

Refinements of the orthogonality relations for blocks

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October 3, 2019

Abstract

For a block B of a finite group G there are well-known orthogonality relations for the generalized decomposition numbers. We refine these relations by expressing the generalized decomposition numbers with respect to an integral basis of a certain cyclotomic field. After that, we use the refinements in order to give upper bounds for the number of irreducible characters (of height 0) in B . In this way we generalize results from [Héthelyi-Külshammer-Sambale, 2014]. These ideas are applied to blocks with abelian defect groups of rank 2. Finally, we address a recent conjecture by Navarro.

Keywords: orthogonality relations, $k(B)$, abelian defect groups, Navarro Conjecture

AMS classification: 20C15, 20C20

1 Introduction

Let B be a p -block of a finite group G with defect group D . Let $k(B) := |\text{Irr}(B)|$ and $l(B) := |\text{IBr}(B)|$. Recall that the *height* $i \geq 0$ of $\chi \in \text{Irr}(B)$ is defined by $\chi(1)_p = p^i |G : D|_p$. We denote the number of irreducible characters of height i in B by $k_i(B)$. For $u \in D$ let b_u be a Brauer correspondent of B in $C_G(u)$, i. e. (u, b_u) is a B -subsection. Let $Q = (d_{\chi\varphi}^u : \chi \in \text{Irr}(B), \varphi \in \text{IBr}(b_u))$ be the matrix of generalized decomposition numbers with respect to (u, b_u) . By a result of Broué [3, Corollary 2], it is known that the rows of Q corresponding to irreducible characters of height 0 do not vanish (see also [26, Proposition 1.36]). Moreover, by the orthogonality relations, $C_u := Q^T \overline{Q}$ is the Cartan matrix of b_u (Q^T is the transpose and \overline{Q} is the complex conjugate of Q). Since Q consists of algebraic integers in a cyclotomic field, it is possible to bound $k_0(B)$ in terms of C_u . If $u \in Z(D)$, then the subsection is *major* which means that B and b_u have the same defect. Then, by a result of Brauer [2, (5H)], every row of Q does not vanish and we get a bound on $k(B)$ in terms of C_u . In many cases these bounds do not rely on the *basic set* chosen for b_u . Recall that a basic set for b_u is a basis of the \mathbb{Z} -module of generalized Brauer characters spanned by $\text{IBr}(b_u)$. If one changes the basic set for b_u , then the matrix C_u is transformed into $SC_u S^T$ where $S \in \text{GL}(l(b_u), \mathbb{Z})$.

We will recall and rephrase two results in that direction which were first published in [12].

Proposition 1.1. *Let (u, b_u) be a B -subsection such that b_u dominates a block $\overline{b_u}$ of $C_G(u)/\langle u \rangle$ with Cartan matrix $C_u = (c_{ij})$ up to basic sets. Then for every positive definite, integral quadratic form $q(x_1, \dots, x_{l(b_u)}) = \sum_{1 \leq i \leq j \leq l(b_u)} q_{ij} x_i x_j$ ($q_{ij} \in \mathbb{Z}$) we have*

$$k_0(B) \leq |\langle u \rangle| \sum_{1 \leq i \leq j \leq l(b_u)} q_{ij} c_{ij}.$$

If (u, b_u) is major, then we can replace $k_0(B)$ by $k(B)$ in this formula.

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Proof. It is well-known that the Cartan matrix of b_u is given by $|\langle u \rangle| \overline{C_u}$ (see for example [18, Theorem 9.10]). Hence, the result follows directly from [26, Theorem 4.2]. \square

Recall that the Brauer characters of b_u are just inflations of the Brauer characters of $\overline{b_u}$. Therefore, we will often identify $\text{IBr}(\overline{b_u})$ by $\text{IBr}(b_u)$.

If $l(b_u) = 1$, then the inequality in Proposition 1.1 becomes $k_0(B) \leq |R|$ where R is a defect group of b_u (after conjugation one may assume that $R = C_D(u)$). In this case it is possible to improve the factor $|\langle u \rangle|$ in Proposition 1.1. In order to do so, one expresses the generalized decomposition numbers $d_{\chi\varphi}^u$ with respect to an integral basis of the cyclotomic field $\mathbb{Q}(e^{2\pi i/|\langle u \rangle|})$. This can be thought of as a discrete Fourier transformation.

Proposition 1.2. *Let (u, b_u) be a B -subsection such that $l(b_u) = 1$ and $\mathcal{N} := N_G(\langle u \rangle, b_u)/C_G(u)$. Suppose that b_u dominates a block $\overline{b_u}$ of $C_G(u)/\langle u \rangle$ with defect d . Then*

$$k_0(B) \leq k_0(\langle u \rangle \rtimes \mathcal{N})p^d.$$

If (u, b_u) is major, then $|\mathcal{N}| \mid p-1$ and

$$k(B) \leq \sum_{i=0}^{\infty} p^{2i} k_i(B) \leq k(\langle u \rangle \rtimes \mathcal{N})|D/\langle u \rangle|.$$

The proof of this result will be given in the next section. The reason that the group $\langle u \rangle \rtimes \mathcal{N}$ appears in Proposition 1.2 is related to the fact that the principal block of this group has a subsection where the generalized decomposition numbers satisfy exactly the same orthogonality relations as in the group G (see Lemma 2.5 below).

The two propositions above are duals of one another in the sense that the first uses $C_G(u)/\langle u \rangle$ while the second uses $\langle u \rangle \rtimes N_G(\langle u \rangle, b_u)/C_G(u)$. The aim of the present paper is to unify both results by making use of the entire group $N_G(\langle u \rangle, b_u)$ (and dropping the assumption $l(b_u) = 1$ in Proposition 1.2). In order to achieve this, we develop two new ideas. The first is to allow also non-integral quadratic forms in Proposition 1.1 (see Lemma 2.3) and secondly we choose a suitable integral basis for certain intermediate fields of cyclotomic fields (see Lemma 2.4). This will simplify the complicated calculations in [26, Section 5.2]. After this has been done, we will apply our main theorem to blocks with abelian defect groups of rank 2 (see Theorem 3.1). In Section 4 we give another application by addressing a recent conjecture by Navarro (see Theorem 4.1). Finally, in the last section some concluding remarks and open questions are presented.

Our notation is mostly standard and can be found in [18] and [26]. In particular, the *inertial quotient* of B is defined by $I(B) := N_G(D, b_D)/C_G(D)D$ where b_D is a Brauer correspondent of B in $C_G(D)$. In the situation above, we will often regard $\mathcal{N} := N_G(\langle u \rangle, b_u)/C_G(u)$ as a subgroup of $\text{Aut}(\langle u \rangle)$. We will even identify \mathcal{N} with a subgroup of $(\mathbb{Z}/p^n\mathbb{Z})^\times$ where p^n is the order of u . Since the Galois group \mathcal{G} of the cyclotomic field of degree p^n is naturally isomorphic to $(\mathbb{Z}/p^n\mathbb{Z})^\times$ as well, we will sometimes consider \mathcal{N} as a subgroup of \mathcal{G} . Finally, we identify the elements of $(\mathbb{Z}/p^n\mathbb{Z})^\times$ with integers where we silently compute modulo p^n .

In accordance with the definition of $k(B)$ we also use $k(G) := |\text{Irr}(G)|$ and $k_i(G)$ in the obvious meaning. We write actions from the left; for instance $({}^g\chi)(h) := \chi(g^{-1}hg)$ where $g, h \in G$ and χ is a character of a subgroup of G . For a finite group H we often use the abbreviation $H^n := H \times \dots \times H$ (n copies). A cyclic group of order n is denoted by Z_n . Moreover, \mathfrak{S}_n is the symmetric group and \mathfrak{A}_n is the alternating group of degree n .

2 Refinements of the orthogonality relations

Proof of Proposition 1.2. We first deal with the case $p = 2$ which is entirely different. Let \mathcal{F} be the fusion system of B . After conjugation, we may assume that $\langle u \rangle$ is a fully \mathcal{F} -normalized subgroup of D . Then $C_D(u)$ is a defect group of b_u and $\mathcal{N} \cong N_D(\langle u \rangle)/C_D(u)$ (see [26, Section 5.1]). Suppose that there exists $\gamma \in \mathcal{N}$ such that $\gamma \equiv -5^n \pmod{|\langle u \rangle|}$ for some $n \in \mathbb{Z}$. Then [26, Theorem 5.2] implies $k_0(B) \leq 2|N_D(\langle u \rangle)/\langle u \rangle| = 2|\mathcal{N}|p^d$. Since

$1 + 5^n \equiv 2 \pmod{4}$, the commutator subgroup of $\langle u \rangle \rtimes \mathcal{N}$ is generated by $\langle u^2 \rangle$. Hence, $k_0(\langle u \rangle \rtimes \mathcal{N}) = 2|\mathcal{N}|$ and the claim follows. Next we suppose that there is no element γ as above. Since

$$(\mathbb{Z}/|\langle u \rangle|\mathbb{Z})^\times \cong \langle -1 + |\langle u \rangle|\mathbb{Z} \rangle \times \langle 5 + |\langle u \rangle|\mathbb{Z} \rangle,$$

we see that \mathcal{N} is generated by some $\gamma \equiv 5^{2^n} \pmod{|\langle u \rangle|}$ where $2^{n+2}|\mathcal{N}| = |\langle u \rangle|$. The commutator subgroup of $\langle u \rangle \rtimes \mathcal{N}$ is generated by $u^{5^{2^n}-1}$. Since $5^{2^n} - 1$ is divisible by 2^{n+2} but not by 2^{n+3} , we derive $|\langle u^{5^{2^n}-1} \rangle| = |\mathcal{N}|$ and $k_0(\langle u \rangle \rtimes \mathcal{N}) = |\langle u \rangle|$. Now the claim already follows from Proposition 1.1.

