Groups of p-central type

Benjamin Sambale*

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Abstract

A finite group G with center Z is of central type if there exists a fully ramified character $\lambda \in \mathrm{Irr}(Z)$, i. e. the induced character λ^G is a multiple of an irreducible character. Howlett–Isaacs have shown that G is solvable in this situation. A corresponding theorem for p-Brauer characters was proved by Navarro–Späth–Tiep under the assumption that $p \neq 5$. We show that there are no exceptions for $p = 5$, i. e. every group of p-central type is solvable. Gagola proved that every solvable group can be embedded in G/Z for some group G of central type. We generalize this to groups of p-central type. As an application we construct some interesting non-nilpotent blocks with a unique Brauer character. This is related to a question by Kessar and Linckelmann.

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1 Introduction

For an irreducible character χ of a finite group G it is easy to show that $\chi(1)^2 \le |G : Z|$ where $Z := \mathrm{Z}(G)$ denotes the center of G (see [13, Corollary 2.30]). If equality holds for some $\chi \in \mathrm{Irr}(G)$, we say that G has *central type*. It was conjectured by Iwahori–Matsumoto and eventually proved by Howlett–Isaacs [11], using the classification of finite simple groups (CFSG), that all groups of central type are solvable (see also [23, Chapter 8]). Apart from this there are no restrictions on the structure of groups of central type. In fact, Gagola [6, Theorem 1.2] has shown that every solvable group can be embedded into G/Z for some group G of central type.

It is well-known that G is of central type if there exists some *fully ramified* character $\lambda \in \mathrm{Irr}(Z)$, i. e. $\lambda^G = e\chi$ for some integer e and some $\chi \in \mathrm{Irr}(G)$ (in this case $\chi(1)^2 = e^2 = |G : Z|$; see [23, Lemma 8.2]). This can be carried over to Brauer characters over an algebraically closed field of characteristic $p > 0$. We call $\lambda \in \mathrm{IBr}(Z)$ *fully ramified* if $\lambda^G = e\varphi$ for some integer e and $\varphi \in \mathrm{IBr}(G)$. In this situation, Navarro–Späth–Tiep [24, Theorem A] proved (again relying on the CFSG) that G is solvable unless $p = 5$. Using the structure of a minimal counterexample for $p = 5$ (as described in [24]), we are able to eliminate this exceptional case.

Theorem 1. *Let G be a finite group, $Z \trianglelefteq G$ and $\lambda \in \mathrm{IBr}(Z)$ be G-invariant. Suppose that $\lambda^G = e\varphi$ for some integer e and $\varphi \in \mathrm{IBr}(G)$. Then G/Z is solvable.*

*Institut für Algebra, Zahlentheorie und Diskrete Mathematik, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany, sambale@math.uni-hannover.de

Notice that Theorem 1 generalizes the Howlett–Isaacs theorem by choosing a prime which does not divide $|G|$. Assuming that G is (p-)solvable, there exists a fully ramified Brauer character in $Z = \mathrm{Z}(G)$ if and only if $\varphi(1)^2 = |G : Z|_{p'}$ for some $\varphi \in \mathrm{IBr}(G)$ where $n_{p'}$ denotes the p'-part of an integer n (see Proposition 6 below). If this is the case, we call G a group of *p-central type*. Our second objective is to carry over Gagola's theorem as follows.

Theorem 2. *Let G be a solvable group and p be a prime. Then there exists a group H of p-central type such that G is isomorphic to a subgroup of $H/\mathrm{Z}(H)$.*

Further properties of the group H in Theorem 2 are stated in Theorem 7 below. Theorems 1 and 2 are proved in the next section.

In Section 3 we apply our results to p-blocks B of G with defect group D. The omnipresent *nilpotent* blocks introduced by Broué–Puig [3] have a unique irreducible Brauer character, i. e. $l(B) := |\mathrm{IBr}(B)| = 1$. We are interested in non-nilpotent blocks with $l(B) = 1$. These occur far less frequent, e. g. they cannot be principal blocks. Indeed, it was speculated by Kessar and Linckelmann that all such blocks are Morita equivalent to their Brauer correspondent in $\mathrm{N}_G(D)$ (if D is abelian, this is equivalent to Broué's conjecture by [27, Theorem 3]). Using Külshammer's reduction [17] and Theorem 1, all such blocks should occur in solvable groups up to Morita equivalence.

