Loewy lengths of centers of blocks

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Abstract

Let B be a block of a finite group with respect to an algebraically closed field F of characteristic $p > 0$. In a recent paper, Otokita gave an upper bound for the Loewy length $LL(ZB)$ of the center ZB of B in terms of a defect group D of B. We refine his methods in order to prove the optimal bound $LL(ZB) \leq LL(FD)$ whenever D is abelian. We also improve Otokita's bound for non-abelian defect groups. As an application we classify the blocks B such that $LL(ZB) \geq |D|/2$.

1 Introduction

We consider a block (algebra) B of FG where G is a finite group and F is an algebraically closed field of characteristic $p > 0$. In general, the structure of B is quite complicated and can only be described in restrictive special cases (e.g. blocks of defect 0). For this reason, we are content here with the study of the center ZB of B. This is a *local F-algebra* in the sense that the *Jacobson radical JZB* has codimension 1. It is well-known that the dimension of ZB itself equals the number $k(B)$ of irreducible complex characters in B. In particular, this dimension is locally bounded by a theorem of Brauer and Feit [3]. Moreover, the number $l(B)$ of irreducible Brauer characters in B is given by the dimension of the *Reynolds ideal $RB := ZB \cap SB$* where SB is the *socle* of B. It follows that the dimension of the quotient ZB/RB is locally determined by Brauer's theory of subsections. Here a *B-subsection* is a pair (u, b) where $u \in G$ is a p-element and b is a Brauer correspondent of B in $\mathrm{C}_G(u)$.

In order to give better descriptions of ZB we introduce the *Loewy length $LL(A)$* of a finite-dimensional F-algebra A as the smallest positive integer l such that $(JA)^l = 0$. A result by Okuyama [22] states that $LL(ZB) \leq |D|$ where $|D|$ is the order of a defect group D of B. In fact, there is an open conjecture by Brauer [2, Problem 20] asserting that even $\dim ZB \leq |D|$. In a previous paper [16] jointly with Shigeo Koshitani, we have shown conversely that $LL(B)$ is bounded from below in terms of $|D|$. There is no such bound for $LL(ZB)$, but again an open question by Brauer [2, Problem 21] asks if $\dim ZB$ can be bounded from below in terms of $|D|$.

Recently, Okuyama's estimate has been improved by Otokita [23]. More precisely, if $\exp(D)$ is the exponent of D, he proved that

$$LL(ZB) \leq |D| - \frac{|D|}{\exp(D)} + 1. \tag{1.1}$$

The present note is inspired by Otokita's methods. Our first result gives a local bound on the Loewy length of ZB/RB. Since $(JZB)(RB) \subseteq (JB)(SB) = 0$, we immediately obtain a bound for $LL(ZB)$. In our main theorem we apply this bound to blocks with abelian defect groups as follows.

Theorem 1. *Let B be a block of FG with abelian defect group D. Then $LL(ZB/RB) < LL(FD)$ and $LL(ZB) \leq LL(FD)$.*

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If, in the situation of Theorem 1, D has type $(p^{a_1}, \ldots, p^{a_r})$, then $LL(FD) = p^{a_1} + \ldots + p^{a_r} - r + 1$ as is well-known. For p-solvable groups G, the stronger assertion $LL(B) = LL(FD)$ holds (see [19, Theorem K]). Similarly, if D is cyclic, one can show more precisely that

$$LL(ZB) = LL(ZB/RB) + 1 = \dim ZB/RB + 1 = \frac{|D| - 1}{l(B)} + 1$$

(see [16, Corollary 2.6]). By Broué-Puig [5], Theorem 1 is best possible for nilpotent blocks. We conjecture conversely that the inequality is strict for non-nilpotent blocks (cf. Corollary 5 and Proposition 7 below).

Arguing inductively, we also improve Otokita's bound for blocks with non-abelian defect groups. More precisely, we show in Theorem 12 that

$$LL(ZB) \leq \frac{|D|}{p} + \frac{|D|}{p^2} - \frac{|D|}{p^3}$$

(see also Proposition 15). Extending Otokita's work again, we use our results to classify all blocks B with $LL(ZB) \geq |D|/2$ in Corollary 16.

It seems that in the non-abelian defect case the inequality $LL(ZB) \leq LL(FD)$ is still satisfied. This holds for example if $D \trianglelefteq G$ (see [23, proof of Lemma 2.4]). We support this observation by computing the Loewy lengths of the centers of some blocks with small defect. Finally, we take the opportunity to improve [23, Proposition 2.2] (see Proposition 3). To do so we recall that the *inertial quotient* $I(B)$ of B is the group $\mathrm{N}_G(D, b_D)/D\,\mathrm{C}_G(D)$ where b_D is a Brauer correspondent of B in $\mathrm{C}_G(D)$. By the Schur-Zassenhaus Theorem, $I(B)$ can be embedded in the automorphism group $\mathrm{Aut}(D)$. Then

$$FD^{I(B)} := \{x \in FD : a^{-1}xa = x \text{ for } a \in I(B)\}$$

is the algebra of fixed points. Moreover, for a subset $U \subseteq G$ we define $U^+ := \sum_{u \in U} u \in FG$. Then RFG has an F-basis consisting of the sums S^+ where S runs through the p'-sections of G (see for example [17]). Note that the trivial p'-section is given by the set G_p of p-elements of G.

2 Abelian defect groups

By the results mentioned in the introduction we may certainly restrict ourselves to blocks with positive defect.

Proposition 2. *Let B be a block of FG with defect group $D \neq 1$. Let $(u_1, b_1), \ldots, (u_k, b_k)$ be a set of representatives for the conjugacy classes of non-trivial B-subsections. Then the map*

$$ZB/RB \to \bigoplus_{i=1}^{k} Zb_i/Rb_i,$$

$$z + RB \mapsto \sum_{i=1}^{k} \mathrm{Br}_{\langle u_i \rangle}(z) 1_{b_i} + Rb_i$$

is an embedding of F-algebras where $\mathrm{Br}_{\langle u_i \rangle} : ZFG \to ZF\,\mathrm{C}_G(u_i)$ denotes the Brauer homomorphism. In particular,

$$LL(ZB/RB) \leq \max\{LL(Zb_i/Rb_i) : i = 1, \ldots, k\}.$$

Proof. First we consider the whole group algebra FG instead of B. For this, let $v_1, \ldots, v_r$ be a set of representatives for the conjugacy classes of non-trivial p-elements of G. Let $z := \sum_{g \in G} \alpha_g g \in ZFG$. Then z is constant on the conjugacy classes of G. It follows that z is constant on the p'-sections of G if and only if $\mathrm{Br}_{\langle v_i \rangle}(z) = \sum_{g \in \mathrm{C}_G(v_i)} \alpha_g g$ is constant on the p'-sections of $\mathrm{C}_G(v_i)$ for $i = 1, \ldots, r$. Therefore, the map $ZFG/RFG \to \bigoplus_{i=1}^{r} ZF\,\mathrm{C}_G(v_i)/RF\,\mathrm{C}_G(v_i)$, $z + RFG \mapsto \sum_{i=1}^{r} \mathrm{Br}_{\langle v_i \rangle}(z) + RF\,\mathrm{C}_G(v_i)$ is a well-defined embedding of F-algebras. Now the first claim follows easily by projecting onto B, i.e. replacing z by $z1_B$. The last claim is an obvious consequence. $\square$