For the remainder of the proof we assume that $p > 2$. With the notation of [26, Theorem 5.12] we have $|\mathcal{N}| = p^s r$ where $r \mid p-1$ and $s \geq 0$. Thus, for the first claim it suffices to show that

$$k_0(\langle u \rangle \rtimes \mathcal{N}) = \frac{|\langle u \rangle| + p^s(r^2 - 1)}{r}.$$

The inflations from \mathcal{N} yield $p^s r$ linear characters in $\langle u \rangle \rtimes \mathcal{N}$, since \mathcal{N} is cyclic (as a subgroup of $\text{Aut}(\langle u \rangle)$). Now let $1 \neq \lambda \in \text{Irr}(\langle u \rangle)$. If the length of the orbit of λ under \mathcal{N} is divisible by p , then the irreducible characters of $\langle u \rangle \rtimes \mathcal{N}$ lying over λ all have positive height. Hence, we may assume that $\lambda^{|\langle u^{p^s} \rangle|} = 1$. Then, by Clifford theory, λ extends in p^s many ways to $\langle u \rangle \rtimes \mathcal{N}_p$ where \mathcal{N}_p is the Sylow p -subgroup of \mathcal{N} . All these extensions induce to irreducible characters of $\langle u \rangle \rtimes \mathcal{N}$ of height 0. We have $(|\langle u^{p^s} \rangle| - 1)/r$ choices for λ . Thus, in total we obtain

$$k_0(\langle u \rangle \rtimes \mathcal{N}) = p^s r + p^s \frac{|\langle u^{p^s} \rangle| - 1}{r} = \frac{|\langle u \rangle| + p^s(r^2 - 1)}{r}.$$

Now suppose that (u, b_u) is major. Then $p^d = |D/\langle u \rangle|$, and it suffices to show that $s = 0$. This follows for example from [1, Proposition I.2.5]. \square

Suppose that we have dropped the assumption $l(b_u) = 1$ in Proposition 1.2. Then the main obstacle in the proof is that $\mathcal{N} := \text{N}_G(\langle u \rangle, b_u)/C_G(u)$ can act non-trivially on $\text{IBr}(b_u)$. This means that \mathcal{N} also acts on the columns of the generalized decomposition matrix corresponding to (u, b_u) . Consequently, we have to be careful when changing basic sets, since this might disturb the action of \mathcal{N} . For these reasons the most general result is quite unhandy:

Theorem 2.1. *Let B be a p -block of a finite group G . Let (u, b_u) be a B -subsection such that b_u dominates a block \bar{b}_u of $C_G(u)/\langle u \rangle$. Let $l := l(b_u)$, and let $\bar{C}_u = (c_{\sigma\tau})$ be the Cartan matrix of \bar{b}_u ($\sigma, \tau \in \text{IBr}(b_u)$). Let $|\langle u \rangle| = p^n$, $m := p^{n-1}(p-1)$ and $\mathcal{N} := \text{N}_G(\langle u \rangle, b_u)/C_G(u)$. For $i \in \mathbb{Z}$ let $i' \in \{1, \dots, p^{n-1}\}$ such that $-i \equiv i' \pmod{p^{n-1}}$. For $i, j \in \mathbb{Z}$, $\gamma \in \mathcal{N}$ and $\tau \in \text{IBr}(b_u)$ let*

$$\begin{aligned} w_{ij}^\tau(\gamma) := & |\{\delta \in \mathcal{N}_\tau : p^n \mid i - j\gamma\delta\}| - |\{\delta \in \mathcal{N}_\tau : p^n \mid i + j'\gamma\delta\}| \\ & + |\{\delta \in \mathcal{N}_\tau : p^n \mid i' - j'\gamma\delta\}| - |\{\delta \in \mathcal{N}_\tau : p^n \mid i' + j\gamma\delta\}| \end{aligned}$$

where \mathcal{N}_τ is the stabilizer of τ in \mathcal{N} . For $\sigma, \tau \in \text{IBr}(b_u)$ we define $A_{\sigma\tau} = (a_{ij})_{i,j=1}^m$ by

$$a_{ij} := \sum_{\gamma \in \mathcal{N}/\mathcal{N}_\tau} c_{\sigma, \gamma\tau} w_{i-1, j-1}^\tau(\gamma).$$

Let $M \in \mathbb{Z}^{ml \times ml}$ be the block matrix with blocks $\{A_{\sigma\tau} : \sigma, \tau \in \text{IBr}(b_u)\}$. Let $k \in \mathbb{N}$ be maximal with the property that there exists a matrix $Q \in \mathbb{Z}^{k \times ml}$ without vanishing rows such that $Q^T Q = M$. Then $k_0(B) \leq k$ and if (u, b_u) is major, we also have $k(B) \leq k$.

Proof. For a fixed $\sigma \in \text{IBr}(b_u)$ we set $d_\sigma^u := (d_{\chi\sigma}^u : \chi \in \text{Irr}(B))$. Let $\zeta := e^{2\pi i/p^n}$. Then $d_{\chi\sigma}^u \in \mathbb{Z}[\zeta]$ and $v := (\zeta^i : i = 0, \dots, m-1)$ is a \mathbb{Z} -basis of $\mathbb{Z}[\zeta]$ (see [20, Proposition I.10.2]). Hence there exists a matrix $A_\sigma \in \mathbb{Z}^{k(B) \times m}$ such that $d_\sigma^u = A_\sigma v$. Let $A := (A_\sigma : \sigma \in \text{IBr}(b_u)) \in \mathbb{Z}^{k(B) \times ml}$. By [26, Proposition 1.36], the rows of A corresponding to irreducible characters of height 0 do not vanish. If (u, b_u) is major, then all the rows of A do not vanish. Consequently, it suffices to show that $A^T A = M$. This matrix is built from blocks of the form $A_\sigma^T A_\tau$. Thus we need to show that $A_\sigma^T A_\tau = A_{\sigma\tau}$.

The orthogonality relation implies

$$v^T A_\sigma^T A_\tau \bar{v} = (d_\sigma^u)^T \bar{d}_\tau^u = p^n c_{\varphi\tau},$$

since $p^n \bar{C}_u$ is the Cartan matrix of b_u . This can be rewritten in the form

$$(v \otimes \bar{v})X = p^n c_{\sigma\tau}$$

where \otimes denotes the Kronecker product and X is the vectorization $A_\sigma^T A_\tau$. If $\gamma, \delta \in (\mathbb{Z}/p^n\mathbb{Z})^\times$ such that $\gamma^{-1}\delta \notin \mathcal{N}$, then $(d_\sigma^u)^\tau d_\tau^{\delta u} = 0$. This gives another equation

$$(\gamma(v) \otimes \delta(\bar{v}))X = 0.$$

Note that we consider γ and δ here as Galois automorphisms of $\mathbb{Q}(\zeta)$. Now let $\gamma^{-1}\delta \in \mathcal{N}$. Then $d_\tau^{\delta u} = d_{\tau'}^u$ with $\tau' := \gamma^{\delta^{-1}}\tau \in \text{IBr}(b_u)$. In this case we obtain

$$(\gamma(v) \otimes \delta(\bar{v}))X = p^n c_{\sigma\tau'}.$$

Let V be the matrix with rows $(\gamma(v) \otimes \delta(\bar{v}) : \gamma, \delta \in (\mathbb{Z}/p^n\mathbb{Z})^\times)$. Then we have a linear system $VX = b$ where b contains zeros and some $p^n c_{\sigma\tau'}$ as above. Observe that $W = (\gamma(v) : \gamma \in (\mathbb{Z}/p^n\mathbb{Z})^\times)$ is a Vandermonde matrix and $V = W \otimes \bar{W}$. By [26, Lemma 5.5], we have $W^{-1} = p^{-n}(\gamma(t_{i-1}))$ with $t_i := \zeta^{-i} - \zeta^{i'}$ where γ now runs through the columns of W^{-1} . Since $X = V^{-1}b = (W^{-1} \otimes \bar{W}^{-1})b$, the entry of $A_\sigma^T A_\tau$ at position $(i+1, j+1)$ is

$$\begin{aligned} p^{-n} \sum_{\substack{\gamma, \delta \in (\mathbb{Z}/p^n\mathbb{Z})^\times, \\ \gamma^{-1}\delta \in \mathcal{N}}} \gamma(t_j) \delta(\bar{t}_i) c_{\sigma, \gamma\delta^{-1}\tau} &= p^{-n} \sum_{\gamma \in \mathcal{N}} \sum_{\mu \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \mu(\bar{t}_i \gamma(t_j)) c_{\sigma, \gamma\tau} \\ &= p^{-n} \sum_{\gamma \in \mathcal{N}/\mathcal{N}_\tau} c_{\sigma, \gamma\tau} \sum_{\delta \in \mathcal{N}_\tau} \sum_{\mu \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \mu(t_i(\gamma\delta)(\bar{t}_j)). \end{aligned} \quad (2.1)$$

Now,

$$t_i(\gamma\delta)(\bar{t}_j) = (\zeta^{-i} - \zeta^{i'}) (\zeta^{\gamma\delta j} - \zeta^{-\gamma\delta j'}) = \zeta^{-i+\gamma\delta j} - \zeta^{-i-\gamma\delta j'} + \zeta^{i'-\gamma\delta j'} - \zeta^{i'+\gamma\delta j}.$$

Since the p^n -th cyclotomic polynomial is given by $\Phi_{p^n}(x) = x^{(p-1)p^{n-1}} + x^{(p-2)p^{n-1}} + \dots + x^{p^{n-1}} + 1$, it follows that

$$\sum_{\mu \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \mu(\zeta^{-i+\gamma\delta j}) = \begin{cases} m & \text{if } p^n \mid i - \gamma\delta j, \\ 0 & \text{if } p^{n-1} \nmid i - \gamma\delta j, \\ -p^{n-1} & \text{otherwise.} \end{cases}$$

An application of this leads us to

$$\sum_{\delta \in \mathcal{N}_\tau} \sum_{\mu \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \mu(\zeta^{-i+\gamma\delta j}) = p^n |\{\delta \in \mathcal{N}_\tau : p^n \mid i - \gamma\delta j\}| - p^{n-1} |\{\delta \in \mathcal{N}_\tau : p^{n-1} \mid i - \gamma\delta j\}|.$$

We get similar expressions for the other terms $i + \gamma\delta j'$, $i' - \gamma\delta j'$ and $i' + \gamma\delta j$ in (2.1). Since $i + i' \equiv 0 \equiv j + j' \pmod{p^{n-1}}$, we have $i - \gamma\delta j \equiv i + \gamma\delta j' \equiv -i' + \gamma\delta j' \equiv -i' - \gamma\delta j \pmod{p^{n-1}}$. Hence, the terms of the form $p^{n-1} |\{\dots\}|$ in (2.1) cancel out each other. It turns out that (2.1) coincides with a_{ij} . This proves the claim. \square

The important point of Theorem 2.1 is the fact that the matrix M is uniquely determined by \bar{C}_u , \mathcal{N} and the action on $\text{IBr}(b_u)$. Therefore, the result can indeed be seen as a refinement of the usual orthogonality relations for blocks. The numbers $w_{ij}^\tau(\gamma)$ can be evaluated individually (see [26, Lemma 5.9]), but the entire matrix M seems to have an extremely opaque shape. Nevertheless, from an algorithmic point of view, it is trivial to construct M . However, the computation of k is a hard problem. There is a sophisticated algorithm by Plesken [21] which can be used if ml is small. In general one can use estimates of k as in Proposition 1.1. Note also that M usually does not have full rank. On the one hand, this is because the generalized decomposition numbers usually lie in proper subfields of $\mathbb{Q}(\zeta)$. On the other hand, $A_{\sigma\tau} = A_{\gamma\sigma, \gamma\tau}$ for $\sigma, \tau \in \text{IBr}(b_u)$ and $\gamma \in \mathcal{N}$. Hence, before one attempts to compute k , one should replace M by a matrix of smaller size. In particular, one can apply the LLL reduction algorithm for lattices. We give a complete example.