Concrete examples can be build as follows: If H is a non-trivial p'-group of central type acting on an elementary abelian p-group V with kernel $\mathrm{Z}(H)$, then $G = V \rtimes H$ has a non-nilpotent p-block B with $l(B) = 1$ (see proof of Theorem 3 below). The smallest instance in terms of $|G|$ is the unique non-principal block of $G \cong C_3^2 \rtimes D_8 \cong \texttt{SmallGroup}(72, 23)$, which first appeared in Kiyota [15]. Kessar [14] has shown that every non-nilpotent block with $l(B) = 1$ and defect group C_3^2 is Morita equivalent to this block (see also [16]). Similar examples with abelian defect group were investigated in [1, 2, 10, 12, 18]. All these blocks arise from groups with a normal abelian Sylow p-subgroup. As an application of Theorem 2, we illustrate that non-nilpotent blocks with $l(B) = 1$ occur in arbitrarily "complicated" solvable groups.

Theorem 3. *Let p be a prime and $l \geq 1$ an integer. Then there exists a solvable group G of p-length $l_p(G) \geq l$ such that G has a non-nilpotent p-block B of maximal defect with $l(B) = 1$.*

On the other hand, we construct some non-solvable examples using the following recipe.

Theorem 4. *Let b be a p-block of a finite group H such that $l(b)$ is not a p-power (in particular $l(b) > 1$). Suppose that there exists an automorphism group $A \leq \mathrm{Aut}(H)$ such that A acts regularly on $\mathrm{IBr}(b)$. Let Q be a p-group upon A acts non-trivially. Then the group $G = (H \times Q) \rtimes A$ where A acts diagonally has a non-nilpotent p-block B with $l(B) = 1$.*

Finally we study lifts of Brauer characters. By the Fong–Swan theorem, every irreducible Brauer character φ of a p-solvable group G is the restriction of an ordinary character to the set of p-regular elements. This is no longer true in non-solvable groups. However, if $\mathrm{IBr}(B) = \{\varphi\}$, then Malle–Navarro–Späth [20] proved (using the CFSG) that φ still has such a lift. For nilpotent blocks with defect group D, the number of these lifts is $|D : D'|$. In particular, φ has at least p^2 lifts unless $|D| \leq p$. For non-nilpotent blocks, it was conjectured by Malle–Navarro [19] (in combination with Olsson's conjecture) that the number of lifts is strictly less than $|D : D'|$. Answering a question of G. Navarro, we construct (non-nilpotent) blocks with $\mathrm{IBr}(B) = \{\varphi\}$ such that φ has a *unique* lift.

2 Groups of p-central type

For a finite group G we denote the Fitting subgroup by $\mathrm{F}(G)$ and the layer by $\mathrm{E}(G)$ (the product of all components of G). Then the generalized Fitting subgroup is a central product $\mathrm{F}^*(G) = \mathrm{F}(G) * \mathrm{E}(G)$. Our notation for characters follows Navarro's book [21].

We need the following lemma from the theory of group extensions (see [8, Theorem 15.3.1]).

Lemma 5. *Let N be a finite group, $\alpha \in \mathrm{Aut}(N)$ and $m \in \mathbb{N}$. Then the following assertions are equivalent:*

(1) There exists a finite group H such that $N \trianglelefteq H$, $H/N = \langle hN \rangle \cong C_m$ and $\alpha(x) = hxh^{-1}$ for all $x \in N$.

(2) There exists $n \in N$ such that $\alpha(n) = n$ and $\alpha^m(x) = nxn^{-1}$ for all $x \in N$.

Proof of Theorem 1. By [24, Theorem A], we may assume that $p = 5$. Let G be a minimal counterexample with respect to $|G : Z|$. Then by [24, Theorem 8.1 and its proof], the following holds:

(i) $Z = \mathrm{Z}(G) = \mathrm{F}(G)$ is a cyclic $\{2, 3\}$-group.

(ii) $\mathrm{E}(G) = T_1 * \ldots * T_m$ where $T_1 \cong \ldots \cong T_m \cong 6.A_6$ and $\mathrm{Z}(T_1) = \ldots = \mathrm{Z}(T_m) \cong C_6$. (In the notation of [24] we have $T_i = S_i'$.)

(iii) $G/\mathrm{F}^*(G)$ is a 2-group, which permutes the T_i transitively. In particular, $m = 2^n$ for some $n \geq 0$.

