $N := E(G)$ is simple. Let B (respectively b) be the principal block of FG (respectively FN). Then $LL(b) \leq 4$ by Corollary 4.1. Moreover, we have $O_p(G) = 1 = O_{p'}(G)$ by Corollary 4.2. Thus $N = S_1 \times \cdots \times S_n$ with simple groups $S_1, \ldots, S_n$. For $i = 1, \ldots, n$, let b_i be the principal block of FS_i. Then $b = b_1 \otimes \cdots \otimes b_n$. By Lemma 4.9, we have

$$4 \geq LL(b) = \sum_{i=1}^{n} LL(b_i) - n + 1.$$

Since $p > 2$, we have $LL(b_i) \geq 3$ for $i = 1, \ldots, n$. Hence $4 \geq 3n - n + 1 = 2n + 1$, so that $n = 1$. This proves that $E(G)$ is simple.

Since by Proposition 2.10 the principal block of $O^{p'}(G)$ does also have Loewy length 4, we may assume $G = O^{p'}(G)$ in order to prove that $O^{p'}(G)$ is simple. Assume that G has a proper normal subgroup $N \neq 1$. Then, by a result of Alperin, Collins and Sibley [1, 1. Introduction], the projective cover P of the trivial FG-module has a filtration $P = P_0 \supseteq P_1 \supseteq \ldots \supseteq P_n = 0$ with $n \geq 2$ such that P/P_1 and P_{n-1} are both isomorphic to the inflation of Q where Q denotes the projective cover of the trivial $F[G/N]$-module. Thus we obtain $c_{11} \geq 4$ which contradicts Proposition 4.6. Hence $O^{p'}(G)$ is simple and we must have $E(G) = O^{p'}(G)$. □

In the situation above, we may assume that G is simple. This allows one to use the classification of the finite simple groups.

Proposition 4.11. *Let B be the principal p-block of a sporadic simple group G where $p \geq 3$. Then $LL(B) \neq 4$.*

Proof. For the blocks of defect 1 we translate Corollary 3.9 into a statement about the decomposition matrix. Then a computation with GAP [13] excludes most possibilities. If the decomposition matrix is not available (in GAP), we check [17]. Finally, for the Monster group in characteristic 29 the possible Brauer trees can be found in [36, Section 2.7]. For the blocks of larger defect we use Proposition 4.6. In case $G = Fi_{24}'$ and $p = 5$ the Cartan matrix is not available (in GAP) and there is indeed a character χ which takes the value -1 on the 5-singular elements. However, the restriction of $1 + \chi$ onto Fi_{23} is not a projective character. Thus, this case cannot occur. In case $G = M$ and $p = 11$ a slightly more involved computation by Jürgen Müller [35] shows $LL(B) \neq 4$. The other cases can be handled similarly. □

Proposition 4.12. *Let B be the principal p-block of a finite simple group G of Lie type in characteristic $p \geq 3$. Then $LL(B) \neq 4$.*

Proof. Let D be a defect group of B. By Lemma 2.2, $r(D) \leq 3$ and $|D| \leq p^{18}$. Moreover, $|D| \leq p^6$ for $p \geq 5$. Now we consider G by going through the classification of finite simple groups. For the order of G we refer to Table 1 on p. 8 in [14]. The p-ranks of G can be found in Table 3.3.1 on p. 108 in [15].

Case 1: $G = \mathrm{PSL}(n, p^s)$.
Here $r(D) = \left[\frac{n^2}{4}\right]s$ implies $(n, s) \in \{(2, 1), (2, 2), (2, 3), (3, 1)\}$. Since $Z(\mathrm{SL}(n, p^s))$ is always a p'-group, B is isomorphic to the principal block of $F[\mathrm{SL}(n, p^s)]$. In particular the first Cartan invariant c_{11} is given in Section 11.12 on p. 108 in [21]. In case $n = 2$ we have $c_{11} = 2^s$, and in case $n = 3$ and $p \geq 5$ we get $c_{11} = 8$. For $n = p = 3$ one can find $c_{11} = 10$ in Section 11.13 on p. 109 in [21]. Now Proposition 4.6 implies $(n, s) = (2, 1)$. Here however, all projective indecomposable modules have Loewy length 3 (see the Proposition on p. 131 in [21]). Since B is the direct sum of some of the projective indecomposable modules, we must have $LL(B) = 3$ in this case.