Example 2.2. Let B be the principal 3-block of $G = {}^2F_4(2)'$. Then B has extraspecial defect group D of order p^3 and exponent p . It can be seen from the Atlas [5] that G has only one conjugacy class of elements of order 3. In particular, there is only one non-trivial B -subsection (u, b_u) which must necessarily be major (in fact, the fusion system of B has been determined in [25]). The group $C_G(u)/\langle u \rangle$ is isomorphic to $Z_3^2 \rtimes Z_4$ and the Cartan matrix \overline{C}_u of \overline{b}_u is given by $\overline{C}_u = (2 + \delta_{ij})_{i,j=1}^4$ where δ_{ij} is the Kronecker delta (in particular, $l(b_u) = 4$). Now b_u is covered only by the principal block of $N := N_G(\langle u \rangle)$, since this subgroup has only one block. Moreover, $|N/C_G(u)| = 2$ and $l(N) = 5$. Clifford theory reveals that b_u has two N -invariant irreducible Brauer characters and the other two characters are conjugate under N . By the shape of \overline{C}_u , we may assume that the last two Brauer characters are stable. Then the matrix M from Theorem 2.1 is given as

$$\begin{pmatrix} 8 & 1 & 7 & -1 & 6 & \cdot & 6 & \cdot \\ 1 & 2 & -1 & -2 & \cdot & \cdot & \cdot & \cdot \\ 7 & -1 & 8 & 1 & 6 & \cdot & 6 & \cdot \\ -1 & -2 & 1 & 2 & \cdot & \cdot & \cdot & \cdot \\ 6 & \cdot & 6 & \cdot & 9 & \cdot & 6 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 6 & \cdot & 6 & \cdot & 6 & \cdot & 9 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

and this reduces to

$$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 5 & 2 & 2 \\ 1 & 2 & 5 & 2 \\ 1 & 2 & 2 & 8 \end{pmatrix}.$$

Now Plesken's algorithm gives $k = 15$. Therefore $k(B) \leq 15$. In comparison, Proposition 1.1 implies only $k(B) \leq 18$ and the actual value is $k(B) = 13$ (Proposition 1.2 is not applicable). The gap between 13 and 15 can be explained as follows: B contains four irreducible characters of height 1. By Brauer's theory, the so-called *contributions* of these characters have positive p -adic valuation (see [26, Proposition 1.36]). This implies that the generalized decomposition numbers corresponding to these characters cannot be too "small". However, note that the expression $\sum_{i=0}^{\infty} p^{2i} k_i(B) \leq 15$ from Proposition 1.2 would not be true here. But we can still say something in this direction: Plesken's algorithm also shows that there is essentially only one solution $Q^T Q = M$ with $Q \in \mathbb{Z}^{15 \times 4}$. For this solution matrix Q we can revert all the calculations to obtain the generalized decomposition numbers. After that we can compute the contributions. This shows that $k(B) = 15$ can only happen if $k(B) = k_0(B)$. Similarly, $k(B) = 14$ would imply $k_0(B) = 12$.

Our next goal is to simplify the situation if the Brauer characters are stable. This will generalize Proposition 1.2. Therefore, we need to revisit and simplify the calculations in [26, Section 5.2].

Lemma 2.3. *Let q be a (not necessarily integral) quadratic form such that $q(x) \geq 1$ for all non-zero integral vectors x . Let $q_A(x) = \sum x_i^2 - \sum x_i x_{i+1}$ be the quadratic form corresponding to a Dynkin diagram of type A . Then we have $(q \otimes q_A)(x) \geq 1$ for every integral $x \neq 0$.*

Proof. Since q_A is positive definite, there is no doubt that $q \otimes q_A$ is positive definite. However, we may have $0 < (q \otimes q_A)(x) < 1$. It is known that $q \otimes q_A$ and $q_A \otimes q$ are isomorphic (the Gram matrices are permutation similar). Thus it suffices to show $(q_A \otimes q)(x) \geq 1$ for all integral $x \neq 0$. If q has rank n and q_A has rank m , then we can write $x = (x_1, \dots, x_m)$ with $x_i \in \mathbb{Z}^n$. It follows that

$$(q_A \otimes q)(x) = \frac{1}{2} \sum_{i=1}^{m-1} q(x_i - x_{i+1}) + \frac{1}{2} (q(x_1) + q(x_m)).$$

The claim follows easily from the hypothesis on q . □

Lemma 2.3 is relevant, because in general a tensor product of two integral quadratic forms is not integral anymore. For, if $q = \sum_{i \leq j} q_{ij} x_i x_j$ is integral, the coefficients q_{ij} with $i \neq j$ lie in $\frac{1}{2}\mathbb{Z}$. Hence, the coefficients of $q_1 \otimes q_2$ only lie in $\frac{1}{4}\mathbb{Z}$. Moreover, the minimum of a tensor product of quadratic forms is not necessarily the product of the minima (but "most" of the time it is, see [13, Section 7.1]).

Lemma 2.4. Let $\zeta \in \mathbb{C}$ be a primitive p^n -th root of unity for an odd prime power p^n , and let \mathcal{G} be the Galois group of $\mathbb{Q}(\zeta)$ over \mathbb{Q} . Let $\mathcal{N} \leq \mathcal{G}$ such that $|\mathcal{N}| = p^s r$ with $r \mid p-1$ and $s \geq 0$. For $i \geq 1$ let S_i be a set of representatives for the $\text{O}_{p^i}(\mathcal{N})$ -orbits of $\{1 \leq j \leq p^i : (j, p) = 1\}$. Let

$$\begin{aligned} T_1 &:= S_1, \\ T_i &:= pT_{i-1} \cup \{s + jp^{i-1} : s \in S_{i-1}, j = 0, \dots, p-2\} \quad \text{for } i \geq 2. \end{aligned}$$

Then

$$\tilde{T}_n := \left\{ \sum_{\gamma \in \text{O}_{p^i}(\mathcal{N})} \gamma(\zeta^t) : t \in p^s T_{n-s} \right\}$$

is an integral basis for the fixed field of \mathcal{N} .

Proof. First we reduce to the case $s = 0$. Thus, assume $s > 0$. Since $p > 2$, \mathcal{G} is cyclic and \mathcal{N} is uniquely determined by r and s . Let F be the fixed field of \mathcal{N} . By Galois theory, it follows that $[\mathbb{Q}(\zeta) : F] = |\mathcal{N}|$ and $F \subseteq \mathbb{Q}(\zeta^{p^s})$. Since $\text{O}_p(\mathcal{N})$ acts trivially on $\mathbb{Q}(\zeta^{p^s})$, F is just the fixed field of $\text{O}_{p^i}(\mathcal{N})$ in $\mathbb{Q}(\zeta^{p^s})$. We may therefore assume that $s = 0$ in the following.

The group \mathcal{N} acts on $\{1 \leq j \leq p^i : (j, p) = 1\}$ via $\gamma j := \gamma j \pmod{p^i}$. Hence, the sets S_i and T_i are well-defined. Since \mathcal{N} is a p' -group, the canonical map $\mathcal{N} \rightarrow (\mathbb{Z}/p^i\mathbb{Z})^\times$ is injective for $i \geq 1$. Therefore, \mathcal{N} acts always semiregularly. Let $m := p^{n-1}(p-1)$. First we show that $V := \{\gamma(\zeta^t) : \gamma \in \mathcal{N}, t \in T_n\}$ is an integral basis of $\mathbb{Q}(\zeta)$. We argue by induction on n . It is well-known that $\zeta, \zeta^2, \dots, \zeta^m$ is an integral basis of $\mathbb{Q}(\zeta)$. Hence, there is nothing to do for $n = 1$. Now let $n \geq 2$. Then by induction we have $|V| = p^{n-2}(p-1) + p^{n-2}(p-1)^2 = m$. Thus, it suffices to show that each ζ^i can be written as an integral linear combination of V . If $p \mid i$, then induction shows that ζ^i can be written in terms of $\gamma(\zeta^t)$ with $t \in pT_{n-1}$. Thus, we may assume that $p \nmid i$. Choose j such that $1 \leq i - jp^{n-1} \leq p^{n-1}$. By the definition of S_{n-1} , there exist $\gamma \in \mathcal{N}$ and $k \in \mathbb{Z}$ such that $\gamma i - kp^{n-1} \in S_{n-1}$. If $0 \leq k \leq p-2$, then $\gamma i \in T_n$ and the claim follows. Now let $k = p-1$. Then $\gamma i + lp^{n-1} \in T_n$ for all $l \in \{1, \dots, p-1\}$. Since $1 = -\sum_{l=1}^{p-1} \zeta^{lp^{n-1}}$, we have $\zeta^{\gamma i} = -\sum_{l=1}^{p-1} \zeta^{\gamma i + lp^{n-1}}$. Therefore, it follows that V is in fact an integral basis for $\mathbb{Q}(\zeta)$.

The elements $\text{tr}(\zeta^t) := \sum_{\gamma \in \mathcal{N}} \gamma(\zeta^t)$ with $t \in T_n$ certainly lie in the ring of integers of the fixed field F of \mathcal{N} . Conversely, let $x \in F$ be an algebraic integer. Then by the arguments above, we can write $x = \sum_{v \in V} \alpha_v v$ with $\alpha_v \in \mathbb{Z}$. It is now obvious that $x = \sum_{t \in T_n} \alpha_{\zeta^t} \text{tr}(\zeta^t)$. Consequently, \tilde{T}_n is a generating set for the ring of integers of F . By Galois theory, $|F : \mathbb{Q}| = p^{n-1}(p-1)/|\mathcal{N}| = |T_n|$. Hence, the elements really form an integral basis and we are done. \square

Lemma 2.5. Let p^n be an odd prime power, and let $\gamma \in (\mathbb{Z}/p^n\mathbb{Z})^\times$ of order $p^s r$ with $r \mid p-1$ and $s \geq 0$. Let B be the principal p -block of

$$G := \langle u, x \mid u^{p^n} = x^{p^s r} = 1, xux^{-1} = u^\gamma \rangle,$$

and let b_u be the principal block of $C_G(u) = \langle u \rangle$. Then $\text{IBr}(b_u) = \{\varphi\}$ and the generalized decomposition numbers $(d_{\chi\varphi}^u : \chi \in \text{Irr}(B))$ can be written in the form $A\tilde{T}_n$ where A is an integral matrix and \tilde{T}_n is the basis from Lemma 2.4. Let $M = (m_{ij}) := A^T A \in \mathbb{Z}^{t \times t}$ with $t = p^{n-s-1}(p-1)/r$. Then for the quadratic form $q(x) = \sum_{1 \leq i \leq j \leq t} q_{ij} x_i x_j$ corresponding to the Dynkin diagram of type A_t we have

$$\sum_{1 \leq i \leq j \leq t} q_{ij} m_{ij} = k_0(B).$$

Proof. Observe that B is the only block of G . It is obvious that $C_G(u) = \langle u \rangle$ and $l(b_u) = 1$. Note that φ is the trivial Brauer character of $\langle u \rangle$. The irreducible characters of B can be obtained as in the proof of Proposition 1.2. If $\chi \in \text{Irr}(G)$ is an inflation from $\langle x \rangle$, then $d_{\chi\varphi}^u = 1$. Now assume that χ lies over a non-trivial character $\lambda \in \text{Irr}(\langle u \rangle)$. Then we have $\chi(u) = \lambda_1(u) + \dots + \lambda_t(u)$ where $\{\lambda_1, \dots, \lambda_t\}$ is the orbit of λ under $\langle x \rangle$. If $p \mid t$, then χ has positive height and $d_{\chi\varphi}^u = 0$ by Lemma 2.4. Otherwise, $d_{\chi\varphi}^u$ has the form

$$\text{tr}(\zeta_i) := \sum_{j=1}^r \zeta_i^{\gamma^j}$$

where ζ_1, \dots, ζ_m is a set of representatives of the non-trivial p^{n-s} -th roots of unity under the action of γ ($m = (p^{n-s} - 1)/r$). Since every λ has p^s extensions to $\langle u, x^r \rangle$, every ζ_i appears p^s times in the generalized decomposition matrix.

