Solution of Brauer's $k(B)$-Conjecture for π-blocks of π-separable groups

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Abstract

Answering a question of Pálfy and Pyber, we first prove the following extension of the $k(GV)$-Problem: Let G be a finite group and let A be a coprime automorphism group of G. Then the number of conjugacy classes of the semidirect product $G \rtimes A$ is at most $|G|$. As a consequence we verify Brauer's $k(B)$-Conjecture for π-blocks of π-separable groups which was proposed by Y. Liu. This generalizes the corresponding result for blocks of p-solvable groups. We also discuss equality in Brauer's Conjecture. On the other hand, we construct a counterexample to a version of Olsson's Conjecture for π-blocks which was also introduced by Liu.

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1 Introduction

One of the oldest outstanding problems in the representation theory of finite groups is *Brauer's $k(B)$-Conjecture* [1]. It asserts that the number $k(B)$ of ordinary irreducible characters in a p-block B of a finite group G is bounded by the order of a defect group of B. For p-solvable groups G, Nagao [12] has reduced Brauer's $k(B)$-Conjecture to the so-called *$k(GV)$-Problem*: If a p'-group G acts faithfully and irreducibly on a finite vector space V in characteristic p, then the number $k(GV)$ of conjugacy classes of the semidirect product $G \ltimes V$ is at most $|V|$. Eventually, the $k(GV)$-Problem has been solved in 2004 by the combined effort of several mathematicians invoking the classification of the finite simple groups. A complete proof appeared in [15].

Brauer himself already tried to replace the prime p in his theory by a set of primes π. Different approaches have been given later by Iizuka, Isaacs, Reynolds and others (see the references in [16]). Finally, Slattery developed in a series of papers [16, 17, 18] a nice theory of π-blocks in π-separable groups (precise definitions are given in the third section below). This theory was later complemented by Laradji [8, 9] and Y. Zhu [20]. The success of this approach is emphasized by the verification of *Brauer's Height Zero Conjecture* and the *Alperin–McKay Conjecture* for π-blocks of π-separable groups by Manz–Staszewski [11, Theorem 3.3] and Wolf [19, Theorem 2.2] respectively. In 2011, Y. Liu [10] put forward a variant of Brauer's $k(B)$-Conjecture for π-blocks in π-separable groups. Since $\{p\}$-separable groups are p-solvable and $\{p\}$-blocks are p-blocks, this generalizes the results mentioned in the first paragraph. Liu verified his conjecture in the special case where G has a nilpotent normal

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Hall π-subgroup. The aim of the present paper is to give a full proof of Brauer's $k(B)$-Conjecture for π-blocks in π-separable groups (see Theorem 3 below). In order to do so, we need to solve a generalization of the $k(GV)$-Problem (see Theorem 1 below). In this way we answer a question raised by Pálfy and Pyber at the end of [13] (see also [6]). The proof relies on the classification of the finite simple groups. Motivated by Robinson's theorem [14] for blocks of p-solvable groups, we also show that equality in Brauer's Conjecture can only occur for π-blocks with abelian defect groups. Finally, we construct a counterexample to a version of *Olsson's Conjecture* which was also proposed by Liu [10].

2 A generalized $k(GV)$-Problem

In the following we use the well-known formula $k(G) \leq k(N)k(G/N)$ where $N \trianglelefteq G$ (see [12, Lemma 1]).