Case 2: $G = \mathrm{PSU}(n, p^s)$.
Here $r(D)$ depends on the parity of n. In any case we get $(n, s) = (3, 1)$ and $G = \mathrm{PSU}(3, p)$. Again the first Cartan invariant of B can be found on p. 109 in [21]. For $p = 3$ we have $c_{11} = 10$, for $p = 5$ we have $c_{11} = 12$ while for $p \geq 7$ it holds that $c_{11} = 8$. Proposition 4.6 gives a contradiction.

Case 3: $G = \mathrm{PSp}(2n, p^s)$.
Here $r(D) = \binom{n+1}{2}s$ gives $(n, s) = (2, 1)$. Hence, $G = \mathrm{PSp}(4, p)$. We have $c_{11} = 7$ (respectively 21, 16 and 14) for $p = 3$ (respectively 5, 7 and ≥ 11). Proposition 4.6 yields to a contradiction.

If $G = \mathrm{P}\Omega(2n + 1, p^s)$, then again $(n, s) = (2, 1)$. Hence, we are in Case 3.

For $G = \mathrm{P}\Omega^+(2n, p^s)$ one gets $(n, s) = (3, 1)$. However, then $G = \mathrm{P}\Omega^+(6, p) \cong \mathrm{PSL}(4, p)$ which was already excluded. For similar reasons $G = \mathrm{P}\Omega^-(2n, p^s)$ is also impossible. Also for $G = {}^3D_4(q)$ the rank of D is too large.

Case 4: $G = G_2(p^s)$.
Here $p \geq 5$ and $s = 1$ by p. 108 in [15]. Hence, $|D| = p^6$ and Proposition 3.3 implies $p \geq 7$. On the other hand, $c_{11} = 168$ for $p \geq 17$ (see p. 109 in [21]). This is also true for $p = 11$ and $p = 13$ by Remark 1 in [20]. Hence, $p = 7$. Here it follows from the generic character table [8] that G does not contain an irreducible character which takes the value -1 on the p-singular elements.

Case 5: $G = {}^2G_2(3^{2n-1})$.
Here $n = 1$ and G' is simple of order 504. Then D is cyclic of order 9 which contradicts Corollary 3.9.

For the other simple groups of Lie type it is easy to see that $|D|$ is too large. □

By a result of Koshitani and Miyachi [30, (0.3) Theorem], we have $LL(B) \geq 5$ for every principal block B with defect group $C_3 \times C_3$. We generalize this result.

Proposition 4.13. *Let B be a principal 3-block with abelian defect groups. Then $LL(B) \neq 4$.*

Proof. Let B be the principal 3-block of a finite group G with abelian Sylow 3-subgroup P. By Proposition 2.10, we may assume $O^{3'}(G) = G$ and $O_{3'}(G) = 1$. Then, by a list due to P. Fong (see [31, Proposition 4.3]), G is a direct product of 3-groups and certain simple groups. Suppose first that G can be written as a direct product $G = G_1 \times G_2$ such that $G_1 \neq 1 \neq G_2$. Then by Lemma 4.9 and Proposition 3.1 we derive the contradiction $LL(B) \geq 5$. By Proposition 3.6, G is not a 3-group. Hence, we conclude that G is one of the simple groups occurring in [31, Proposition 4.3]. By Corollary 3.9, P is not cyclic. It follows from Proposition 4.11 that G is not a sporadic group. According to Remark 4.4 in [31], Lemma 2.2 and the remark above, we may assume that $P \cong C_9 \times C_9$. In particular, G is not of type (i)–(iv). In cases (v) and (vi) the first Cartan invariant is 4 contradicting Proposition 4.6 (see [49, Theorem 4.1 and 4.2] and [51, Theorem 2.2 and 3.2]). Assume next that G is of type (vii) or (ix). Since $(q-1, 3) = 1$, we may assume that $G = \mathrm{GL}(4, q)$ or $G = \mathrm{GL}(5, q)$ respectively. Then the first Cartan invariant for G is at least 3 (see [23, Appendix 1]). Contradiction. If G is of type (viii) or (x), then [31, Lemma 3.7] says that $LL(b) = LL(B) = 4$ where b is the principal block of $N_G(P)$. However, this contradicts Proposition 3.7. Finally, the case (xi) for G was already excluded by Proposition 4.12. □

The next result is in the same spirit.