Now we need to express the numbers $d_{\chi\varphi}^u$ in terms of \widetilde{T}_n from Lemma 2.4. With the notation from Lemma 2.4 we have $1 = -\sum_{i=1}^{p-1} \zeta^{p^{n-1}i}$. Hence the first rows of A are given by $(-1, \dots, -1, 0, \dots, 0)$ with $(p-1)/r$ entries -1 . If ζ_i above has order p , then $\text{tr}(\zeta_i)$ is just an element of \widetilde{T}_n . Now suppose that ζ_i has order greater than p . Then it can happen that $\text{tr}(\zeta_i)$ does not belong to \widetilde{T}_n . But in this case we have $\zeta_i = -\sum_{j=1}^{p-1} \zeta_i \zeta^{jp^{n-1}}$ and by construction all the elements $\text{tr}(\zeta_i \zeta^{jp^{n-1}})$ lie in \widetilde{T}_n . Moreover, for $i' \neq i$, the sets $\{\text{tr}(\zeta_i \zeta^{jp^{n-1}}) : j = 1, \dots, p-1\}$ and $\{\text{tr}(\zeta_{i'} \zeta^{jp^{n-1}}) : j = 1, \dots, p-1\}$ are disjoint. Consequently, we may order these basis elements in such way that the corresponding rows of A have the form

$$\begin{pmatrix} 1 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & & & & & & & & & & 1 \\ -1 & \dots & -1 & & & & & & & & 0 \\ & & & -1 & \dots & -1 & & & & & \\ & & & & & & \ddots & & & & \\ & & & & & & & -1 & \dots & -1 & \end{pmatrix}.$$

Each of these rows appear p^s times. Now we put all the ingredients together and it follows that $A^T A$ is a block diagonal matrix. The first block is $(p^s r + p^s \delta_{ij})_{i,j=1}^{(p-1)/r}$ and corresponds to the basis elements of order p . All other blocks are given by $p^s(1 + \delta_{ij})_{i,j=1}^{p-1}$. The number of these blocks is $(p^{n-s-1} - 1)/r$. Hence, we have

$$\begin{aligned} \sum_{1 \leq i \leq j \leq t} q_{ij} m_{ij} &= \left(\frac{p-1}{r} (p^s r + p^s) - \left(\frac{p-1}{r} - 1 \right) p^s r \right) + \frac{p^{n-1} - p^s}{r} (2(p-1) - (p-2)) \\ &= \frac{p-1}{r} p^s + p^s r + \frac{p^n - p^{s+1}}{r} = \frac{p^n + p^s (r^2 - 1)}{r} = k_0(B) \end{aligned}$$

(cf. proof of Proposition 1.2). □

Theorem 2.6. *Let B be a p -block of a finite group G . Let (u, b_u) be a B -subsection such that b_u dominates a block \overline{b}_u of $C_G(u)/\langle u \rangle$. Suppose that all irreducible Brauer characters of b_u are stable under $\mathcal{N} := N_G(\langle u, b_u \rangle)/C_G(u)$. Let $l := l(b_u)$, and let \overline{C}_u be the Cartan matrix of \overline{b}_u up to basic sets. Then for every positive definite, integral quadratic form $q(x_1, \dots, x_l) = \sum_{1 \leq i \leq j \leq l} q_{ij} x_i x_j$ we have*

$$k_0(B) \leq k_0(\langle u \rangle \rtimes \mathcal{N}) \sum_{1 \leq i \leq j \leq l} q_{ij} c_{ij}.$$

If (u, b_u) is major, then

$$k(B) \leq k(\langle u \rangle \rtimes \mathcal{N}) \sum_{1 \leq i \leq j \leq l} q_{ij} c_{ij}.$$

Proof. First of all, note that we only need to know \overline{C}_u up to basic sets, because changing the basic set has the same effect as changing the quadratic from q accordingly (see [17, p. 83]). Now we discuss the special case $p = 2$. Similarly as in the proof of Proposition 1.2, we need to distinguish two cases. In the first case \mathcal{N} contains an element $\gamma \equiv -5^n \pmod{|\langle u \rangle|}$. Then [26, Theorem 5.2] implies

$$k_0(B) \leq 2|\mathcal{N}| \sum_{1 \leq i \leq j \leq l} q_{ij} c_{ij}$$

and the claim follows as in Proposition 1.2. In the second case there is no such element γ . Here, $k_0(\langle u \rangle \rtimes \mathcal{N}) = |\langle u \rangle|$ and the claim follows directly from Proposition 1.1.

In the following we will assume $p > 2$. Let $|\langle u \rangle| = p^n$ and $m := p^{n-1}(p-1)$. In the definition of a_{ij} in Theorem 2.1 we may take $\gamma = 1$. It follows that there is a matrix $T = (t_{ij})_{i,j=1}^m$ such that $A_{\sigma\tau} = c_{\sigma\tau}T$ for all $\sigma, \tau \in \text{IBr}(b_u)$. This gives $M = (m_{ij}) = \overline{C_u} \otimes T$. It is easy to see that T is exactly the matrix we would get from Theorem 2.1 applied to the group $\langle u \rangle \rtimes \mathcal{N}$. By using the integral basis from Lemma 2.4, we can replace T by a matrix of size $\frac{m}{|\mathcal{N}|} \times \frac{m}{|\mathcal{N}|}$ (constructed in the proof of Lemma 2.5). We still denote this smaller matrix by T . If $q_A(x) = \sum_{1 \leq i \leq j \leq \frac{m}{|\mathcal{N}|}} f_{ij} x_i x_j$ is the quadratic form corresponding to the Dynkin diagram of type $A_{m/|\mathcal{N}|}$, then Lemma 2.5 gives

$$\sum_{1 \leq i \leq j \leq \frac{m}{|\mathcal{N}|}} f_{ij} t_{ij} = k_0(\langle u \rangle \rtimes \mathcal{N}).$$

Now let $U^T U = M$ for a matrix $U = (u_{ij}) \in \mathbb{Z}^{k \times \frac{m}{|\mathcal{N}|}}$ without vanishing rows. We need to bound k . By Lemma 2.3, $q^* := q \otimes q_A$ satisfies $q^*(x) \geq 1$ for every $x \in \mathbb{Z}^{\frac{m}{|\mathcal{N}|}} \setminus \{0\}$. Recall that the Gram matrix of q^* is given by

$$(z_{ij}) = \frac{1}{2} \begin{pmatrix} 2q_{11} & q_{12} & \cdots & q_{1l} \\ q_{21} & 2q_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & q_{l-1,l} \\ q_{l1} & \cdots & q_{l,l-1} & 2q_{ll} \end{pmatrix} \otimes \frac{1}{2} \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

This implies

$$k \leq \sum_{i=1}^k q^*(u_{i1}, \dots, u_{i, \frac{m}{|\mathcal{N}|}}) = \sum_{i=1}^k \sum_{1 \leq r, s \leq \frac{m}{|\mathcal{N}|}} z_{rs} u_{ir} u_{is} = \sum_{1 \leq r, s \leq \frac{m}{|\mathcal{N}|}} z_{rs} m_{rs} = k_0(\langle u \rangle \rtimes \mathcal{N}) \sum_{1 \leq i \leq j \leq l} q_{ij} c_{ij}. \quad \square$$

The hypothesis of Theorem 2.6 is met surprisingly often. This can be explained by the following argument. By [18, Corollary 9.21], $B_u := b_u^{\text{N}_G(\langle u \rangle, b_u)}$ is the only block of $\text{N}_G(\langle u \rangle, b_u)$ covering b_u . Clifford theory gives strong relations between $k(b_u)$ and $l(b_u)$ on the one hand and $k(B_u)$ and $l(B_u)$ on the other hand. In particular, all characters in $\text{IBr}(b_u)$ are $\text{N}_G(\langle u \rangle, b_u)$ -stable if and only if $l(B_u) = l(b_u)r$ where r is the p' -part of $|\mathcal{N}|$ (see [18, Theorems 8.11, 8.12 and Corollary 8.20]). In general, the numbers $l(b_u)$, $l(B_u)$ and $|\mathcal{N}|$ do not determine the orbit lengths on $\text{IBr}(b_u)$ uniquely. For example, $l(b_u) = l(B_u) = 18$ and $|\mathcal{N}| = 24$ allow orbit lengths 3, 3, 12, but also 2, 8, 8.

In the next section we will discuss a special case where not all Brauer characters are stable.

3 Abelian defect groups of rank 2

Suppose that the defect group D of the block B is metacyclic. If $p = 2$, then Alperin's Weight Conjecture holds for B (see [26, Corollary 8.2]). Thus, let $p > 2$. Then by Watanabe [33] (cf. [26, Theorem 8.8]), Alperin's Weight Conjecture also holds provided D is non-abelian (in addition). Hence, it is of interest to study the case $D \cong Z_{p^n} \times Z_{p^m}$ more closely. Here even in the smallest case $D \cong Z_3^2$ Alperin's Weight Conjecture is open (cf. Kiyota [14] and Watanabe [32]). Therefore, any new information is valuable. In this section we will prove the following.