(iv) Every Sylow 2-subgroup of G is of central type.

Let $N := \bigcap_{i=1}^m \mathrm{N}_G(T_i) \trianglelefteq G$ and $a \in N$. Suppose by way of contradiction that $a \notin \mathrm{F}^*(G)$. Since $\mathrm{F}^*(G)$ is self-centralizing and $\mathrm{Out}(T_i) \cong C_2^2$, we have $a^2 \in \mathrm{F}^*(G)$. Let $t_i \in T_i$ and $z \in Z$ such that $a^2 = t_1 \ldots t_m z$. If a induces an inner automorphism on every T_i, then we have the contradiction $a \in \mathrm{F}^*(G)\mathrm{C}_G(\mathrm{F}^*(G)) \leq \mathrm{F}^*(G)$. Thus, without loss of generality we may assume that a induces an outer automorphism α of T_1. Then $\alpha(t_1) = t_1$ and α^2 is the inner automorphism on T_1 induced by t_1. Hence, by Lemma 5, there exists a group H with $T_1 \trianglelefteq H$ and $H/T_1 = \langle hT_1 \rangle \cong C_2$ where h acts as α on T_1. Since $\mathrm{Z}(T_1) = \mathrm{Z}(\mathrm{E}(G)) \leq \mathrm{F}(G) = \mathrm{Z}(G)$, we have $\mathrm{Z}(Z_1) \leq \mathrm{Z}(H)$. Moreover, $H/\mathrm{Z}(T_1)$ is isomorphic to one of the three subgroups of $\mathrm{Aut}(A_6)$ of index 2, namely S_6, $\mathrm{PGL}(2, 9)$ or the Mathieu group M_{10}. However, none of these three groups has a 6-fold Schur extension as can be checked with GAP [7]. This contradiction shows that $N = \mathrm{F}^*(G)$.

Now the 2-group G/N is a transitive permutation group of degree m. Therefore, $|G : N|$ is bounded by the order of a Sylow 2-subgroup of the symmetric group S_m. This yields

$$w := |G : N| \leq 2^{\sum_{i=1}^n \frac{m}{2^i}} = 2^{1+2+\ldots+2^{n-1}} = 2^{m-1}. \tag{2.1}$$

Let P be a Sylow 2-subgroup of G and $Z_2 = P \cap Z \leq \mathrm{Z}(P)$. Observe that $\mathrm{C}_P(P \cap N) \leq N$. Since T_1 has Sylow 2-subgroups isomorphic to Q_{16}, we have $P \cap N \cong Q_{16} * \ldots * Q_{16} * Z_2$ and consequently $\mathrm{Z}(P) \leq \mathrm{C}_P(P \cap N) = Z_2$. Thus, $\mathrm{Z}(P) = Z_2$. Choose $v_i \in T_i \cap P$ of order 8 for $i = 1, \ldots, m$. Since $[T_i, T_j] = 1$ for $i \neq j$, $A := \langle v_1, \ldots, v_m \rangle Z_2 \leq P$ is abelian with $|P : A| = |G : N||P \cap N : A| = 2^m w$. By part (iv), there exists $\chi \in \mathrm{Irr}(P)$ such that $\chi(1)^2 = |P : Z_2|$. For a (linear) constituent $\mu \in \mathrm{Irr}(A)$ of the restriction χ_A we have $\chi(1) \leq \mu^P(1) = |P : A|$. Hence,

$$2^{3m} w = |G : N||N : Z|_2 = |P : Z_2| = \chi(1)^2 \leq |P : A|^2 = 2^{2m} w^2$$

and $w \geq 2^m$. This contradicts (2.1). $\square$

The original conjecture by Iwahori–Matsumoto has been generalized by J. F. Humphreys (as mentioned in [9]) and independently by Navarro [22, Conjecture 11.1] to the following statement: Let $N \trianglelefteq G$ and $\lambda \in \mathrm{Irr}(N)$ be G-invariant such that all irreducible constituents of λ^G have the same degree. Then G/N is solvable. This conjecture is still open, but the corresponding version for Brauer characters does not hold. Indeed the Schur cover $6.A_6$ considered in the proof above is a counterexample for $p = 5$.

Next, we prove the equivalent characterization of groups of p-central type mentioned in the introduction.