Theorem 1. *Let G be a finite group, and let $A \leq \mathrm{Aut}(G)$ such that $(|G|, |A|) = 1$. Then $k(G \rtimes A) \leq |G|$.*

Proof. We argue by induction on $|G|$. The case $G = 1$ is trivial and we may assume that $G \neq 1$. Suppose first that G contains an A-invariant normal subgroup $N \trianglelefteq G$ such that $1 < N < G$. Let $B := \mathrm{C}_A(G/N) \trianglelefteq A$. Then B acts faithfully on N and by induction we obtain $k(NB) \leq |N|$. Similarly we have $k((G/N) \rtimes (A/B)) \leq |G/N|$. It follows that

$$k(GA) \leq k(NB)k(GA/NB) \leq |N|k((G/N)(A/B)) \leq |N||G/N| = |G|.$$

Hence, we may assume that G has no proper non-trivial A-invariant normal subgroups. In particular, G is characteristically simple, i. e. $G = S_1 \times \ldots \times S_n$ with simple groups $S := S_1 \cong \ldots \cong S_n$. If S has prime order, then G is elementary abelian and the claim follows from the solution of the $k(GV)$-Problem (see [15]). Therefore, we assume in the following that S is non-abelian.

We discuss the case $n = 1$ (that is G is simple) first. Since $(|A|, |G|) = 1$, A is isomorphic to a subgroup of $\mathrm{Out}(G)$. If G is an alternating group or a sporadic group, then $|\mathrm{Out}(G)|$ divides 4 and $A = 1$ as is well-known. In this case the claim follows since $k(GA) = k(G) \leq |G|$. Hence, we may assume that S is a group of Lie type over a field of size p^f for a prime p. According to the Atlas [2, Table 5], the order of $\mathrm{Out}(G)$ has the form dfg. Here d divides the order of the Schur multiplier of G and therefore every prime divisor of d divides $|G|$. Moreover, $g \mid 6$ and in all cases g divides $|G|$. Consequently, $|A| \leq f \leq \log_2 p^f \leq \log_2 |G|$. On the other hand, [5, Theorem 9] shows that $k(G) \leq \sqrt{|G|}$. Altogether, we obtain

$$k(GA) \leq k(G)|A| \leq \sqrt{|G|} \log_2 |G| \leq |G|$$

(note that $|G| \geq |\mathfrak{A}_5| = 60$ where $\mathfrak{A}_5$ denotes the alternating group of degree 5).

It remains to handle the case $n > 1$. Here $\mathrm{Aut}(G) \cong \mathrm{Aut}(S) \wr \mathfrak{S}_n$ where $\mathfrak{S}_n$ is the symmetric group of degree n. Let $B := \mathrm{N}_A(S_1) \cap \ldots \cap \mathrm{N}_A(S_n) \trianglelefteq A$. Then $B \leq \mathrm{Out}(S_1) \times \ldots \times \mathrm{Out}(S_n)$ and the arguments from the $n = 1$ case yield

$$k(GB) \leq k(G)|B| = k(S)^n |B| \leq \big(\sqrt{|S|} \log_2 |S|\big)^n. \tag{2.1}$$

By Feit–Thompson, $|G|$ has even order and $A/B \leq \mathfrak{S}_n$ has odd order since $(|G|, |A|) = 1$. A theorem of Dixon [3] implies that $|A/B| \leq \sqrt{3}^n$. If $|G| = 60$, then $G \cong \mathfrak{A}_5$, $B = 1$ and

$$k(GA) \leq k(\mathfrak{A}_5)^n |A| \leq (5\sqrt{3})^n \leq 60^n = |G|.$$

Therefore, we may assume that $|G| \geq |\mathrm{PSL}(3, 2)| = 168$. Then (2.1) gives

$$k(GA) \leq k(GB)|A/B| \leq (\sqrt{3|S|} \log_2 |S|)^n \leq |S|^n = |G|. \qquad \square$$

3 π-Blocks of π-separable groups

Let π be a set of primes. Recall that a finite group G is called *π-separable* if G has a normal series

$$1 = N_0 \unlhd \ldots \unlhd N_k = G$$

such that each quotient N_i/N_{i-1} is a π-group or a π'-group. The following consequence of Theorem 1 generalizes and proves the conjecture made in [6].