Theorem 3.1. *Let B be a p -block of a finite group G with defect group $D \cong Z_{p^n} \times Z_{p^m}$ and inertial quotient $I(B) = I \times J$ such that $D \rtimes I(B) \cong (Z_{p^n} \rtimes I) \times (Z_{p^m} \rtimes J)$. Let b be the Brauer correspondent of B in $\text{N}_G(D)$. Then $k(B) \leq k(b)$ and $l(B) \leq l(b) \leq |I(B)|$.*

Before we begin with the proof we make a few remarks. Alperin's Weight Conjecture for blocks with abelian defect groups asserts that $l(B) = l(b)$ (with the notation from Theorem 3.1). Hence, $l(B) \leq l(b)$ is a sharp bound and so is $k(B) \leq k(b)$ by standard arguments (see proof of Theorem 3.1). Our result is trivial if $p = 2$, because B is nilpotent in this case. If $p = 3$, then we have $I(B) \leq Z_2^2$. Hence, results by Usami and Puig [30, 23]

already imply Alperin's Weight Conjecture. For $p \geq 5$, Theorem 3.1 is something new (at least to author's knowledge).

More generally, let B be a block with abelian defect group D of rank 2 such that $I(B)$ acts decomposably on D . Then $I(B) \leq Z_{p-1}^2$. Apart from Theorem 3.1, it may happen that $I(B)$ acts semiregularly on $D \setminus \{1\}$. Then $I(B)$ is cyclic and [27, Theorem 5] implies the optimal bound $l(B) \leq |I(B)| = l(b)$. Unfortunately, there are other "mixed" cases where we cannot say much at the moment. For example, let $D \cong Z_7^2$, and let $I(B) \cong Z_6$ act on the first factor faithfully and as an inversion on the second factor. Then the ideas of the present paper show $k(B) \leq 25$, but actually $k(B) = 22$. Moreover, in general it seems vastly more difficult to prove the converse inequality $l(B) \geq l(b)$ (this is a main obstacle for $D \cong Z_3^2$).

Now we begin with the proof of Theorem 3.1. The key is to show that for blocks with cyclic defect groups, the action on the Brauer characters is not arbitrary. We say that an action is $\frac{1}{2}$ -transitive, if all its orbits have the same length. Note that we also consider the trivial action as $\frac{1}{2}$ -transitive.

Proposition 3.2. *Let G be a finite group with $N \trianglelefteq G$ such that G/N is cyclic. Let B be a G -stable block of N with cyclic defect group. Then the action of G on $\text{IBr}(B)$ is $\frac{1}{2}$ -transitive. In particular, the action is trivial if B is the principal block.*

Proof. Let us assume the contrary. By replacing G with a group H such that $N \leq H \leq G$, we may assume that the action of G on $\text{IBr}(B)$ has fixed points and all non-trivial orbits have the same length q . We may assume further that q is a prime and $|G/N|$ is a power of q . By Dade's theory of cyclic defect groups [6] (see also [26, Theorem 8.6]), $l(B) < p$. In particular, $q \neq p$. Let $N \leq H \leq G$ such that $|G/H| = q$. Then H acts trivially on $\text{IBr}(B)$. Since H/N is cyclic, it is well-known that all irreducible Brauer characters of B extend to H (see [18, Theorem 8.12 and Corollary 8.20]). By [18, Theorem 9.4], there is a block B_H of H covering B such that $\text{IBr}(B_H)$ contains an extension of every irreducible Brauer character of B . In particular, $\text{IBr}(B_H)$ also contains a G -invariant character and B_H is G -stable. Moreover, B and B_H have the same defect group. Since $\text{IBr}(B_H)$ also contains non-stable characters, we may replace N by H and B by B_H . Thus we may assume that $|G/N| = q$ in the following.

Let D be a defect group of B . Since G acts on the defect groups of B , the Frattini argument implies $G = NN_G(D)$. Hence, we may regard G/N as a quotient of $N_G(D)$. Let b be a Brauer correspondent of B in $N_N(D)$. By work of Dade [7] (see also [15, Lemma 4.1]), the G/N -sets $\text{IBr}(B)$ and $\text{IBr}(b)$ are isomorphic. Therefore, we may assume that $D \trianglelefteq G$. Let $e \mid p-1$ be the inertial index of B . By Külshammer [16] (see also [26, Theorem 1.19]), B is Morita equivalent to the group algebra of $D \rtimes Z_e$. In particular, the decomposition matrix of B is given by

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ 1 & \cdots & \cdots & 1 \\ \vdots & & & \vdots \\ 1 & \cdots & \cdots & 1 \end{pmatrix} \quad (3.1)$$

where the first rows correspond to the non-exceptional characters $\chi_1, \dots, \chi_e \in \text{Irr}(B)$ (see for instance, [8, Theorem VII.2.12]). Since $|G/N|$ is a prime, there exists a $\varphi \in \text{IBr}(B)$ such that $\varphi^G \in \text{IBr}(G)$. This shows that B is covered by a unique block B_G of G (see [18, Theorem 9.4]). Since G/N is p' -group, B_G also has defect group D . Let $e = rq + s$ where s is the number of G -invariant Brauer characters of B . Then Clifford theory gives $l(B_G) = r + sq$. Now we study the relation between $k(B) = e + \frac{|D|-1}{e}$ and $k(B_G)$. If $G = NC_G(D)$, then, by considering the generalized decomposition numbers, it is easy to see that G acts trivially on the exceptional characters of B . Hence, $k(B_G) = r + sq + q \frac{|D|-1}{e}$. On the other hand, we have $k(B_G) = l(B_G) + \frac{|D|-1}{l(B_G)}$. This implies $rq + s = e = q(r + sq)$ and we derive the contradiction $s = 0$. We conclude that $C_G(D) \leq N$. Here G/N acts semiregularly on the exceptional characters. This gives $k(B_G) = r + sq + \frac{|D|-1}{eq}$, $(rq + s)q = eq = r + sq$ and $r = 0$. Again a contradiction. This proves the first claim. The second claim is obvious, since the trivial character is always G -invariant. \square

In the situation of Proposition 3.2, it is clear that an appropriate quotient G/M with $N \leq M$ acts semiregularly on $\text{IBr}(B)$. The action on $\text{IBr}(B)$ induces a graph automorphism on the Brauer tree of B . Therefore, a non-trivial action greatly restricts the possible Brauer trees. We will see later in Lemma 3.4 that all possible $\frac{1}{2}$ -transitive actions actually occur.

Lemma 3.3. *Let G be a finite group with $N \trianglelefteq G$ such that G/N is cyclic. Let B be a G -stable block of N with cyclic defect group D . Then there exists a basic set \mathcal{B} for B which is G/N -isomorphic to $\text{IBr}(B)$ and the Cartan matrix of B with respect to \mathcal{B} is given by $(m + \delta_{ij})$ with $m = \frac{|D|-1}{l(B)}$.*

Proof. Let $Q = (q_{ij})$ be the decomposition matrix of B (with respect to $\text{IBr}(B)$). We may assume that the first $l := l(B)$ rows correspond to the non-exceptional characters χ_1, \dots, χ_l while the last m rows correspond to the exceptional characters and therefore are all equal. Moreover, we order the non-exceptional characters and the Brauer characters $\varphi_1, \dots, \varphi_l$ in such a way that Q has unitriangular shape (pick leaves from the Brauer tree successively). Let $S := (q_{ij})_{i,j=1}^l$. Then $\det(S) = 1$ and $S \in \text{GL}(l, \mathbb{Z})$. Hence, we may define $\mathcal{B} = \{\tilde{\varphi}_1, \dots, \tilde{\varphi}_l\}$ with $\tilde{\varphi}_i := q_{i1}\varphi_1 + \dots + q_{il}\varphi_l$ for $i = 1, \dots, l$.

For $g \in G$ we write ${}^g\varphi_i = \varphi_{g_i}$. We will show that ${}^g\chi_i = \chi_{g_i}$. This is obvious if $i = 1$, because χ_1 is a lift of φ_1 . Now assume that we have shown the claim for all $k < i$. We have $({}^g\chi_i)|_{G_{p'}} = q_{i1}\varphi_{g_1} + \dots + q_{ii}\varphi_{g_i}$ where $G_{p'}$ is the set of p -regular elements of G . By way of contradiction, suppose that ${}^g i < {}^g j$ for some $j < i$ with $q_{ij} \neq 0$. Then it follows that S has two rows of the form

$$\begin{pmatrix} * & \cdots & * & 1 & 0 & \cdots & 0 \\ * & \cdots & * & 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Hence, Q is not unitriangular and we have a contradiction. Therefore, ${}^g\chi_i = \chi_{g_i}$ for $i = 1, \dots, l$.

We conclude that

$${}^g\tilde{\varphi}_i = q_{i1}\varphi_{g_1} + \dots + q_{il}\varphi_{g_l} = (\chi_{g_i})|_{G_{p'}} = q_{g_i,1}\varphi_1 + \dots + q_{g_i,l}\varphi_l = \tilde{\varphi}_{g_i}.$$

Hence, $\text{IBr}(B)$ and \mathcal{B} are G/N -isomorphic. Let \mathcal{Q} be the decomposition matrix of B with respect to \mathcal{B} . Then the first l rows of \mathcal{Q} form an identity matrix. On the other hand, B is perfectly isometric to the group algebra of $D \rtimes Z_l$ (see for example [24]). This implies that \mathcal{Q} is given by (3.1) above. From that, the last claim follows easily. \square

Next we prove a variant of Lemma 2.5.