Proposition 6. *For a p-solvable group G with center $Z := \mathrm{Z}(G)$ the following assertions are equivalent:*

(1) There exists some $\varphi \in \mathrm{IBr}(G)$ such that $\varphi(1)^2 = |G : Z|_{p'}$.

(2) There exists a fully ramified Brauer character $\lambda \in \mathrm{IBr}(Z)$.

Proof. It has been shown in [24, Theorem 2.1] that (2) implies (1). The other implication was claimed in [24] without proof. So assume that (1) holds and choose a constituent $\lambda \in \mathrm{IBr}(Z)$ of φ_Z. Since G is p-solvable, there exists a p-complement $H \leq G$. Since $\varphi(1)$ has p'-degree, the restriction φ_H is irreducible by [21, Theorem 10.9]. We may consider φ_H as an ordinary character. Let $Z = Z_p \times Z_{p'}$ and $\lambda = \lambda_p \times \lambda_{p'}$ where Z_p is the Sylow p-subgroup of Z and $\lambda_p \in \mathrm{Irr}(Z_p)$. Then $\varphi_H(1)^2 = |H : Z_{p'}|$ and $\lambda_{p'}^H = e\varphi_H$ where $e = \varphi(1)$ by [23, Lemma 8.2]. In particular, φ_H is the only character of H lying over $\lambda_{p'}$. Let $\varphi' \in \mathrm{IBr}(G)$ be a constituent of λ^G. Then φ'_H is a multiple of φ_H. But since H is a p-complement, φ' is uniquely determined by φ'_H. It follows that φ' is a multiple of φ, but then of course $\varphi' = \varphi$. Therefore, λ is fully ramified in G. □

It has been remarked in [24] that Proposition 6 does not hold for the non-solvable group $G = \mathrm{SL}(2, 17)$ when $p = 17$. Another counterexample with a different flavor is the direct product $G = \mathrm{SL}(2, 5)^2$ for $p = 5$. Indeed, $\mathrm{SL}(2, 5)$ has Brauer characters of degree 3 and 4. So G has a Brauer character of degree 12.

Now we prove a strong version of Theorem 2.

Theorem 7. *Let G be a solvable group and p be a prime. Then there exists a solvable H with the following properties:*

(i) $Z := \mathrm{Z}(H)$ has square-free p'-order.

(ii) There exists a faithful Brauer character $\varphi \in \mathrm{IBr}(H)$ with $\varphi(1)^2 = |H : Z|_{p'}$. In particular, H has p-central type.

(iii) G is isomorphic to a subgroup of H/Z.

(iv) $|G|$, $|H|$ and $|Z|$ have the same prime divisors apart from p.

Proof. We follow closely Gagola's construction [6, Theorem 1.2]. Since the trivial group is of p-central type, we may assume that $G \neq 1$. Let $M \trianglelefteq G$ such that $q := |G : M|$ is a prime. By induction on $|G|$, there exists a solvable group K and a faithful $\mu \in \mathrm{IBr}(K)$ with $\mu(1)^2 = |K : \mathrm{Z}(K)|_{p'}$ fulfilling the conclusion for M instead of G. Let

$$W := \{(x_1, \ldots, x_q) \in \mathrm{Z}(K)^q : x_1 \ldots x_q = 1\} \leq \mathrm{Z}(K^q)$$

and define $U := K^q/W$. Let $z := (x_1, \ldots, x_q)W \in \mathrm{Z}(U)$. For $g \in K$ and $1 \leq i \leq q$ we have

$$1 = [(1, \ldots, 1, g, 1, \ldots, 1)W, z] = (1, \ldots, 1, [g, x_i], 1, \ldots, 1)W$$

and thus $[g, x_i] = 1$. This shows that $\mathrm{Z}(U) = \mathrm{Z}(K)^q/W \cong \mathrm{Z}(K)$ and $U/\mathrm{Z}(U) \cong (K/Z(K))^q$. Since μ is faithful, we have $\mathrm{O}_p(K) = 1 = \mathrm{O}_p(U)$ by [21, Lemma 2.32].