Proposition 4.14. *Let $p > 2$, and let G be a group with Sylow p-subgroup p_+^{1+2}. Then the principal block of G does not have Loewy length 4.*

Proof. First we reduce to simple groups G. For $p \geq 5$ this is clear by Proposition 4.10. Now let $p = 3$. If we follow the proof of Proposition 4.10 carefully, it turns out that the only thing which can happen is $|O_3(G)| = 3$. In this case the principal block B of G dominates the principal block $\overline{B}$ of $G/O_3(G)$ which has defect group $C_3 \times C_3$. Here Proposition 2.9 and the remark above give the contradiction $LL(B) \geq LL(\overline{B}) \geq 5$.

Hence, for the remainder of the proof we may assume that G is simple (and $p > 2$). Then the possibilities for G are listed in Theorem 31 in [37]. For $p \geq 5$ we only get sporadic groups and groups of Lie type in defining characteristic. These were handled in Propositions 4.11 and 4.12. For $p = 3$ it remains to consider $G_2(q)$ and ${}^2F_4(q)$. The decomposition matrix of the principal 3-block of $G_2(q)$ can be found in Table I and II in [19]. By Corollary 4.7 this block does not have Loewy length 4. For the groups ${}^2F_4(q)$ we look up the character table of the principal 3-block in Appendix A in [37]. Here it turns out that no irreducible character takes the value -1 on the 3-singular element t_4. Therefore, this case cannot occur either. □

5 Examples

We tracked down the following principal blocks of Loewy length 4:

(i) $p = 2$ and $G = C_4$, $C_2 \times C_2 \times C_2$, $C_2 \times \mathrm{PSL}(2, q)$ and $\mathrm{PGL}(2, q)$ for $q \equiv 3 \pmod 8$ (see Theorem 4.5).

(ii) $p \equiv 1 \pmod 3$, $n := (p-1)/3$ and $G = \mathrm{PSL}(n, q)$ if q has order n modulo p, but not modulo p^2 (see [12]).

(iii) $p = 5$ and $G = \mathrm{Sz}(2^{2n+1})$ if $n \equiv k \pmod{20}$ for some $k \in \{1, 5, 6, 9, 10, 13, 14, 18\}$ (see [7, (2.1)]).

Let $p \equiv 1 \pmod 3$ be a prime, and let ω be a primitive root modulo p. Then by Dirichlet's Theorem, there is always a prime q such that $q \equiv p + \omega^3 \pmod{p^2}$. Hence, we get examples for infinitely many primes p from (ii). The same conclusion might be true for other groups of Lie type.

We also found arbitrary p-blocks of Loewy length 4 and defect 1 in the following groups:

(i) $p = 7$ and $G = E_6(q)$ if $7 \mid q+1$, but $49 \nmid q+1$ (see [18, Theorem 3.1(2)]).

(ii) $p = 13$ and $G = E_6(q)$ if $13 \mid q^2+1$, but $13^2 \nmid q^2+1$ (see [18, Theorem 3.1(4)]).

(iii) $p = 7$ and $G = G_2(q)$ if $7 \mid q+1$, but $49 \nmid q+1$ (see [46, Section 3.3] and [45, Section 2.2]).

(iv) $p = 7$ and $G = {}^2G_2(3^{2n+1})$ if $n \equiv k \pmod{21}$ for some $k \in \{1, 4, 7, 13, 16, 19\}$ (see [16, Theorem 4.2(b)]).

(v) $p = 7$ and $G = \mathrm{Sp}(4, q)$ if $7 \mid q+1$, but $49 \nmid q+1$ (see [50, Theorem 2.6]).

(vi) $p \in \{5, 7, 11\}$ and $G = 12.M_{22}$ (see [17, Section 6.4]).

(vii) $p \in \{5, 7\}$ and $G = 6.A_7$ (see GAP [13]).

(viii) $p = 5$ and $G = 3.O'N$ (see [17, Section 6.14]).

(ix) $p = 7$ and $G = Ru$ or $G = 2.Ru$ (see [17, Section 6.12]).

(x) $p = 7$ and $G = 2.\mathrm{Sz}(8)$ (see GAP [13]).

(xi) $p = 7$ and $G = 12.\,\mathrm{PSL}(3, 4)$ (= `AtlasGroup("12_1.L3(4)")` in GAP).

We do not expect that this is the exhaustive list of examples among (quasi)simple groups. However, it seems not unreasonable that for $p > 2$ all p-blocks of Loewy length 4 have defect 1. We also note that all three types of Brauer trees in Corollary 3.9 occur. Moreover, we do not know a single example for $p = 3$.

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