Proof of Theorem 1. Let $D \cong C_{p^{a_1}} \times \ldots \times C_{p^{a_r}}$. It is well-known that

$$LL(FD) = LL(FC_{p^{a_1}} \otimes \ldots \otimes FC_{p^{a_r}}) = p^{a_1} + \ldots + p^{a_r} - r + 1.$$

Hence, it suffices to show that $LL(ZB/RB) \leq p^{a_1} + \ldots + p^{a_r} - r$.

We argue by induction on r. If $r = 0$, then we have $D = 1$, $ZB = RB$ and $LL(ZB/RB) = 0$. Thus, we may assume that $r \geq 1$. Let $I := I(B)$ be the inertial quotient of B. In order to apply Proposition 2, we consider a B-subsection (u, b) with $1 \neq u \in D$. Then b has defect group D and inertial quotient $\mathrm{C}_I(u)$. Since I is a p'-group, we have $D = Q \times [D, \mathrm{C}_I(u)]$ with $Q := \mathrm{C}_D(\mathrm{C}_I(u)) \neq 1$. Let β be the Brauer correspondent of b in $\mathrm{C}_G(Q) \subseteq \mathrm{C}_G(u)$. By Watanabe [37, Theorem 2], the Brauer homomorphism Br_D induces an isomorphism between Zb and $Z\beta$. Since the intersection of a p'-section of G with $\mathrm{C}_G(D)$ is a union of p'-sections of $\mathrm{C}_G(D)$, it follows that $\mathrm{Br}_D(Rb) \subseteq R\beta$. On the other hand, $\dim_F Rb = l(b) = l(\beta) = \dim_F R\beta$ by [38, Theorem 1]. Thus, we obtain $Zb/Rb \cong Z\beta/R\beta$ and it suffices to show that

$$LL(Z\beta/R\beta) \leq p^{a_1} + \ldots + p^{a_r} - r.$$

Let $\overline{\beta}$ be the unique block of $\mathrm{C}_G(Q)/Q$ dominated by β. By [18, Theorem 7] (see also [8, Theorem 1.2]), it follows that the source algebra of β is isomorphic to a tensor product of FQ and the source algebra of $\overline{\beta}$. Since β is Morita equivalent to its source algebra, we may assume in the following that $\beta = FQ \otimes \overline{\beta}$. Let $Q \cong C_{p^{a_1}} \times \ldots \times C_{p^{a_s}}$ with $1 \leq s \leq r$. Since the defect group D/Q of $\overline{\beta}$ has rank $r - s < r$, induction implies that

$$LL(Z\overline{\beta}/R\overline{\beta}) \leq p^{a_{s+1}} + \ldots + p^{a_r} - r + s =: l.$$

In particular, $(JZ\overline{\beta})^l \subseteq R\overline{\beta}$. Since Q is an abelian p-group, we have $RFQ = SFQ \cong F$. Consequently, $LL(FQ/RFQ) = p^{a_1} + \ldots + p^{a_s} - s =: l'$, i.e. $(JFQ)^{l'} \subseteq RFQ$. Moreover, $SFQ \otimes S\overline{\beta} \subseteq S(FQ \otimes \overline{\beta})$. Hence,

$$RFQ \otimes R\overline{\beta} \subseteq Z(FQ \otimes \overline{\beta}) \cap S(FQ \otimes \overline{\beta}) = R(FQ \otimes \overline{\beta}) = R\beta.$$

Since $JZ\beta = J(FQ \otimes Z\overline{\beta}) = JFQ \otimes Z\overline{\beta} + FQ \otimes JZ\overline{\beta}$, we see that $(JZ\beta)^{l+l'}$ is a sum of terms of the form $(JFQ)^i \otimes (JZ\overline{\beta})^j$ with $i + j = l + l'$. If $i > l'$, then $(JFQ)^i = 0$. Similarly, if $j > l$, then $(JZ\overline{\beta})^j = 0$. It follows that

$$(JZ\beta)^{l+l'} = (JFQ)^{l'} \otimes (JZ\overline{\beta})^l \subseteq RFQ \otimes R\overline{\beta} \subseteq R\beta.$$

This proves the theorem, because $l + l' = p^{a_1} + \ldots + p^{a_r} - r$. $\qquad\square$

Our theorem shows that Otokita's bound (1.1) is only optimal for nilpotent blocks with cyclic defect groups or defect group $C_2 \times C_2$ (see [23, Corollary 3.1]).

The next result strengthens [23, Proposition 2.2].

Proposition 3. *Let B be a block of FG with defect group D. Moreover, let $c := \dim F\,\mathrm{Z}(D)^{I(B)}$ and $z := LL(F\,\mathrm{Z}(D)^{I(B)})$. Then $LL(ZB/RB) \leq k(B) - l(B) + z - c$ and in particular*

$$LL(ZB) \leq k(B) - l(B) + z - c + 1.$$

Proof. Let $K := \mathrm{Ker}(\mathrm{Br}_D) \cap ZB \trianglelefteq ZB$. Since ZB is local, we have $K \subseteq JZB$. Furthermore, $RB + K/K$ annihilates the radical JZB/K of ZB/K. It follows that $RB + K/K$ is contained in the socle of ZB/K. By Broué [4, Proposition (III)1.1], it is known that Br_D induces an isomorphism between ZB/K and the symmetric F-algebra $F\,\mathrm{Z}(D)^{I(B)}$. The socle of the latter algebra has dimension 1. Hence,

$$\dim RB + K/K \leq 1.$$

On the other hand, $G_p^+ \in RFG$. Therefore, $1_B G_p^+ \in RB$ and

$$\mathrm{Br}_D(1_B G_p^+) = \mathrm{Br}_D(1_B)\,\mathrm{Br}_D(G_p^+) = \mathrm{Br}_D(1_B)\,\mathrm{C}_G(D)_p^+.$$