Lemma 3.4. *Let p be an odd prime, $n_1, n_2 \in \mathbb{N}$, $l_1, l_2 \mid p-1$, $d \mid (l_1, l_2)$, and let $\gamma_i \in (\mathbb{Z}/p^{n_i}\mathbb{Z})^\times$ of order l_i for $i = 1, 2$. Let B be a block of*

$$G := \langle u, v, x, y \mid u^{p^{n_1}} = v^{p^{n_2}} = [u, v] = x^{d l_1} = y^{l_2} = [u, y] = [v, x] = 1, \\ x u x^{-1} = u^{\gamma_1}, y v y^{-1} = v^{\gamma_2}, y x y^{-1} = x^{1+l_1} \rangle$$

lying over a generator λ of $\text{Irr}(\langle x^{l_1} \rangle)$. Let (u, b_u) be a B -subsection. Then $l(b_u) = l_2$ and the generalized decomposition matrix $Q_u = (d_{\chi\varphi}^u : \chi \in \text{Irr}(B), \varphi \in \text{IBr}(b_u))$ is given as follows

$$Q_u = \left(\begin{array}{cccccccc}
\overbrace{1 \cdots 1}^d & & & & & & & \\
\vdots & & & & & & & \\
1 & \cdots & 1 & & & & & \\
& & & \ddots & & & & \\
& & & & 1 & \cdots & 1 & \\
& & & & \vdots & & \vdots & \\
& & & & 1 & \cdots & 1 & \\
& & \text{tr}_d(\zeta_1^{\gamma_1}) & \cdots & \text{tr}_d(\zeta_1^{\gamma_1^d}) & & & \\
& & \vdots & & \vdots & & & \\
& & \text{tr}_d(\zeta_m^{\gamma_1}) & \cdots & \text{tr}_d(\zeta_m^{\gamma_1^d}) & & & \\
& & \vdots & & \vdots & & & \\
& & \text{tr}_d(\zeta_1^{\gamma_1^d}) & \cdots & \text{tr}_d(\zeta_1^{\gamma_1^{d^2}}) & & & \\
& & \vdots & & \vdots & & & \\
& & \text{tr}_d(\zeta_m^{\gamma_1^d}) & \cdots & \text{tr}_d(\zeta_m^{\gamma_1^{d^2}}) & & & \\
& & & & \ddots & & & \\
& & & & & \text{tr}_d(\zeta_1^{\gamma_1}) & \cdots & \text{tr}_d(\zeta_1^{\gamma_1^d}) \\
& & & & & \vdots & & \vdots \\
& & & & & \text{tr}_d(\zeta_m^{\gamma_1}) & \cdots & \text{tr}_d(\zeta_m^{\gamma_1^d}) \\
& & & & & \vdots & & \vdots \\
& & & & & \text{tr}_d(\zeta_1^{\gamma_1^d}) & \cdots & \text{tr}_d(\zeta_1^{\gamma_1^{d^2}}) \\
& & & & & \vdots & & \vdots \\
& & & & & \text{tr}_d(\zeta_m^{\gamma_1^d}) & \cdots & \text{tr}_d(\zeta_m^{\gamma_1^{d^2}}) \\
& & & & & & & \\
& & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\
& & \vdots & & & & & \vdots \\
& & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\
& & \text{tr}(\zeta_1) & \cdots & \cdots & \cdots & \cdots & \text{tr}(\zeta_1) \\
& & \vdots & & & & & \vdots \\
& & \text{tr}(\zeta_1) & \cdots & \cdots & \cdots & \cdots & \text{tr}(\zeta_1) \\
& & \text{tr}(\zeta_2) & \cdots & \cdots & \cdots & \cdots & \text{tr}(\zeta_2) \\
& & \vdots & & & & & \vdots \\
& & \text{tr}(\zeta_2) & \cdots & \cdots & \cdots & \cdots & \text{tr}(\zeta_2) \\
& & \vdots & & & & & \vdots \\
& & \vdots & & & & & \vdots \\
& & \text{tr}(\zeta_m) & \cdots & \cdots & \cdots & \cdots & \text{tr}(\zeta_m) \\
& & \vdots & & & & & \vdots \\
& & \text{tr}(\zeta_m) & \cdots & \cdots & \cdots & \cdots & \text{tr}(\zeta_m)
\end{array} \right) \left. \begin{array}{l} \left. \right\} \frac{l_1}{d} \\ \left. \right\} m = \frac{p^{n_1} - 1}{l_1} \\ \left. \right\} l_1 \frac{p^{n_2} - 1}{l_2} \\ \left. \right\} \frac{p^{n_2} - 1}{l_2} \end{array} \right.$$

where ζ_1, \dots, ζ_m is a set of representatives of the non-trivial p^{n_1} -th roots of unity under the action of γ_1 , $\text{tr}(\zeta_i) = \sum_{j=1}^{l_1} \zeta_i^{\gamma_1^j}$ and $\text{tr}_d(\zeta_i) = \sum_{j=1}^{l_1/d} \zeta_i^{\gamma_1^{jd}}$.

Proof. Since $x^{l_1} \leq Z(G)$, λ is G -invariant. Hence, by the extended first main theorem (see [18, Theorem 9.7]) the characters of $\text{Irr}(B)$ lying over λ indeed form a block. Observe that $C_G(u) = \langle u, v, x^{l_1}, y \rangle$. Hence, $\text{IBr}(b_u) = \{\lambda\rho : \rho \in \text{Irr}(\langle y \rangle)\}$ and $l(b_u) = l_2$. Since λ is G -invariant, so is b_u . Now we will use Clifford theory to construct $\text{Irr}(B)$. We will see that all the ramification indices of characters in $\text{Irr}(B)$ are trivial. Since ${}^x(\lambda\rho)(y) = (\lambda\rho)(x^{l_1}y) =$

$\lambda(x^{l_1})\rho(y)$, it follows that the orbits of $\text{IBr}(b_u)$ under G all have length d . Every character $\lambda\rho \in \text{IBr}(b_u)$ extends in l_1/d many ways to $\langle x^d, y \rangle$. Every such extension induces to $\text{Irr}(B) \cup \text{IBr}(B)$ (considered as inflation from $G/\langle u, v \rangle$). This gives $l_1 l_2 / d^2$ irreducible (Brauer) characters of B . Let $\chi \in \text{Irr}(B)$ be one of them, and let $s \in \langle x^{l_1}, y \rangle$. Then $\chi(us) = \chi(s) = (\lambda\rho_1)(s) + \dots + (\lambda\rho_d)(s)$ where $\lambda\rho_1, \dots, \lambda\rho_d \in \text{IBr}(b_u)$ is an orbit under G . Hence, the corresponding lines of Q_u have the form $(1, \dots, 1, 0, \dots, 0)$.

Now we compute the characters lying over $\theta\lambda \in \text{Irr}(\langle u, v, x^{l_1} \rangle)$ where $1 \neq \theta \in \text{Irr}(\langle u \rangle)$. Obviously, $\theta\lambda$ extends in l_2 many ways to $\langle u, v, x^{l_1}, y \rangle$. Each such extension induces to $\text{Irr}(B)$. This gives l_2 new characters. For one of them we have

$$\chi(us) = (\theta_1(u) + \dots + \theta_{l_1/d}(u))(\lambda\rho_1)(s) + \dots + (\theta_{l_1 - l_1/d + 1}(u) + \dots + \theta_{l_1}(u))(\lambda\rho_d)(s) = \sum_{j=1}^d \text{tr}_d(\zeta_i^{\gamma_1^j})(\lambda\rho_j)(s)$$

for some i . We have $(p^{n_1} - 1)/l_1$ essentially different choices for θ . Thus in total we get $l_2(p^{n_1} - 1)/l_1$ characters in this way.

Next consider $\theta\lambda \in \text{Irr}(\langle u, v, x^{l_1} \rangle)$ where $1 \neq \theta \in \text{Irr}(\langle v \rangle)$. This character extends in l_1 many ways to $\langle u, v, x \rangle$ and the extensions induce to $\text{Irr}(B)$. In total there are $l_1(p^{n_2} - 1)/l_2$ such characters χ . Since $\langle u, v, x^{l_1} \rangle \trianglelefteq \langle u, v, x^{l_1}, y \rangle$, it is easy to see that $\chi(us) = \chi(s) = (\sum_{i=1}^{l_2} \lambda\rho_i)(s)$.

Finally, let $1 \neq \theta \in \text{Irr}(\langle u \rangle)$ and $1 \neq \eta \in \text{Irr}(\langle v \rangle)$. Then the characters $\theta\eta\lambda$ induce directly to $\text{Irr}(B)$. If χ is one of them, we get

$$\chi(us) = \left(\sum_{i=1}^{l_1} x^i (\theta\eta\lambda)^{\langle u, v, x^{l_1}, y \rangle} \right)(us) = \sum_{i=1}^{l_1} \theta_i(u) \sum_{j=1}^{l_2} (\lambda\rho_j)(s) = \text{tr}(\zeta_i) \sum_{j=1}^{l_2} (\lambda\rho_j)(s).$$

It is easy to see that this amounts all the characters in $\text{Irr}(B)$. □

Lemma 3.5. *Let Q_u be the matrix from Lemma 3.4 with columns q_i . Then we can write q_i with respect to the integral basis \tilde{T}_n from Lemma 2.4 with $n = n_1$ and $|\mathcal{N}| = l_1/d$. Hence, let $A_i \in \mathbb{Z}^{k \times p^{n_1-1}(p-1)d/l_1}$ such that $q_i = A_i \tilde{T}_n$. Let $A = (A_{ij})_{i,j=1}^{l_2/d}$ the block matrix with blocks*

$$A_{ij} = \begin{cases} A_1^T A_1 & \text{if } i = j, \\ A_1^T A_{d+1} & \text{if } i \neq j. \end{cases}$$

Let $U \in \mathbb{Z}^{r \times s}$ without vanishing rows such that $U^T U = A$. Then

$$r \leq \frac{p^{n_1} - 1}{l_1} \frac{p^{n_2} - 1}{l_2} + l_1 \frac{p^{n_2} - 1}{l_2} + l_2 \frac{p^{n_1} - 1}{l_1} + \frac{l_1 l_2}{d^2}.$$

Proof. First observe that $1 = -\sum_{t \in p^{n_1-1} T_1} \text{tr}_d(\zeta^t)$ where $\zeta \in \mathbb{C}$ is a primitive p^n -th root of unity and tr_d as in Lemma 3.4. Thus, the first rows of A_1 have the form $(-1, \dots, -1, 0, \dots, 0)$ with $(p-1)d/l_1$ entries -1 . Now we use induction on n to choose the elements ζ_i from Lemma 3.4 in such a way that A_1 has a nice shape. If $n = 1$, then all the entries $\text{tr}_d(\zeta_i^{\gamma_1^j})$ in q_1 are just elements of \tilde{T}_n . So the corresponding rows of A_1 consist of zeros and exactly one entry 1. Now let $n \geq 2$. Then, by induction, the entries $\text{tr}_d(\zeta_i^{\gamma_1^j})$ where ζ_i is not primitive form a part $A_1^{(p)}$ of A_1 which we already know. There are $\frac{p^n - 1}{l_1} - \frac{p^{n-1} - 1}{l_1} = p^{n-1} \frac{p-1}{l_1}$ remaining entries (counting without multiplicities). We choose these ζ_i such that they form blocks $\{\zeta^{s+jp^{n-1}} : j = 0, \dots, p-1\}$ for some $s \in S_{n-1}$ with the notation from Lemma 2.4. It is crucial here to see that these elements are not conjugate under γ_1 . Moreover, only the elements $\zeta^{s+(p-1)p^{n-1}}$ do not lie in \tilde{T}_n . Here however, $\zeta^{s+(p-1)p^{n-1}} = -\sum_{j=0}^{p-2} \zeta^{s+jp^{n-1}}$.