Let $\tau := \mu \times \ldots \times \mu \in \mathrm{IBr}(K^q)$. Let $\lambda \in \mathrm{IBr}(\mathrm{Z}(K))$ be a constituent of $\mu_{\mathrm{Z}(K)}$. Then for $(x_1, \ldots, x_q) \in W$ we have $\tau(x_1, \ldots, x_q) = \tau(1)\lambda(x_1 \ldots x_q) = \tau(1)$ (note that $\mathrm{Z}(K)$ and W are p'-groups by induction). Hence, $W \le \mathrm{Ker}(\tau)$ by [21, Lemma 6.11]. Conversely, let $x := (x_1, \ldots, x_q) \in \mathrm{Ker}(\tau)$. Suppose that $x_i \notin \mathrm{Z}(K)$. Since μ is faithful, we obtain $|\mu(x_i)| < \mu(1)$ and $|\tau(x)| < \tau(1)$, a contradiction. Hence, $x \in \mathrm{Z}(K)^q$ and $\tau(1)\lambda(x_1 \ldots x_q) = \tau(x) = \tau(1)$. Since μ and λ are faithful, it follows that $x \in W$. Consequently, we may consider τ as a faithful Brauer character of U.

Case 1: $q = p$.
Let $\alpha \in \mathrm{Aut}(K^p)$ be the shift automorphism such that $\alpha(x_1, \ldots, x_p) := (x_p, x_1, \ldots, x_{p-1})$. Clearly, $\alpha(W) = W$ and we may define $H := U \rtimes \langle\alpha\rangle$. Let $z := ((x_1, \ldots, x_p)W, \alpha^i) \in \mathrm{Z}(H)$ with $0 \le i < p$. For $g \in K \setminus \mathrm{Z}(K)$ we compute

$$1 = [(g, 1, \ldots, 1)W, z] = (g, 1, \ldots, 1, x_{i+1}g^{-1}x_{i+1}^{-1}, 1, \ldots, 1)W$$

and thus $i = 0$ and $x_1, \ldots, x_p \in \mathrm{Z}(K)$. Conversely, it is easy to see that $\mathrm{Z}(U) \le \mathrm{Z}(H)$. Hence, $Z = \mathrm{Z}(H) = Z(U)$ and

$$H/Z \cong (U/\mathrm{Z}(U)) \rtimes \langle\alpha\rangle \cong (K/\mathrm{Z}(K)) \wr C_p.$$

Moreover, $|G|$, $|H|$ and $|Z|$ have the same prime divisors apart from p. By the universal embedding theorem (see [4, Theorem 2.6A]), G is isomorphic to a subgroup of

$$M \wr (G/M) \le (K/\mathrm{Z}(K)) \wr C_p \cong H/Z.$$

Observe that τ is H-invariant. Hence, by Green's theorem τ has a (unique) extension $\varphi \in \mathrm{IBr}(H)$ (see [21, Theorem 8.11]). Then

$$\varphi(1)^2 = \tau(1)^2 = \mu(1)^{2p} = |K/\mathrm{Z}(K)|_{p'}^p = |H : Z|_{p'}$$

and H is of p-central type by Proposition 6.

Case 2: $q \ne p$.
Here we need to modify the construction along the lines of [6]. Suppose first that $|\mathrm{Z}(K)|$ is not divisible by q. Let $Q := \langle x\rangle \cong C_q$ and $\rho \in \mathrm{Irr}(Q)$ be faithful. We replace (K, μ) by $(K_1, \mu_1) := (K \times Q, \mu \times \rho)$. Then $\mathrm{Z}(K_1) = \mathrm{Z}(K) \times Q$ has square-free p'-order and $\mu_1(1)^2 = |K_1 : \mathrm{Z}(K_1)|_{p'}$. Since μ is faithful, $\mathrm{O}_p(K_1) = \mathrm{O}_p(K) = 1$. Suppose that $(k, x^i) \in \mathrm{Ker}(\mu_1)$ with $k \in K$. Then $|\mu(k)| = \mu(1)$. Thus, $k \in \mathrm{Z}(K)$ and $\mu(k)$ is an integral multiple of a root of unity. On the other hand,

$$\mu(k) = \mu(1)/\rho(x^i) \in \mathbb{Q}_{|\mathrm{Z}(K)|} \cap \mathbb{Q}_q = \mathbb{Q},$$

where $\mathbb{Q}_n$ denotes the n-th cyclotomic field. This implies $\mu(k) = \pm\mu(1)$. In the case $\mu(k) = -\mu(1)$ we have $q \ne 2$ and therefore $\rho(x^i) = 1$. This contradicts $\mu_1(k, x^i) = \mu_1(1)$. Hence, $\mu(k) = \mu(1)$, $\rho(x^i) = 1$ and $(k, x^i) = 1$. Thus, μ_1 is faithful. From now on we may assume that q divides $|\mathrm{Z}(K)|$.