Here, $\mathrm{Br}_D(1_B)$ is the block idempotent of $b_D^{\mathrm{N}_G(D)}$ where b_D is a Brauer correspondent of B in $\mathrm{C}_G(D)$. In particular, $1_{b_D}\mathrm{Br}_D(1_B) = 1_{b_D}$ and

$$0 \neq 1_{b_D}\mathrm{C}_G(D)_p^+ = 1_{b_D}\mathrm{Br}_D(1_B)\mathrm{C}_G(D)_p^+ = 1_{b_D}\mathrm{Br}_D(1_B G_p^+).$$

From that we obtain $1_B G_p^+ \notin K$ and $\dim RB + K/K = 1$. This implies $RB + K/K = S(ZB/K)$ and

$$LL(ZB/RB + K) = z - 1.$$

Now we consider the lower section of ZB. Here we have

$$\dim RB + K/RB = \dim RB + K - \dim RB = 1 + \dim K - l(B)$$
$$= 1 + \dim ZB - \dim ZB/K - l(B) = 1 + k(B) - c - l(B).$$

The claim follows easily. $\qquad\square$

The invariant c in Proposition 3 is just the number of orbits of $I(B)$ on $\mathrm{Z}(D)$. Moreover, if D and $I(B)$ are given, the number z can be calculated by computer. It happens frequently that $I(B)$ acts trivially on $\mathrm{Z}(D)$. In this case, $c = |\mathrm{Z}(D)|$ and z is determined by the isomorphism type of $\mathrm{Z}(D)$ as explained earlier. In particular, $LL(ZB) \le k(B) - l(B)$ whenever $\mathrm{Z}(D)$ is non-cyclic. Now we give a general upper bound on z.

Lemma 4. *Let P be a finite abelian p-group, and let I be a p'-subgroup of $\mathrm{Aut}(P)$. Then*

$$LL(FP^I) \le LL(F\,\mathrm{C}_P(I)) + \frac{LL(F[P,I]) - 1}{2}.$$

Proof. Since $FP^I = F\,\mathrm{C}_P(I) \otimes F[P,I]^I$, we may assume that $\mathrm{C}_P(I) = 1$. It suffices to show that $JFP^I \subseteq (JFP)^2$. It is well-known that JFP is the augmentation ideal of FP and $JFP^I = JFP \cap FP^I$. In particular, I acts naturally on JFP and on $JFP/(JFP)^2$. We regard $P/\Phi(P)$ as a vector space over $\mathbb{F}_p$. By [10, Remark VIII.2.11] there exists an isomorphism of $\mathbb{F}_p$-spaces

$$\Gamma : JFP/(JFP)^2 \to F \otimes_{\mathbb{F}_p} P/\Phi(P)$$

sending $1 - x + (JFP)^2$ to $1 \otimes x\Phi(P)$ for $x \in P$. After choosing a basis, it is easy to see that $\Gamma(w^\gamma) = \Gamma(w)^\gamma$ for $w \in JFP/(JFP)^2$ and $\gamma \in I$. Let $w \in JFP^I \subseteq JFP$. Then $\Gamma(w + (JFP)^2)$ is invariant under I. It follows that $\Gamma(w + (JFP)^2)$ is a linear combination of elements of the form $\lambda \otimes x$ where $\lambda \in F$ and $x \in \mathrm{C}_{P/\Phi(P)}(I)$. However, by hypothesis, $\mathrm{C}_{P/\Phi(P)}(I) = \mathrm{C}_P(I)\Phi(P)/\Phi(P) = \Phi(P)$ and therefore $\Gamma(w + (JFP)^2) = 0$. This shows $w \in (JFP)^2$ as desired. $\qquad\square$

We describe a special case which extends Theorem 1. Here, the action of $I(B)$ on D is called *semiregular* if all orbits on $D \setminus \{1\}$ have length $|I(B)|$.

Corollary 5. *Let B be a block of FG with abelian defect group D such that $I := I(B)$ acts semiregularly on $[D, I]$. Then*

$$LL(ZB) = LL(ZF[D \rtimes I]) = LL(FD^I) \le LL(F\,\mathrm{C}_D(I)) + \frac{LL(F[D,I]) - 1}{2}.$$

Proof. Let $Q := \mathrm{C}_D(I)$ and let b be a Brauer correspondent of B in $\mathrm{C}_G(Q)$. By [37, Theorem 2], $ZB \cong Zb$. Moreover, by [18, Theorem 7] we have $ZB \cong FQ \otimes Z\bar{b}$ where $\bar{b}$ is the block of $\mathrm{C}_G(Q)/Q$ dominated by b. As usual, $\bar{b}$ has defect group $D/Q \cong [D, I]$ and inertial quotient $I(\bar{b}) \cong I(B)$. It follows that $LL(ZB) = LL(FQ) + LL(Z\bar{b}) - 1$. On the other hand, $FD^I \cong FQ \otimes F[D,I]^I$ and $F[D \rtimes I] \cong FQ \otimes F[[D,I] \rtimes I]$. Hence, we may assume that $Q = 1$ and $[D, I] = D \ne 1$.

Let $(u_1, b_1), \ldots, (u_k, b_k)$ be a set of representatives for the G-conjugacy classes of non-trivial B-subsections. Since I acts semiregularly on D, every block b_i has inertial quotient $I(b_i) \cong \mathrm{C}_I(u_i) = 1$. Hence, b_i is nilpotent and $l(b_i) = 1$. With the notation of Proposition 3, it follows that

$$k(B) - l(B) = \sum_{i=1}^k l(b_i) = \frac{|D| - 1}{|I|} = c - 1$$

and $LL(ZB) \le LL(FD^I)$. By the proof of Proposition 3, we also have the opposite inequality $LL(ZB) \ge LL(FD^I)$. It is easy to see that $ZF[D \rtimes I] = FD^I \oplus \Gamma$ where Γ is the subspace spanned by the non-trivial p'-class sums of $D \rtimes I$. By hypothesis, every non-trivial p'-conjugacy class is a p'-section of $D \rtimes I$. Hence, we obtain $\Gamma \subseteq RF[D \rtimes I]$. The claim $LL(FD^I) = LL(ZF[D \rtimes I])$ follows. The last claim is a consequence of Lemma 4. $\qquad\square$

Corollary 5 applies for instance whenever I has prime order. For example, if $|I| = 2$, we have equality

$$LL(ZB) = LL(F\,\mathrm{C}_D(I)) + \frac{LL(F[D, I]) - 1}{2}$$

by [16, Proposition 2.6]. However, in general for a block B with abelian defect group D it may happen that $LL(ZB) > LL(FD^I)$. An example is given by the principal 3-block of $G = (C_3 \times C_3) \rtimes SD_{16}$. Here $LL(ZB) = 3$ and $\dim FD^I = 2$.