Corollary 2. *For every π-separable group G we have $k(G/\operatorname{O}_{\pi'}(G)) \le |G|_\pi$.*

Proof. We may assume that $\operatorname{O}_{\pi'}(G) = 1$ and $N := \operatorname{O}_\pi(G) \ne 1$. We argue by induction on $|N|$. By the Schur–Zassenhaus Theorem, N has a complement in $\operatorname{O}_{\pi\pi'}(G)$ and Theorem 1 implies $k(\operatorname{O}_{\pi\pi'}(G)) \le |N|$. Now induction yields

$$k(G) \le k(\operatorname{O}_{\pi\pi'}(G))k(G/\operatorname{O}_{\pi\pi'}(G)) \le |N||G/N|_\pi = |G|_\pi. \qquad \square$$

A *π-block* of a π-separable group G is a minimal non-empty subset $B \subseteq \operatorname{Irr}(G)$ such that B is a union of p-blocks for every $p \in \pi$ (see [16, Definition 1.12 and Theorem 2.15]). In particular, the $\{p\}$-blocks of G are the p-blocks of G. In accordance with the notation for p-blocks we set $k(B) := |B|$ for every π-block B.

A *defect group* D of a π-block B of G is defined inductively as follows. Let $\chi \in B$ and let $\lambda \in \operatorname{Irr}(\operatorname{O}_{\pi'}(G))$ be a constituent of the restriction $\chi_{\operatorname{O}_{\pi'}(G)}$ (we say that B *lies over* λ). Let G_λ be the inertial group of λ in G. If $G_\lambda = G$, then D is a Hall π-subgroup of G (such subgroups always exist in π-separable groups). Otherwise we take a π-block b of G_λ lying over λ. Then D is a defect group of b up to G-conjugation (see [17, Definition 2.2]). It was shown in [17, Theorem 2.1] that this definition agrees with the usual definition for p-blocks.

The following theorem verifies Brauer's $k(B)$-Conjecture for π-blocks of π-separable groups (see [10]).

Theorem 3. *Let B be a π-block of a π-separable group G with defect group D. Then $k(B) \le |D|$.*

Proof. We mimic Nagao's reduction [12] of Brauer's $k(B)$-Conjecture for p-solvable groups. Let $N := \operatorname{O}_{\pi'}(G)$, and let $\lambda \in \operatorname{Irr}(N)$ lying under B. By [16, Theorem 2.10] and [17, Corollary 2.8], the Fong–Reynolds Theorem holds for π-blocks. Hence, we may assume that λ is G-stable and B is the set of irreducible characters of G lying over λ (see [16, Theorem 2.8]). Then D is a Hall π-subgroup of G by the definition of defect groups. By [7, Problem 11.10] and Corollary 2, it follows that $k(B) \le k(G/N) \le |G|_\pi = |D|$. $\square$

In the situation of Theorem 1 it is known that GA contains only one π-block where π is the set of prime divisors of $|G|$ (see [16, Corollary 2.9]). Thus, in the proof of Theorem 3 one really needs to full strength of Theorem 1.

Liu [10] has also proposed the following conjecture (cf. [17, Definition 2.13]):

Conjecture 4 (Olsson's Conjecture for π-blocks)**.** *Let B be a π-block of a π-separable group G with defect group D. Let $k_0(B)$ be the number of characters $\chi \in B$ such that $\chi(1)_\pi|D| = |G|_\pi$. Then $k_0(B) \le |D : D'|$.*

This conjecture however is false. A counterexample is given by $G = \operatorname{PSL}(2, 2^5) \rtimes C_5$ where C_5 acts as a field automorphism on $\operatorname{PSL}(2, 2^5)$. Here $|G| = 2^5 \cdot 3 \cdot 5 \cdot 11 \cdot 31$ and we choose $\pi = \{2, 3, 11, 31\}$. Then $\operatorname{O}_\pi(G) = \operatorname{PSL}(2, 2^5)$ and [16, Corollary 2.9] implies that G has only one π-block B which must contain the five linear characters of G. Moreover, B has defect group $D = \operatorname{O}_\pi(G)$ by [17, Lemma 2.3]. Hence, $k_0(B) \ge 5 > 1 = |D : D'|$ since D is simple.