Let $z \in \mathrm{Z}(K)$ be an element of order q and let $Y := \langle y\rangle \cong C_q$. It is easy to check that the map

$$\alpha : U \times Y \to U \times Y, \qquad ((x_1, \ldots, x_q)W, y^i) \mapsto ((z^i x_q, x_1, x_2, \ldots, x_{q-1})W, y^i)$$

is a well-defined automorphism of order q. We define $H := (U \times Y) \rtimes \langle\alpha\rangle$. Let

$$h := ((x_1, \ldots, x_q)W, y^i, \alpha^j) \in \mathrm{Z}(H).$$

Then α^j commutes with y and therefore, $\alpha^j = 1$. Moreover,

$$1 = [\alpha, h] = (z^i x_q x_1^{-1}, x_1 x_2^{-1}, \ldots, x_{q-1} x_q^{-1})W$$

implies $z^i = 1 = y^i$. This shows that $h \in \mathrm{Z}(U)$. Conversely, it is easy to see that $\mathrm{Z}(U) \leq \mathrm{Z}(H) = Z$. Therefore, $Z = \mathrm{Z}(U) \cong \mathrm{Z}(K)$ has square-free p'-order and

$$|H : Z| = \frac{|U|q^2}{|\mathrm{Z}(K)|} = q^2 |K : \mathrm{Z}(K)|^q. \tag{2.2}$$

It follows that $|G|$, $|H|$ and $|Z|$ have the same prime divisors apart from p. A closer look shows that

$$H/Z \cong \big((K/\mathrm{Z}(K))^q \rtimes \langle\alpha\rangle\big) \times Y \cong (K/\mathrm{Z}(K) \wr C_q) \times C_q.$$

By the universal embedding theorem, G is isomorphic to a subgroup of $M \wr (G/M) \leq K/\mathrm{Z}(K) \wr C_q \leq H/Z$.

Since $\alpha(y) = ((z, 1, \ldots, 1)W, y)$ and $\mu(z) \neq \mu(1)$, the Brauer character $\tau \times 1_Y \in \mathrm{IBr}(U \times Y)$ is not invariant under α. By Clifford's theorem, $\varphi := (\tau \times 1_Y)^H \in \mathrm{IBr}(H)$ and $\mathrm{Ker}(\varphi) \leq Y \cap \alpha(Y) = 1$. Moreover,

$$\varphi(1)^2 = \tau(1)^2 q^2 = \mu(1)^{2q} q^2 = |K : \mathrm{Z}(K)|_{p'}^q q^2 \overset{(2.2)}{=} |H : Z|_{p'}. \qquad \square$$

3 Applications to blocks

For $N \trianglelefteq G$ and $\lambda \in \mathrm{IBr}(N)$ we define $\mathrm{IBr}(G|\lambda)$ as the set of irreducible constituents of λ^G as usual. Recall from [21, Corollary 8.7] that $\varphi \in \mathrm{IBr}(G|\lambda)$ if and only if λ is a constituent of φ_N.

Proof of Theorem 3. We may assume that $l \geq 2$. By Theorem 7, there exists a solvable group H of p-central type with $l_p(H) \geq l$ and $Z := \mathrm{Z}(H)$ a p'-group. Let H/Z act faithfully on an elementary abelian p-group V (for instance, the regular module) and define $G := V \rtimes H$. Then $Z = \mathrm{O}_{p'}(G)$. Let $\lambda \in \mathrm{IBr}(Z)$ be fully ramified in H. Since $\mathrm{IBr}(G) = \mathrm{IBr}(H)$, λ is also fully ramified in G. By a theorem of Fong (see [21, Theorem 10.20]), there exists a block B of G of maximal defect with $\mathrm{IBr}(B) = \mathrm{IBr}(G|\lambda)$. In particular, $l(B) = 1$. Let (V, b) be a B-subpair, i. e. b is a Brauer correspondent of B in $\mathrm{C}_G(V) = V \times Z$. Then $\mathrm{IBr}(b) = \{1_V \times \lambda\}$ and $\mathrm{N}_G(V, b) = G$. Since $G/\mathrm{C}_G(V) \cong H/Z$ is not a p-group (as $l_p(H/Z) = l_p(H) \geq l \geq 2$), B is not nilpotent. $\square$

We give a concrete example starting with $H := \texttt{SmallGroup}(54, 8) \cong 3_+^{1+2} \rtimes C_2$ of 2-central type. This group acts on $V \cong C_2^4$ and

$$G := V \rtimes H \cong \texttt{SmallGroup}(864, 3996)$$

has 2-length 2. Moreover, G has a 2-block B of maximal defect and $l(B) = 1$. We remark that B covers a non-principal block of $V \rtimes 3^{1+2}$, which was investigated in [18].