In the situation of Corollary 5, I is a complement in the Frobenius group $[D, I] \rtimes I$. In particular, the Sylow subgroups of I are cyclic or quaternion groups. It follows that I has trivial Schur multiplier. By a result of the first author [20], the Brauer correspondent of B in $\mathrm{N}_G(D)$ is Morita equivalent to $F[D \rtimes I]$. In this way we see that Corollary 5 is in accordance with Broué's Abelian Defect Group Conjecture. Moreover, Alperin's Weight Conjecture predicts $l(B) = k(I)$ in this situation. By a result of the second author (see [33, Lemma 9] and [32, Theorem 5]), we also have

$$\dim ZB \leq |\mathrm{C}_D(I)| \left(\frac{|[D, I]| - 1}{l(B)} + l(B) \right) \leq |D|.$$

Further properties of this class of blocks have been obtained in Kessar-Linckelmann [14]. Nevertheless, it seems difficult to express $LL(FD^I)$ explicitly in terms of D and I. Some special cases have been considered in [35, Section 6.3].

Our next aim concerns the sharpness of Theorem 1. For this we need to discuss twisted group algebras of the form $F_\alpha[D \rtimes I(B)]$.

Lemma 6. *Let P be a finite abelian p-group, and let I be a non-trivial p'-subgroup of $\mathrm{Aut}(P)$. Then*

$$LL(ZF_\alpha[P \rtimes I]) < LL(FP)$$

for every $\alpha \in \mathrm{H}^2(I, F^\times)$.

Proof. For the sake of brevity we write PI instead of $P \rtimes I$. We may normalize α such that $x \cdot y$ in $F_\alpha[PI]$ equals $xy \in PI$ for all $x \in P$ and $y \in PI$. By Passman [24, Theorem 1.6],

$$JZF_\alpha[PI] = JF_\alpha[PI] \cap ZF_\alpha[PI] = (JFP \cdot F_\alpha[PI]) \cap ZF_\alpha[PI].$$

It is known that $ZF_\alpha[PI]$ has a basis consisting of the α-regular class sums (see for example [7, Remark 4 on p. 155]). Hence, let K be an α-regular conjugacy class of PI. If $K \subseteq P$, then clearly $|K|1 - K^+ \in ZF_\alpha[PI] \cap JFP \subseteq JZF_\alpha[PI]$, since JFP is the augmentation ideal of FP. Now assume that $K \subseteq PI \setminus P$ and $x \in K$. Then the P-orbit of x (under conjugation) is the coset $x[x, P]$. Hence, K is a disjoint union of cosets $x_1[x_1, P], \ldots, x_m[x_m, P]$. Since I acts faithfully on P, we have $[x_i, P] \neq 1$ and $[x_i, P]^+ \in JFP$ for $i = 1, \ldots, m$. It follows that $K^+ \in (JFP \cdot F_\alpha[PI]) \cap ZF_\alpha[PI] = JZF_\alpha[PI]$. In this way we obtain an F-basis of $JZF_\alpha[PI]$. Let $l := LL(ZF_\alpha[PI])$. Then there exist conjugacy classes $K_1, \ldots, K_s \subseteq P$ and elements $x_1, \ldots, x_t \in PI \setminus P$ such that $s + t = l - 1$ and

$$(|K_1|1 - K_1^+) \ldots (|K_s|1 - K_s^+)x_1[x_1, P]^+ \ldots x_t[x_t, P]^+ \neq 0$$

in $F_\alpha[PI]$. Since $x_i[x_i, P] = [x_i, P]x_i$, we conclude that

$$0 \neq (|K_1|1 - K_1^+) \ldots (|K_s|1 - K_s^+)[x_1, P]^+ \ldots [x_t, P]^+ \in FP. \tag{2.1}$$

At this point, α does not matter anymore and we may assume that $\alpha = 1$ in the following. Since $ZF[PI] = F\,\mathrm{C}_P(I) \otimes ZF[[P, I] \rtimes I]$ and $FP = F\,\mathrm{C}_P(I) \otimes F[P, I]$, we may assume that $\mathrm{C}_P(I) = 1$. By Lemma 4 we have

$$s \leq LL(FP^I) - 1 \leq \frac{LL(FP) - 1}{2} < LL(FP) - 1.$$

Thus, we may assume that $t > 0$. Since x_1 acts non-trivially on $[x_1, P]$, we obtain $|[x_1, P]| \geq 3$ and $[x_1, P]^+ \in (JF[x_1, P])^2 \subseteq (JFP)^2$. Also, $|K_i|1 - K_i^+ \in JFP$ for $i = 1, \ldots, s$. Therefore, (2.1) shows that $(JFP)^l \neq 0$ and the claim follows. $\square$

Proposition 7. *Let B be a block of FG with abelian defect group D. Suppose that the character-theoretic version of Broué's Conjecture holds for B. Then $LL(ZB) = LL(FD)$ if and only if B is nilpotent.*

Proof. A nilpotent block B satisfies $LL(ZB) = LL(FD)$ by Broué-Puig [5]. Thus, we may assume conversely that $LL(ZB) = LL(FD)$. Broué's Conjecture implies $ZB \cong Zb$ where b is the Brauer correspondent of B in $N_G(D)$. By Külshammer [20], $Zb \cong ZF_\alpha[D \rtimes I(B)]$ for some $\alpha \in H^2(I(B), F^\times)$. Now Lemma 6 shows that $I(B) = 1$. Hence, B must be nilpotent. $\qquad\square$

3 Non-abelian defect groups

We start with a result about nilpotent blocks which might be of independent interest.

Proposition 8. *For a non-abelian p-group P we have $JZFP \subseteq JF[P' Z(P)] \cdot FP$ and*

$$LL(ZFP) \leq LL(FP' Z(P)) < LL(FP).$$

Proof. We have already used that JFP is the augmentation ideal of FP and $JZFP = ZFP \cap JFP$. Hence, $JZFP$ is generated as an F-space by the elements $1 - z$ and K^+ where $z \in Z(P)$ and $K \subseteq P \setminus Z(P)$ is a conjugacy class. Each such K has the form $K = xU$ with $x \in P$ and $U \subseteq P'$. Since $|U| = |K|$ is a multiple of p, we have $U^+ \in JFP'$. On the other hand, $1 - z \in JF Z(P)$ for $z \in Z(P)$. Setting $N := P' Z(P)$ we obtain $JZFP \subseteq FP \cdot JFN$. Since P acts on FN preserving the augmentation, we also have $FP \cdot JFN = JFN \cdot FP$. This shows $LL(ZFP) \leq LL(FN)$.