4 Abelian defect groups

In this section we prove that the equality $k(B) = |D|$ in Theorem 3 can only hold if D is abelian. We begin with Gallagher's observation [4] that $k(G) = k(N)k(G/N)$ for $N \trianglelefteq G$ implies $G = \mathrm{C}_G(x)N$ for all $x \in N$. Next we analyze equality in our three results above.

Lemma 5. *Let G be a finite group and $A \le \mathrm{Aut}(G)$ such that $(|G|, |A|) = 1$. If $k(G \rtimes A) = |G|$, then G is abelian.*

Proof. We assume that $k(GA) = |G|$ and argue by induction on $|G|$. Suppose first that there is an A-invariant normal subgroup $N \trianglelefteq G$ such that $1 < N < G$. As in the proof of Theorem 1 we set $B := \mathrm{C}_A(G/N)$ and obtain $k(GA) = k(NB)k(GA/NB)$. By induction, N and G/N are abelian and $GA = \mathrm{C}_{GA}(x)NB = \mathrm{C}_{GA}(x)B$ for every $x \in N$. Hence $G \le \mathrm{C}_{GA}(x)$ and $N \le \mathrm{Z}(G)$. Therefore, G is nilpotent (of class at most 2). Then every Sylow subgroup of G is A-invariant and we may assume that G is a p-group. In this case the claim follows from [14, Theorem 1'].

Hence, we may assume that G is characteristically simple. If G is non-abelian, then we easily get a contradiction by following the arguments in the proof of Theorem 1. □

Lemma 6. *Let G be a π-separable group such that $\mathrm{O}_{\pi'}(G) = 1$ and $k(G) = |G|_\pi$. Then $G = \mathrm{O}_{\pi\pi'}(G)$.*

Proof. Let $N := \mathrm{O}_{\pi\pi'}(G)$. Since $\mathrm{O}_{\pi'}(N) \le \mathrm{O}_{\pi'}(G) = 1$, we have $k(N) \le |N|_\pi$ by Corollary 2. Moreover, $\mathrm{O}_{\pi'}(G/N) = 1$, $k(G/N) \le |G/N|_\pi$ and $k(G) = k(N)k(G/N)$. In particular, $G = \mathrm{C}_G(x)N$ for every $x \in N$. Let $g \in G$ be a π-element. Then g is a class-preserving automorphism of N and also of $N/\mathrm{O}_\pi(G)$. Since $N/\mathrm{O}_\pi(G) = \mathrm{O}_{\pi'}(G/\mathrm{O}_\pi(G))$ is a π'-group, it follows that g acts trivially on $N/\mathrm{O}_\pi(G)$. By the Hall–Higman Lemma 1.2.3, $N/\mathrm{O}_\pi(G)$ is self-centralizing and therefore $g \in N$. Thus, G/N is a π'-group and $N = G$. □

Theorem 7. *Let B be a π-block of a π-separable group with non-abelian defect group D. Then $k(B) < |D|$.*

Proof. We assume that $k(B) = |D|$. Following the proof of Theorem 3, we end up with a π-separable group G such that $D \le G$, $\mathrm{O}_{\pi'}(G) = 1$ and $k(G) = |G|_\pi = |D|$. By Lemma 6, $D \trianglelefteq G$ and by Lemma 5, D is abelian. □

Similar arguments imply the following π-version of [14, Theorem 3] which also extends Corollary 2.

Theorem 8. *Let G be a π-separable group such that $\mathrm{O}_{\pi'}(G) = 1$ and $H \le G$. Then $k(H) \le |G|_\pi$ and equality can only hold if $|H|_\pi = |G|_\pi$.*

The proof is left to the reader.

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