Proof of Theorem 4. Let b_0 be the principal p-block of Q. Then $b \otimes b_0$ is a G-invariant block of $H \times Q$. By [21, Corollary 9.21], $B := (b \otimes b_0)^G$ is the unique block covering $b \otimes b_0$. Since A acts regularly on $\mathrm{IBr}(b \otimes b_0)$, Clifford theory implies that $l(B) = 1$. Note that $(Q, b \otimes b_0)$ is a B-subpair. Since, $G/HQ \cong A$ is not a p-group (as $|A| = l(b)$ is not a p-power), B cannot be nilpotent. $\square$

It is not so easy to find groups H with blocks b where $\mathrm{Aut}(H)$ acts transitively on $\mathrm{IBr}(b)$. The quasisimple groups H with the desired property were investigated and partially classified in [20]. For alternating and sporadic groups, b must have defect 1. Here, $l(b)$ divides $p-1$ and one can choose $A \cong C_{l(b)}$ and $Q \cong C_p$. The smallest case, $p = 3$ and $H = \mathrm{SL}(2,5)$, leads to the group

$$G := (H \times C_3) \rtimes C_2 = H \rtimes S_3 = \texttt{SmallGroup}(720, 414)$$

with a non-nilpotent 3-block B with defect 2 and $l(B) = 1$. The same construction works with the simple group $H = \mathrm{PSL}(2,11)$. Similarly examples can be obtained by wreath products like

$$G = (H \times H) \rtimes \langle (\alpha, 1)\sigma \rangle \cong H^2 \rtimes C_4 \le \mathrm{Aut}(H^2)$$

where $\alpha \in \mathrm{Aut}(H)$ and σ interchanges the two copies of H. By Kessar [14], such blocks are Morita equivalent to blocks of solvable groups. For $p = 5$, one can take $H = \mathrm{PSL}(2,19)$ (there is certainly an infinite family). As long as we take blocks b of H with cyclic defect group and conjugate Brauer characters in $\mathrm{Aut}(H)$, the Brauer tree of b is a star (otherwise there is no graph automorphism permuting the Brauer characters). Then b is a so-called *inertial* block, i. e. b is basically Morita equivalent to its Brauer correspondent in the normalizer of a defect group (see [25]). The same must be true for the block $b \otimes Q$ of $H \times Q$. Now a theorem of Zhou [26, Corollary] implies that B is inertial. In particular, B is Morita equivalent to a block of a solvable group.

In general, the block of $H \rtimes A$ covering b is often nilpotent. In this case, a theorem of Puig [25] shows that b is inertial.

There are examples where b has non-cyclic defect groups. For $p = 2$, we start with $H := \mathrm{PSU}(3,5)$ where b has defect group C_2^2 and $l(b) = 3$ (b is Morita equivalent to the principal block of S_4). We take $A \cong C_3$ and $Q = C_2^2$. Then $G := H \rtimes A_4$ has a 2-block with defect group C_2^4. By Eaton [5] this block is again Morita equivalent to a block of a solvable group.

We now construct a block B with $\mathrm{IBr}(B) = \{\varphi\}$ such that φ has a unique lift. If a group H acts on a group V, we denote the stabilizer of $v \in V$ in H by H_v.

Theorem 8. *Let H be a p'-group of central type. Suppose that $H/\mathrm{Z}(H)$ acts faithfully on an elementary abelian p-group V such that $|H : H_v| > |H_v : \mathrm{Z}(H)|$ for all $v \in V \setminus \{1\}$. Then $G := V \rtimes H$ has a p-block B such that $\mathrm{IBr}(B) = \{\varphi\}$ and φ has a unique lift to $\mathrm{Irr}(B)$.*