For the second inequality, note that $N \leq Z(P)\Phi(P) < P$. Hence, $FN^+ = (JFN)^{LL(FN)-1} \subseteq (JFP)^{LL(FN)-1}$ and $(JFP)^{LL(FP)-1} = FP^+ \neq FN^+$. Therefore, we must have $LL(FN) < LL(FP)$. $\qquad\square$

If P has class 2, we have $P' \leq Z(P)$ and $JF Z(P) \subseteq JZFP$. Hence, Proposition 8 implies $LL(ZFP) = LL(F Z(P))$ in this case.

In the following we improve (1.1) for non-abelian defect groups. We make use of Otokita's inductive method:

$$LL(ZB) \leq \max\{(|\langle u \rangle| - 1)LL(Z\bar{b}) : (u, b) \ B\text{-subsection}\} + 1 \qquad (3.1)$$

(see [23, proof of Theorem 1.3]). Here $\bar{b}$ denotes the block of $C_G(u)/\langle u \rangle$ dominated by b. By [30, Lemma 1.34], we may assume that $\bar{b}$ has defect group $C_D(u)/\langle u \rangle$ where D is a defect group of B.

We start with a detailed analysis of the defect groups of large exponent.

Lemma 9. *Let P be a p-group such that $Z(P)$ is cyclic and $|P : Z(P)| = p^2$. Then one of the following holds:*

(i) $P \cong \langle x, y \mid x^{p^{d-1}} = y^p = 1, \ y^{-1}xy = x^{1+p^{d-2}} \rangle =: M_{p^d}$ *for some $d \geq 3$.*

(ii) $P \cong \langle x, y, z \mid x^{p^{d-2}} = y^p = z^p = [x,y] = [x,z] = 1, \ [y,z] = x^{p^{d-3}} \rangle =: W_{p^d}$ *for some $d \geq 3$.*

(iii) $P \cong Q_8$.

Proof. Let $|P| = p^d$ with $d \geq 3$. If $\exp(P) = p^{d-1}$, then the result is well-known. Thus, we may assume that $\exp(P) = p^{d-2}$. Let $Z(P) = \langle x \rangle$ and $D = \langle x, y, z \rangle$. Since $\langle x, y \rangle \cong \langle x, z \rangle \cong C_{p^{d-2}} \times C_p$, we may assume that $y^p = z^p = 1$. Since P is non-abelian, we have $1 \neq [y,z] \in P' \leq Z(P)$. In particular, P has nilpotency class 2. It follows that $[y,z]^p = [y^p, z] = 1$ and therefore $[y,z] = x^{p^{d-3}}$. Consequently, the isomorphism type of P is uniquely determined. Conversely, one can construct such a group as a central product of $C_{p^{d-2}}$ and an extraspecial group of order p^3. $\qquad\square$

Proposition 10. *Let B be a block of FG with defect group $D \cong M_{p^d}$ or $D \cong Q_8$. Then one of the following holds:*

(i)

$$LL(ZB) = \frac{p^{d-2} - 1}{l(B)} + 1 \leq p^{d-2} = LL(ZFD) \leq LL(FD).$$

(ii) $|D| = 8$ *and* $LL(ZB) \leq 3$.

Proof. Suppose first that $p = 2$. If $|D| = 8$, then there are in total five possible fusion systems for B and none of them is exotic (see [30, Theorem 8.1]). By [6], the fusion system of B determines the perfect isometry class of B. Since perfect isometries preserve the isomorphism type of ZB, we may assume that B is the principal block of FH where $H \in \{D_8, Q_8, S_4, \mathrm{SL}(2,3), \mathrm{GL}(3,2)\}$. A computation with GAP [9] reveals that $LL(ZB) \leq 3$ in all cases. Note that we may work over the field with two elements, since the natural structure constants of ZFH (and of ZB) lie in the prime field of F. (The fusion system corresponding to $H = \mathrm{GL}(3,2)$ can be handled alternatively with Proposition 3.) If $D \cong M_{2^d}$ with $d \geq 4$, then B is nilpotent ([30, Theorem 8.1]) and the result follows from the remark after Proposition 8.

Now assume that $p > 2$. By [36], B is perfectly isometric to its Brauer correspondent in $\mathrm{N}_G(D)$. Hence, we may assume that $D \trianglelefteq G$. It is known that B has cyclic inertial quotient $I(B)$ of order dividing $p - 1$ (see [30, proof of Theorem 8.8]). Hence, by [20] we may assume that $G = D \rtimes I(B)$. Then G has only one block and $ZB = ZFG$. Moreover, $l(B) = |I(B)|$. After conjugation, we may assume that $I(B) = \langle a \rangle$ acts non-trivially on $\langle x \rangle$ and trivially on $\langle y \rangle$. Since $|D'| = p$, the conjugacy classes of D are either singletons in $\mathrm{Z}(D)$ or cosets of D'. Some of these classes are fused in G. The classes in $G \setminus D$ are cosets of $\langle x \rangle$. As usual, ZFG is generated by the class sums and $JZFG$ is the augmentation ideal (intersected with ZFG). In particular, $JZFG$ contains the class sums of conjugacy classes whose length is divisible by p. Let $U_1, \ldots, U_k$ be the non-trivial orbits of $I(B)$ on $\mathrm{Z}(D)$. Then $JZFG$ also contains the sums $l(B)1_G - U_i^+$ for $i = 1, \ldots, k$. For $u, v \in D$ we have

$$u(D')^+ \cdot v(D')^+ = uv((D')^+)^2 = 0,$$
$$u(D')^+ \cdot v\langle x \rangle^+ = uv(D')^+ \langle x \rangle^+ = 0,$$
$$u\langle x \rangle^+ \cdot v\langle x \rangle^+ = uv(\langle x \rangle^+)^2 = 0,$$
$$u(D')^+ \cdot (l(B)1_G - U_i^+) = l(B)u(D')^+ - l(B)u(D')^+ = 0,$$
$$u\langle x \rangle^+ \cdot (l(B)1_G - U_i^+) = l(B)u\langle x \rangle^+ - l(B)u\langle x \rangle^+ = 0.$$

It follows that $(JZFG)^2 = (JZF\langle x, a \rangle)^2$. Now the claim can be shown with [16, Corollary 2.8]. $\square$

Lemma 11. *Let B be a block of FG with defect group $D \cong W_{p^d}$. Then $LL(ZB) \leq p^{d-1} - p + 1$.*

Proof. If $|D| = 8$, then the claim holds by Proposition 10. Hence, we may exclude this case in the following. We consider B-subsections (u, b) with $1 \neq u \in D$. As usual, we may assume that b has defect group $\mathrm{C}_D(u)$.