Hence, the rows in A_1 corresponding to $\text{tr}_d(\zeta_1^{\gamma_1^j}), \dots, \text{tr}_d(\zeta_m^{\gamma_1^j})$ for a fixed j have the following form:

$$\left(\begin{array}{cccccccc} A_1^{(p)} & & & & & & & \\ & 1 & \dots & & & & & \\ & & \dots & \dots & & & & \\ & & & \dots & \dots & & & \\ & & & & & \dots & & \\ & -1 & \dots & -1 & & & & \\ & & & & -1 & \dots & -1 & \\ & & & & & \dots & & \\ & & & & & & \dots & \\ & & & & & & & -1 \dots -1 \end{array} \right) \left. \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \right\} \begin{array}{l} p^{n-2} \frac{(p-1)^2}{l_1} \\ \\ \\ \\ \\ 0 \\ \\ \\ \\ p^{n-2} \frac{p-1}{l_1} \end{array}$$

It remains to handle the numbers $\text{tr}(\zeta_i)$. But this follows directly from what we have done, since $\text{tr}(\zeta_i) = \sum_{j=1}^d \text{tr}_d(\zeta_i^{\gamma_1^j})$. In case $n = 1$ the matrix $A_1^T A_1$ is given as follows

$$1_{d \frac{p-1}{l_1}} + \left(\begin{array}{ccc} \overbrace{1 \dots 1}^d \\ \vdots \\ 1 \dots 1 \end{array} \right) \otimes \left(\begin{array}{cccc} \overbrace{\alpha \ \beta \ \dots \ \beta}^{\frac{p-1}{l_1}} \\ \beta \ \ddots \ \ddots \ \vdots \\ \vdots \ \ddots \ \ddots \ \beta \\ \beta \ \dots \ \beta \ \alpha \end{array} \right) \quad \begin{array}{l} \alpha := \frac{l_1}{d} + (l_1 + 1) \frac{p^{n_2} - 1}{l_2}, \\ \beta := \frac{l_1}{d} + l_1 \frac{p^{n_2} - 1}{l_2}. \end{array}$$

Now let $n \geq 2$. Suppose that the matrix $A_1^T A_1$ for $n - 1$ is given by B_{n-1} . Then $A_1^T A_1$ for n has the form

$$A_1^T A_1 = B_{n-1} \oplus \left(1_{p^{n-2} \frac{p-1}{l_1}} \otimes \left(\begin{array}{cccc} \overbrace{\delta + 1 \ \delta \ \dots \ \delta}^d \\ \delta \ \ddots \ \ddots \ \vdots \\ \vdots \ \ddots \ \ddots \ \delta \\ \delta \ \dots \ \delta \ \delta + 1 \end{array} \right) \otimes \left(\begin{array}{cccc} \overbrace{2 \ 1 \ \dots \ 1}^{p-1} \\ 1 \ \ddots \ \ddots \ \vdots \\ \vdots \ \ddots \ \ddots \ 1 \\ 1 \ \dots \ 1 \ 2 \end{array} \right) \right) \quad \delta := \frac{p^{n_2} - 1}{l_2}.$$

Here the integral basis vectors in the second summand are ordered in the following way: $s_1, s_1 + p^{n-1}, \dots, s_1 + (p-2)p^{n-1}, \gamma_1(s_1), \dots, \gamma_1(s_1 + (p-2)p^{n-1}), \dots, \gamma_1^{d-1}(s_1 + (p-2)p^{n-1}), s_2, \dots$ where $s_i \in S_{n-1}$ with the notation from Lemma 2.4.

Now assume that $d < l_2$, so that A_{d+1} really exists. In case $n = 1$ we have

$$A_1^T A_{d+1} = \delta \left(\begin{array}{ccc} \overbrace{1 \ \dots \ 1}^d \\ \vdots \\ 1 \ \dots \ 1 \end{array} \right) \otimes \left(\begin{array}{cccc} \overbrace{l_1 + 1 \ l_1 \ \dots \ l_1}^{\frac{p-1}{l_1}} \\ l_1 \ \ddots \ \ddots \ \vdots \\ \vdots \ \ddots \ \ddots \ l_1 \\ l_1 \ \dots \ l_1 \ l_1 + 1 \end{array} \right).$$

As before denote the matrix $A_1^T A_{d+1}$ for $n - 1$ by B'_{n-1} . Then for $n \geq 2$ we obtain

$$A_1^T A_{d+1} = B'_{n-1} \oplus \delta \left(1_{p^{n-2} \frac{p-1}{l_1}} \otimes \left(\begin{array}{ccc} \overbrace{1 \ \dots \ 1}^d \\ \vdots \\ 1 \ \dots \ 1 \end{array} \right) \otimes \left(\begin{array}{cccc} \overbrace{2 \ 1 \ \dots \ 1}^{p-1} \\ 1 \ \ddots \ \ddots \ \vdots \\ \vdots \ \ddots \ \ddots \ 1 \\ 1 \ \dots \ 1 \ 2 \end{array} \right) \right).$$

Finally, we consider $A = U^T U$. In order to estimate r , we may replace A by SAS^T where $S \in \text{GL}(s, \mathbb{Z})$ with $s = p^{n-1}(p-1)l_2/l_1$. In particular, we may permute the rows and columns of A , so that A has block diagonal form. The first block corresponds to the integral basis elements $p^{n-1}T_1$ and so on. It is enough to handle each of these blocks separately, i. e. we may assume that $U^T U$ equals one of these blocks. We use induction on n . If $n = 1$, then there is only one block. Let y_i be the quadratic form corresponding to the Dynkin diagram of type A_i (not the matrix here). By Lemma 2.3, we can use the quadratic form $y_{l_2/d} \otimes y_d \otimes y_{(p-1)/l_1}$ in order to estimate r like in Theorem 2.6. For $n = 1$ we get

$$\begin{aligned} r &\leq \frac{l_2}{d} \left(d \left((\alpha + 1) \frac{p-1}{l_1} - \beta \left(\frac{p-1}{l_1} - 1 \right) \right) - (d-1) \left(\alpha \frac{p-1}{l_1} - \beta \left(\frac{p-1}{l_1} - 1 \right) \right) \right) - \left(\frac{l_2}{d} - 1 \right) \delta \left(\frac{p-1}{l_1} + l_1 \right) \\ &= \frac{l_2}{d} \left(\alpha \frac{p-1}{l_1} - \beta \left(\frac{p-1}{l_1} - 1 \right) + d \frac{p-1}{l_1} \right) - \left(\frac{l_2}{d} - 1 \right) \delta \left(\frac{p-1}{l_1} + l_1 \right) \quad (\text{use } \alpha = \beta + \delta) \\ &= \frac{l_2}{d} \left(\beta + (\delta + d) \frac{p-1}{l_1} \right) - \left(\frac{l_2}{d} - 1 \right) \delta \left(\frac{p-1}{l_1} + l_1 \right) \quad (\text{use definition of } \beta) \\ &= \frac{l_1 l_2}{d^2} + \delta \frac{p-1}{l_1} + l_2 \frac{p-1}{l_1} + l_1 \delta \end{aligned}$$

and this is exactly what we wanted to have. Now let $n \geq 2$. By induction the block of A corresponding to the indices $t \in pT_{n-1}$ gives

$$r \leq \frac{p^{n_1-1} - 1}{l_1} \frac{p^{n_2} - 1}{l_2} + l_1 \frac{p^{n_2} - 1}{l_2} + l_2 \frac{p^{n_1-1} - 1}{l_1} + \frac{l_1 l_2}{d^2}. \quad (3.2)$$

For the remaining block corresponding to the indices $t \in T_n \setminus pT_{n-1}$ we use the quadratic form $y_{l_2/d} \otimes y_{p^{n-2}(p-1)/l_1} \otimes y_d \otimes y_{p-1}$. This gives

$$\begin{aligned} r &\leq \frac{l_2}{d} \left(p^{n-2} \frac{p-1}{l_1} \left(d((p-1)2(\delta+1) - (p-2)(\delta+1)) - (d-1)((p-1)2\delta - (p-2)\delta) \right) \right) \\ &\quad - \left(\frac{l_2}{d} - 1 \right) \left(p^{n-1} \frac{p-1}{l_1} \delta \right) \\ &= \frac{l_2}{d} p^{n-2} \frac{p-1}{l_1} (d(\delta+1)p - (d-1)\delta p) - \left(\frac{l_2}{d} - 1 \right) \left(p^{n-1} \frac{p-1}{l_1} \delta \right) \\ &= \frac{l_2}{d} p^{n-1} \frac{p-1}{l_1} (d + \delta) - \left(\frac{l_2}{d} - 1 \right) \left(p^{n-1} \frac{p-1}{l_1} \delta \right) \\ &= p^{n-1} \frac{p-1}{l_1} \delta + l_2 p^{n-1} \frac{p-1}{l_1}. \end{aligned}$$

Adding this to (3.2) gives precisely our claim. \square

Proof of Theorem 3.1. Let $D = \langle u \rangle \times \langle v \rangle$ such that I acts on $\langle u \rangle$ and J acts on $\langle v \rangle$. If $D \trianglelefteq G$, then there is nothing to prove. Hence, we may assume that $\langle u \rangle \not\trianglelefteq G$. Let b_D be a Brauer correspondent of B in $C_G(D)$. By Fong-Reynolds (see [18, Theorem 9.14]), we may assume that b is a block of $N := N_G(D, b_D)$. In particular, $\langle u \rangle, \langle v \rangle \trianglelefteq N$. By Külshammer [16] (see also [26, Theorem 1.19]), b is Morita equivalent to a twisted group algebra $\mathcal{O}_\gamma[D \rtimes I(B)]$ where \mathcal{O} is a suitable complete discrete valuation ring. Let $|I| = l_1$ and $|J| = l_2$. By the Künneth formula, $H^2(I(B), \mathbb{C}^\times) \cong Z_{(l_1, l_2)}$. Hence, we may assume that b is Morita equivalent to the block in Lemma 3.4 such that $l(b) = l_1 l_2 / d^2$ for some $d \mid (l_1, l_2)$ (see [26, Proposition 1.20]; this does not depend on the choice of the covering group by [22, Proposition 5.15]).

Let $b_u := b_D^{C_G(u)}$ so that (u, b_u) is a major B -subsection. Let $B_u := b_u^{N_G(\langle u \rangle, b_u)}$. Then

$$B_u = b_D^{N_G(\langle u \rangle, b_u)} = b^{N_G(\langle u \rangle, b_u)}.$$

Observe that B_u has defect group D and inertial quotient $I(B)$. Since $\langle u \rangle \not\trianglelefteq G$, induction on $|G|$ shows that $l(B_u) \leq l(b)$. Now b_u dominates a block \bar{b}_u with cyclic defect group $\langle v \rangle$ and inertial index l_2 (see [28, Lemma 3]). In particular, $l(b_u) = l_2$. By Proposition 3.2, $N_G(\langle u \rangle, b_u)$ acts $\frac{1}{2}$ -transitively on $\text{IBr}(\bar{b}_u) = \text{IBr}(b_u)$. Suppose that the orbits have length α . Then Clifford theory shows that $l(B_u) = l(b_u)l_1/\alpha^2 = l_1 l_2 / \alpha^2$. Therefore, $\alpha \geq d$.