Proof. Let $Z := \mathrm{Z}(H)$ and choose a fully ramified character $\lambda \in \mathrm{Irr}(Z)$. Let $\lambda^H = e\varphi$ for some $\varphi \in \mathrm{Irr}(H)$. By Fong's theorem, there exists a block B of G such that $\mathrm{Irr}(B) = \mathrm{Irr}(G|\lambda)$ and $\mathrm{IBr}(B) = \mathrm{IBr}(G|\lambda)$. We may consider the inflation of φ as an ordinary character and as a Brauer character of B. It suffices to show that every $\chi \in \mathrm{Irr}(B)\setminus\{\varphi\}$ has degree $\chi(1) > \varphi(1) = e$. There exists some non-trivial character $\mu \in \mathrm{Irr}(V)$ such that $\mu \times \lambda$ is a constituent of χ_{VZ}. Since H is a p'-group, the actions of H on V and on $\mathrm{Irr}(V)$ are permutation-isomorphic (see [23, Corollary 2.12]). Thus, by Clifford theory and by the hypothesis we have

$$\chi(1) \ge |G : G_{\mu\times\lambda}| = |H : H_\mu| > |H_\mu : Z|.$$

Multiplying the last inequality by $|H : H_\mu|$ further yields $|H : H_\mu| > \sqrt{|H : Z|} = e$. Hence, $\chi(1) > \varphi(1)$ as desired. □

The property $|H : H_v| > |H_v : \mathrm{Z}(H)|$ in Theorem 8 is fulfilled whenever $H/\mathrm{Z}(H)$ acts semiregularly on V, i. e. $H_v = 1$ for all $v \in V \setminus \{1\}$. In this case, $V \rtimes (H/\mathrm{Z}(H))$ is a Frobenius group with complement $H/\mathrm{Z}(H)$. In particular, the Sylow subgroups of $H/\mathrm{Z}(H)$ are cyclic or quaternion groups. But then

$H/\mathrm{Z}(H)$ has trivial Schur multiplier and there is no group H of central type. Hence, $|H/\mathrm{Z}(H)|$ is a product of at least four (possibly equal) primes (keeping in mind that $|H/\mathrm{Z}(H)|$ is a square). Suitable groups H can be found with GAP [7]. For instance, $H = \mathtt{SmallGroup}(128, 144)$ is of central type and acts on $V \cong C_5^4$ such that $V \rtimes (H/\mathrm{Z}(H)) \cong \mathtt{PrimitiveGroup}(625, 166)$ with the desired property.

The condition in Theorem 8 can be relaxed by taking the degrees of the characters in H_μ (where $\mu \in \mathrm{Irr}(V)$) into account. This leads to a smaller example where $H \cong \mathtt{SmallGroup}(128, 138)$, $V \cong C_3^4$ and $V \rtimes (H/\mathrm{Z}(H)) \cong \mathtt{PrimitiveGroup}(81, 33)$. An example for $p = 2$ is given by

$$H/\mathrm{Z}(H) \cong \mathtt{IrredSolMatrixGroup}(18, 2, 6, 163) \cong \mathtt{SmallGroup}(81, 9)$$

acting on $V \cong C_2^{18}$.

We use the opportunity to construct yet another unusual class of blocks. Let $Z \trianglelefteq H$ and $\lambda \in \mathrm{Irr}(Z)$. A theorem of Gallagher asserts that $|\mathrm{Irr}(H|\lambda)|$ equals the number of so-called λ-*good* conjugacy classes of H/Z (see [23, Section 5.5]). If λ is fully ramified, then 1 is the unique λ-good conjugacy class. One may ask when the opposite situation occurs, where all conjugacy classes of H/Z are good. This is holds for instance whenever λ extends to G (then $|\mathrm{Irr}(H|\lambda)| = k(H/Z)$ by Gallagher's theorem). But the converse is not true. For instance, $H := \mathtt{SmallGroup}(128, 731)$ has a subgroup $Z \le \mathrm{Z}(H) \cap H'$ of order 2 such that $k(H) = 2k(H/Z)$. The non-trivial character of Z cannot extend to H since $Z \le H'$. Now H/Z acts non-trivially on $V \cong C_3^6$. It turns out that the solvable group $G := V \rtimes H$ has two 3-blocks B_0 and B_1 with defect group V and inertial quotient H/Z such that $k(B_0) = k(B_1) = 84$ and $l(B_0) = l(B_1) = 16$ (B_0 is the principal block of G). However, B_0 and B_1 are not Morita equivalent, because they have distinct decomposition matrices (the former matrix contains a 4, but the latter does not).

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