Suppose first that $I(B)$ acts faithfully on $\mathrm{Z}(D)$. We apply Proposition 2. If $u \notin \mathrm{Z}(D)$, then $\mathrm{C}_D(u) \cong C_{p^{d-2}} \times C_p$. Thus, Theorem 1 implies $LL(Zb/Rb) \leq p^{d-2} + p - 2$. Now assume that $u \in \mathrm{Z}(D)$. The centric subgroups in the fusion system of b are maximal subgroups of D. In particular, they are abelian of rank 2. Now by [30, Proposition 6.11], it follows that b is a controlled block. Since $I(b) \cong \mathrm{C}_{I(B)}(u) = 1$, b is nilpotent and $Zb \cong ZFD$. By Proposition 8 we obtain $LL(Zb/Rb) \leq LL(Zb) = LL(ZFD) \leq LL(F\mathrm{Z}(D)) = p^{d-2}$. Hence, Proposition 2 gives

$$LL(ZB) \leq LL(ZB/RB) + 1 \leq p^{d-2} + p - 1 \leq p^{d-1} - p + 1.$$

Now we deal with the case where $I(B)$ is non-faithful on $\mathrm{Z}(D)$. We make use of (3.1). Let $|\langle u \rangle| = p^s$. The dominated block $\bar{b}$ has defect group $\mathrm{C}_D(u)/\langle u \rangle$. If $u \notin \mathrm{Z}(D)$, then $|\mathrm{C}_D(u)/\langle u \rangle| = p^{d-s-1} \geq p$ and

$$(p^s - 1)LL(Z\bar{b}) \leq (p^s - 1)p^{d-s-1} \leq p^{d-1} - p.$$

Next suppose that $u \in \mathrm{Z}(D)$. Then $D' \subseteq \langle u \rangle$ and $\bar{b}$ has defect group $D/\langle u \rangle \cong C_{p^{d-s-2}} \times C_p \times C_p$. In case $\langle u \rangle < \mathrm{Z}(D)$, we have $s \leq d - 3$ and Theorem 1 implies

$$(p^s - 1)LL(Z\bar{b}) \leq (p^s - 1)(p^{d-s-2} + 2p - 2) \leq p^{d-2} + 2p^{d-2} - 2p^{d-3} - 3p + 2 \leq p^{d-1} - p.$$

Finally, assume that $\langle u \rangle = \mathrm{Z}(D)$. By [33, Lemma 3], we have

$$I(\bar{b}) \cong I(b) \cong \mathrm{C}_{I(B)}(u) \neq 1.$$

We want to show that $I(\bar{b})$ acts semiregularly on $D/\mathrm{Z}(D)$. Let $D = \langle x, y, z \rangle$ as in Lemma 9, and let $\gamma \in I(\bar{b})$. Then $y^\gamma \equiv y^i z^j \pmod{\mathrm{Z}(D)}$ and $z^\gamma \equiv y^k z^l \pmod{\mathrm{Z}(D)}$ for some $i, j, k, l \in \mathbb{Z}$. Since D has nilpotency class 2, we have

$$[y, z] = [y, z]^\gamma = [y^\gamma, z^\gamma] = [y^i z^j, y^k z^l] = [y, z]^{il-jk}.$$

It follows that $il - jk \equiv 1 \pmod{p}$ and $I(\bar{b}) \leq \mathrm{SL}(2,p)$. As a p'-subgroup of $\mathrm{SL}(2,p)$, $I(\bar{b})$ acts indeed semiregularly on $D/\mathrm{Z}(D)$. Thus, Corollary 5 shows that

$$(p^s - 1)LL(Z\bar{b}) \leq (p^{d-2} - 1)p = p^{d-1} - p$$

Therefore, the claim follows from (3.1). $\qquad\square$

We do not expect that Lemma 11 is sharp. In fact, Jennings's Theorem [11] shows that $LL(FW_{p^3}) = 4p - 3$. Even in this small case the perfect isometry classes are not known (see for example [27]).

We are now in a position to deal with all non-abelian defect groups.

Theorem 12. *Let B be a block of FG with non-abelian defect group of order p^d. Then*

$$LL(ZB) \leq p^{d-1} + p^{d-2} - p^{d-3}.$$

Proof. We argue by induction on d. Let D be a defect group of B. Again we will use (3.1). Let (u,b) be a B-subsection with $u \in D$ of order $p^s \neq 1$. As before, we may assume that the dominated block $\bar{b}$ has defect group $\mathrm{C}_D(u)/\langle u \rangle$. If $\mathrm{C}_D(u)/\langle u \rangle$ is cyclic, then $\mathrm{C}_D(u)$ is abelian and therefore $\mathrm{C}_D(u) < D$. Hence,

$$(p^s - 1)LL(Z\bar{b}) \leq (p^s - 1)p^{d-s-1} \leq p^{d-1} - 1 \leq p^{d-1} + p^{d-2} - p^{d-3} - 1.$$

Suppose next that $\mathrm{C}_D(u)/\langle u \rangle$ is abelian of type $(p^{a_1}, \ldots, p^{a_r})$ with $r \geq 2$. If $s = d - 2$, then D fulfills the assumption of Lemma 9. Hence, by Proposition 10 and Lemma 11 we conclude that

$$LL(ZB) \leq p^{d-1} - p + 1 \leq p^{d-1} + p^{d-2} - p^{d-3}.$$

Consequently, we can restrict ourselves to the case $s \leq d - 3$. Theorem 1 shows that

$$LL(Z\bar{b}) \leq p^{a_1} + \ldots + p^{a_r} - r + 1 \leq p^{a_1 + \ldots + a_{r-1}} + p^{a_r} - 1 \leq \frac{|\mathrm{C}_D(u)|}{p^{s+1}} + p - 1.$$

Hence, one gets

$$(p^s - 1)LL(Z\bar{b}) \leq (p^s - 1)(p^{d-s-1} + p - 1) \leq p^{d-1} + p^{s+1} - p^s - 1 \leq p^{d-1} + p^{d-2} - p^{d-3} - 1.$$

It remains to consider the case where $\mathrm{C}_D(u)/\langle u \rangle$ is non-abelian. Here induction gives

$$(p^s - 1)LL(Z\bar{b}) \leq (p^s - 1)(p^{d-s-1} + p^{d-s-2} - p^{d-s-3}) \leq p^{d-1} + p^{d-2} - p^{d-3} - 1.$$

Now the claim follows with (3.1). $\qquad\square$

In the situation of Theorem 12 we also have

$$\dim ZFD \leq |\mathrm{Z}(D)| + \frac{p^d - |\mathrm{Z}(D)|}{p} \leq p^{d-1} + p^{d-2} - p^{d-3},$$

but it is not clear if $LL(ZB) \leq \dim ZFD$.

Doing the analysis in the proof above more carefully, our bound can be slightly improved, but this does not affect the order of magnitude. Note also that Theorem 12 improves Eq. (1.1) even in case $p = 2$, because then $\exp(D) \geq 4$. Nevertheless, we develop a stronger bound for $p = 2$ in the following. We begin with the 2-blocks of defect 4. The definition of the minimal non-abelian group $MNA(2,1)$ can be found in [30, Theorem 12.2]. The following proposition covers all non-abelian 2-groups of order 16.

Proposition 13. *Let B be a block of FG with defect group D. Then*

$$LL(ZB) \leq \begin{cases} 3 & \text{if } D \cong C_4 \rtimes C_4, \\ 4 & \text{if } D \in \{M_{16}, D_8 \times C_2, Q_8 \times C_2, MNA(2,1)\}, \\ 5 & \text{if } D \in \{D_{16}, Q_{16}, SD_{16}, W_{16}\}. \end{cases}$$

In all cases we have $LL(ZB) \leq LL(FD)$.

Proof. The case $D \cong M_{16}$ has already been done in Proposition 10. For the metacyclic group $D \cong C_4 \rtimes C_4$, B is nilpotent (see [30, Theorem 8.1]) and the result follows from Proposition 8. For the dihedral, quaternion, semidihedral and minimal non-abelian groups the perfect isometry class is uniquely determined by the fusion system of B (see [6, 34]). Moreover, all these fusion systems are non-exotic (see [30, Theorem 10.17]). In particular, $LL(ZB) \leq LL(ZFH)$ for some finite group H. More precisely, if B is non-nilpotent, we may consider the following groups H:

- $\mathrm{PGL}(2,7)$ and $\mathrm{PSL}(2,17)$ if $D \cong D_{16}$,
- $\mathrm{SL}(2,7)$ and $\mathtt{SmallGroup}(240,89) \cong 2.S_5$ if $D \cong Q_{16}$,
- M_{10} (Mathieu group), $\mathrm{GL}(2,3)$ and $\mathrm{PSL}(3,3)$ if $D \cong SD_{16}$,
- $\mathtt{SmallGroup}(48,30) \cong A_4 \rtimes C_4$ if $D \cong MNA(2,1)$.

For all these groups H the number $LL(ZFH)$ can be determined with GAP [9].

Finally, for $D \in \{D_8 \times C_2, Q_8 \times C_2, W_{16}\}$ one can enumerate the possible generalized decomposition matrices of B up to basic sets (see [29, Propositions 3, 4 and 5]). In each case the isomorphism type of ZB can be determined with a result of Puig [26]. We omit the details. Observe that we improve Lemma 11 for $D \cong W_{16}$. Finally, the claim $LL(ZB) \leq LL(FD)$ can be shown with Jennings's Theorem [11] or one consults [12, Corollary 4.2.4 and Table 4.2.6]. $\qquad \square$

Next we elaborate on Lemma 9.

Lemma 14. *Let B be a 2-block of FG with non-abelian defect group D such that there exists a $z \in \mathrm{Z}(D)$ with $D/\langle z \rangle \cong C_{2^n} \times C_2$ where $n \geq 2$. Then $LL(ZB) < |D|/2$.*

Proof. By hypothesis there exist two maximal subgroups M_1 and M_2 of D containing z such that $M_1/\langle z \rangle \cong M_2/\langle z \rangle \cong C_{2^n}$. It follows that M_1 and M_2 are abelian. Since $D = M_1 M_2$, we obtain $\mathrm{Z}(D) = M_1 \cap M_2$ and $|D : \mathrm{Z}(D)| = 4$. This implies $|D'| = 2$ (see e.g. [1, Lemma 1.1]). Obviously, $D' \leq \langle z \rangle$. By Lemma 9 we may assume that $\mathrm{Z}(D)$ is abelian of rank 2.

Suppose for the moment that B is nilpotent. Since $\mathrm{Z}(D)$ is not cyclic, $D \not\cong M_{2^m}$ for all m. Now a result of Koshitani-Motose [21, Theorems 4 and 5] shows that

$$LL(ZB) = LL(ZFD) \leq LL(FD) < \frac{|D|}{2}.$$

For the remainder of the proof we may assume that B is not nilpotent. Suppose that $\mathrm{Z}(D) = \Phi(D)$. Then D is minimal non-abelian and it follows from [30, Theorem 12.4] that $D \cong MNA(r,1)$ for some $r \geq 2$. By Proposition 13 we can assume that $r \geq 3$. By the main result of [34], B is isotypic to the principal block of $H := A_4 \rtimes C_{2^r}$. In particular, $LL(ZB) \leq LL(FH)$. Note that H contains a normal subgroup $N \cong C_{2^{r-1}} \times C_2 \times C_2$ such that $H/N \cong S_3$ (see [34, Lemma 2]). By Passman [25, Theorem 1.6], $(JFH)^2 \subseteq (JFN)(FH) = (FH)(JFN)$. It follows that

$$LL(FH) \leq 2LL(FN) = 2(2^{r-1} + 2) < 2^{r+1} = \frac{|D|}{2}.$$

Thus, we may assume $|D : \Phi(D)| = 8$ in the following. Let $\mathcal{F}$ be the fusion system of B. Suppose that there exists an $\mathcal{F}$-essential subgroup $Q \leq D$ (see [30, Definition 6.1]). Then $z \in \mathrm{Z}(D) \leq \mathrm{C}_D(Q) \leq Q$ and Q is abelian. Moreover, $|D : Q| = 2$. It is well-known that $\mathrm{Aut}_{\mathcal{F}}(Q)$ acts faithfully on $Q/\Phi(Q)$ (see [30, p. 64]). Since $D/Q \leq \mathrm{Aut}_{\mathcal{F}}(Q)$, we obtain $D' \not\subseteq \Phi(Q)$. On the other hand, $z^2 \in \Phi(Q)$. This shows that $D' = \langle z \rangle$ and D/D' has rank 2. However, this contradicts $|D : \Phi(D)| = 8$.

Therefore, B is a controlled block and $\mathrm{Aut}(D)$ is not a 2-group. Let $1 \neq \alpha \in \mathrm{Aut}(D)$ be of odd order. Then α acts trivially on D' and on $\Omega(\mathrm{Z}(D))/D'$, since $\mathrm{Z}(D)$ has rank 2. Hence, α acts trivially on $\Omega(\mathrm{Z}(D))$ and also on $\mathrm{Z}(D)$. But then α acts non-trivially on $D/\langle z \rangle \cong C_{2^n} \times C_2$ which is impossible. This contradiction shows that there are no more blocks with the desired property. $\qquad \square$

Proposition 15. *Let B be a 2-block of FG with non-abelian defect group of order 2^d. Then $LL(ZB) < 2^{d-1}$.*

Proof. We mimic the proof of Theorem 12. Let D be a defect group of B, and let (u, b) be a B-subsection such that u has order $2^s > 1$. It suffices to show that $(2^s - 1)LL(Z\bar{b}) \leq 2^{d-1} - 2$. If $C_D(u)/\langle u \rangle$ is cyclic, then $C_D(u)$ is abelian and $C_D(u) < D$. Then we obtain

$$(2^s - 1)LL(Z\bar{b}) \leq (2^s - 1)2^{d-s-1} = 2^{d-1} - 2^{d-s-1}.$$

We may assume that $s = d - 1$. Then by Proposition 10, we may assume that D is dihedral, semidihedral or quaternion. Moreover, by Proposition 13, we may assume that $d \geq 5$. Then [30, Theorem 8.1] implies

$$LL(ZB) \leq \dim ZB = k(B) \leq 2^{d-2} + 5 < 2^{d-1}.$$

Now suppose that $C_D(u)/\langle u \rangle$ is abelian of type $(2^{a_1}, \ldots, 2^{a_r})$ with $r \geq 2$. As in Theorem 12, we may assume that $s \leq d - 3$. If $a_1 = 1$ and $r = 2$, then by Lemma 14, we may assume that $C_D(u) < D$. Hence, we obtain

$$(2^s - 1)LL(Z\bar{b}) \leq (2^s - 1)(2^{d-s-2} + 1) \leq 2^{d-2} + 2^{d-3} \leq 2^{d-1} - 2$$

in this case. Now suppose that $r \geq 3$ or $a_i > 1$ for $i = 1, 2$. If $r = 3$ and $a_1 = a_2 = a_3 = 1$, we have $(2^s - 1)LL(Z\bar{b}) \leq 2^{d-1} - 4$. In the remaining cases we have $s \leq d - 4$ and

$$(2^s - 1)LL(Z\bar{b}) \leq (2^s - 1)(2^{d-s-2} + 3) \leq 2^{d-2} + 3 \cdot 2^{d-4} \leq 2^{d-1} - 2.$$

Finally, suppose that $C_D(u)/\langle u \rangle$ is non-abelian. Then the claim follows by induction on d. $\square$

Corollary 16. *Let B be a block of FG with defect group D. Then $LL(ZB) \geq |D|/2$ if and only if one of the following holds:*

(i) *D is cyclic and $l(B) \leq 2$,*

(ii) *$D \cong C_{2^n} \times C_2$ for some $n \geq 1$,*

(iii) *$D \cong C_2 \times C_2 \times C_2$ and B is nilpotent,*

(iv) *$D \cong C_3 \times C_3$ and B is nilpotent.*

Proof. Suppose that $LL(ZB) \geq |D|/2$. Then by Theorem 12 and Proposition 15, D is abelian. If D is cyclic, we have $LL(ZB) = \frac{|D|-1}{l(B)} + 1$. If additionally $l(B) \geq 3$, then we get the contradiction $|D| \leq 4$. Now suppose that D is not cyclic. Then

$$\frac{|D|}{2} \leq LL(ZB) \leq \frac{|D|}{p} + p - 1$$

by Theorem 1 and we conclude that

$$p^2 \leq |D| \leq \frac{2p(p-1)}{p-2}.$$

This yields $p \leq 3$. Suppose first that $p = 3$. Then we have $D \cong C_3 \times C_3$ and $5 = LL(ZB) \leq k(B) - l(B) + 1$ by Proposition 3. It follows from [15] that $I(B) \notin \{C_4, C_8, Q_8, SD_{16}\}$ (note that $k(B) - l(B)$ is determined locally). The case $I(B) \cong C_2$ is excluded by Corollary 5. Hence, we may assume that $I(B) \in \{C_2 \times C_2, D_8\}$. By [31, Theorem 3] and [28, Lemma 2], B is isotypic to its Brauer correspondent in $N_G(D)$. This gives the contradiction $LL(ZB) \leq 3$. Therefore, B must be nilpotent and $LL(ZB) = 5$.

Now let $p = 2$. Then D has rank at most 3. If the rank is 3, we obtain $LL(ZB) \leq 2^{d-2} + 2$ and $d = 3$. In this case, B is nilpotent or $I(B) \cong C_7 \rtimes C_3$ by Corollary 5. By [13], B is isotypic to its Brauer correspondent in $N_G(D)$. From that we can derive that B is nilpotent and $LL(ZB) = 4$. It remains to handle defect groups of rank 2. Here, $D \cong C_{2^n} \times C_2$ for some $n \geq 1$. If $n \geq 2$, then B is always nilpotent and $LL(ZB) = 2^n + 1$. If $n = 1$, then both possibilities $l(B) \in \{1, 3\}$ give $LL(ZB) \geq 2$.

Conversely, we have seen that all our examples actually satisfy $LL(ZB) \geq |D|/2$. $\square$

The following approach gives more accurate results for a given arbitrary defect group. For a finite p-group P we define a recursive function $\mathcal{L}$ as follows:

$$\mathcal{L}(P) := \begin{cases} p^{a_1} + \ldots + p^{a_r} - r + 1 & \text{if } P \cong C_{p^{a_1}} \times \ldots \times C_{p^{a_r}}, \\ p^{d-2} & \text{if } P \cong M_{p^d} \text{ with } p^d \neq 8, \\ p^{d-1} - p + 1 & \text{if } P \cong W_{p^d} \text{ with } p^d \neq 16, \\ 3 & \text{if } P \in \{D_8, Q_8, C_4 \rtimes C_4\}, \\ 4 & \text{if } P \in \{D_8 \times C_2, Q_8 \times C_2, MNA(2,1)\}, \\ 5 & \text{if } P \in \{D_{16}, Q_{16}, SD_{16}, W_{16}\}, \\ \max\big\{(|\langle u \rangle| - 1)\mathcal{L}(\mathrm{C}_P(u)/\langle u \rangle) : 1 \neq u \in P\big\} + 1 & \text{otherwise.} \end{cases}$$

Then, by the results above, every block B of FG with defect group D satisfies $LL(ZB) \leq \mathcal{L}(D)$. For example, there are only three non-abelian defect groups of order 3^6 giving the worst case estimate $LL(ZB) \leq 287$.

In general, it is difficult to give good lower bounds on $LL(ZB)$ (cf. [16, Corollary 2.7]). Assume for instance that $\mathbb{F}_{p^n}$ is the field with p^n elements and $G = \mathbb{F}_{p^n} \rtimes \mathbb{F}_{p^n}^\times$ for some $n \geq 1$. Then G has only one block B and $k(B) - l(B) = 1$. It follows that $LL(ZB) = 2$. In particular, the defect of B is generally not bounded in terms of $LL(ZB)$.

Acknowledgment

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