Next we show that $k(B) - l(B) = k(b) - l(b)$ by using a formula of Brauer (see [26, Theorem 1.35]). Let $1 \neq w \in D$ and $b_w := b_D^{C_G(w)}$. If $w = u^i v^j$ such that $u^i \neq 1$ and $v^j \neq 1$, then $l(b_w) = 1$. Now assume that $w \in \langle u \rangle$. As usual, $\overline{b_w}$ has defect group $D/\langle w \rangle$. A result by Watanabe [31, Theorem 1] (see also [26, Theorem 1.39]) implies that $l(b_w) = l(\overline{b_w}) = l(\overline{b_u}) = l_2$. The case $w \in \langle v \rangle$ is similar. Altogether we have proved that

$$k(B) - l(B) = \frac{p^n - 1}{l_1} \frac{p^m - 1}{l_2} + l_1 \frac{p^m - 1}{l_2} + l_2 \frac{p^n - 1}{l_1}. \quad (3.3)$$

The same arguments apply to b and it follows that $k(B) - l(B) = k(b) - l(b)$. Hence, it suffices to show that $l(B) \leq l(b)$.

In order to apply Theorem 2.1 we need to know the Cartan matrix $\overline{C_u}$ of $\overline{b_u}$. By Lemma 3.3 we can choose a basic set such that $\overline{C_u} = (t + \delta_{ij})$ with $t = (p^m - 1)/l_2$. Since it is still complicated to use the formulas in Theorem 2.1, we investigate an alternative approach to compute the matrix M in Theorem 2.1. Let \tilde{B} be the block from Lemma 3.4 where we choose the parameter d in that lemma to be α . Let (u, β_u) be the corresponding \tilde{B} -subsection. Then the irreducible Brauer characters of β_u also split in orbits of length α . Moreover, since β_u has a normal defect group, the Cartan matrix of $\overline{\beta_u}$ is the same as $\overline{C_u}$ above. Therefore, the generalized decomposition numbers with respect to (u, β_u) satisfy exactly the same refined orthogonality relations as the corresponding numbers with respect to (u, b_u) . Furthermore, we only need to consider one irreducible Brauer character from each orbit. By the special shape of $\overline{C_u}$, the scalar product of two columns d_i and d_j of the generalized decomposition matrix only depends on whether $i = j$ or $i \neq j$. Hence, the matrix M in Theorem 2.1 is equivalent to the matrix A we have computed in Lemma 3.5. It follows that $l(B) \leq l_1 l_2 / \alpha^2 \leq l(b)$ and we are done. \square

4 A conjecture by Navarro

Very recently Gabriel Navarro [19] proposed the following conjecture as a generalization of Brauer's $k(B)$ -Conjecture for principal blocks.

Conjecture. *Let G be a finite group with $N \trianglelefteq G$, and let p be a prime. Let B_G and $B_{G/N}$ be the principal p -blocks of G and G/N respectively. Then $k(B_G) \leq k(B_{G/N})|N|_p$.*

Here, as usual $|N|_p$ denotes the p -part of the order of N . We use Theorem 2.6 above to prove this conjecture in a special case.

Theorem 4.1. *Let G be a finite group with $N \trianglelefteq G$ such that the Sylow p -subgroups of N and G/N are cyclic for a prime p . Then for the principal blocks we have $k(B_G) \leq k(B_{G/N})|N|_p$.*

Proof. Let P be a Sylow p -subgroup of G . Then $P \cap N$ and $P/P \cap N \cong PN/N$ are Sylow subgroups of N and G/N respectively. By hypothesis, these groups are cyclic and therefore P is metacyclic. If B_G is nilpotent, then G is p -nilpotent and $B_{G/N}$ must be nilpotent too. In this case the well-known theorem by Broué-Puig [4] says that $k(B_G) = k(P) \leq k(P/P \cap N)|P \cap N| = k(B_{G/N})|N|_p$. Thus, we may assume that B is non-nilpotent.

Now we deal with the case where P is non-abelian. If $p = 2$, then [26, Theorem 8.1] shows that P is a dihedral group, a semidihedral group or a quaternion group. This implies that $|P/P \cap N| = 2$ and $k(B_G) \leq |P| = k(B_{G/N})|N|_p$ follows from [26, Corollary 8.2]. Hence, let $p > 2$. Here [26, Theorem 8.8] shows that P is a split extension of two cyclic groups. From the proof of this result, we see further that $N_G(P)$ acts non-trivially on P' . Hence, $N_G(P)$ acts non-trivially on $P \cap N$ as well. Moreover, the focal subgroup $G' \cap P$ is cyclic. This implies that $N_G(P)$ acts trivially on $P/P \cap N$ (see [26, proof of Theorem 8.8]). Since $N_G(PN/N)$ permutes the Sylow p -subgroups of PN , the Frattini argument implies that $N_G(PN/N) = N_G(P)N$. Therefore, $N_G(PN/N)$ acts trivially on PN/N and $B_{G/N}$ is nilpotent. Again the claim follows from $k(B_G) \leq |P|$.

Finally, it remains to handle the abelian groups P . We may assume that $1 < P \cap N < P$. Since $N_G(P)$ normalizes $P \cap N$, it also normalizes a non-trivial subgroup of $\Omega(P) = \{x \in P : x^p = 1\}$. By Maschke's Theorem, $\Omega(P)$ decomposes into a direct sum of $N_G(P)$ -invariant cyclic subgroups. We show that this is also true for P . By [11, Theorem 5.2.2], we may assume that P is a direct product of two isomorphic cyclic groups in order to show this claim. Choose an element $x \in P$ of maximal order such that $\Omega(\langle x \rangle)$ is $N_G(P)$ -invariant. Suppose that there

exists $g \in N_G(P)$ such that $gxg^{-1} \notin \langle x \rangle$. Then Burnside's Basis Theorem shows $P = \langle x, gxg^{-1} \rangle$ and we derive the contradiction

$$\Omega(\langle x \rangle) = \Omega(\langle x \rangle) \cap g\Omega(\langle x \rangle)g^{-1} = \Omega(\langle x \rangle) \cap \Omega(\langle gxg^{-1} \rangle) = \Omega(\langle x \rangle \cap \langle gxg^{-1} \rangle) = 1.$$

Hence, we may assume that $\langle x \rangle$ is $N_G(P)$ -invariant. By [11, Theorem 3.3.2], $\langle x \rangle$ contains an $N_G(P)$ -invariant complement in P .

Thus, let $P = \langle u \rangle \times \langle v \rangle$ with $N_G(P)$ -invariant subgroups $\langle u \rangle$ and $\langle v \rangle$ (not necessarily of the same order). Let $P \cap N = \langle u^i v^j \rangle$ for some $i, j \in \mathbb{Z}$. Since $P/P \cap N$ is cyclic, so is $P/\langle u^i, v^j \rangle$. Thus, we may assume that $j = 1$ without loss of generality. Let $l = |N_G(P)/C_G(u)|$. Then l is the inertial index of $B_{G/N}$. Hence, by Dade [6] (see also [26, Theorem 8.6]), we have $k(B_{G/N}) = \frac{|P/P \cap N| - 1}{l} + l$. Now let b_u be the principal block of $C_G(u)$, so that (u, b_u) is a B_G -subsection. Then b_u dominates the principal block $\overline{b_u}$ of $C_G(u)/\langle u \rangle$ with cyclic defect group $P/\langle u \rangle \cong \langle v \rangle$. By Proposition 3.2, all irreducible Brauer characters of b_u are stable under $N_G(\langle u \rangle)$. Hence, Theorem 2.6 (applied with the now familiar quadratic form corresponding to the Dynkin diagram of type A) implies

$$k(B_G) \leq \left(\frac{|\langle u \rangle| - 1}{l} + l \right) |\langle v \rangle|.$$

It suffices to show that

$$\left(\frac{|\langle u \rangle| - 1}{l} + l \right) |\langle v \rangle| \leq \left(\frac{|P/P \cap N| - 1}{l} + l \right) |P \cap N|.$$

This reduces easily to $|\langle v \rangle| \leq |P \cap N|$ which is true. The claim follows. \square

The arguments of the proof of Theorem 4.1 can be used to verify the conjecture for many groups with metacyclic Sylow p -subgroups P . However, at the moment not all of these groups can be handled. For instance, imagine that $P \cong Z_{p^2}^2$ and $P \cap N \cong Z_p^2$. In these cases, the claim would follow from Alperin's Weight Conjecture. By [26, Section 13.1], it is not hard to prove the conjecture in case $p = 2$ and $|P| \leq 16$. We have also used GAP [9] to check the conjecture for all groups in the libraries of small groups and perfect groups.

5 Concluding remarks

In view of Example 2.2 and Theorem 3.1, it appears that the inequalities for $k(B)$ get better if not all irreducible Brauer characters are stable. This seems to be related to the fact that the number of simple modules of a twisted group algebra is not larger than the number for the corresponding untwisted group algebra. Consequently, one might hope that Theorem 2.6 also holds without the hypothesis on the Brauer characters. It would then be a direct generalization of both Propositions 1.1 and 1.2. The case $p = 2$ has already been done in [26, Theorem 5.2]. The odd prime case seems much harder to prove and we did not find a counterexample either, but we can provide a partial answer which generalizes [26, Proposition 5.13].

Proposition 5.1. *Let B be a p -block of a finite group G . Let (u, b_u) be a major B -subsection such that b_u dominates a block $\overline{b_u}$ of $C_G(u)/\langle u \rangle$. Suppose that $l(b_u) = 2$, and let $\overline{C_u} = (c_{ij})$ be the Cartan matrix of $\overline{b_u}$ up to basic sets. Let $r := |N_G(\langle u \rangle, b_u)/C_G(u)|$. Then*

$$k(B) \leq \left(\frac{|\langle u \rangle| - 1}{r} + r \right) (c_{11} + c_{22} - c_{12}).$$

Proof. By Theorem 2.6, we may assume that $N_G(\langle u \rangle, b_u)$ interchanges the two irreducible Brauer characters of b_u . We suppose first that $\overline{C_u}$ is the actual Cartan matrix (not up to basic sets). Then it follows that $c_{11} = c_{22}$. Since the determinant of $\overline{C_u}$ is a p -power, we have

$$(c_{11} - c_{12})(c_{11} + c_{12}) = c_{11}c_{22} - c_{12}^2 = \det(\overline{C_u}) = p^d$$

for some $d \in \mathbb{N}$. Let $c_{11} - c_{12} = p^{d_1}$ and $c_{11} + c_{12} = p^{d_2}$. Then $c_{11} = (p^{d_1} + p^{d_2})/2$ and $c_{12} = (p^{d_2} - p^{d_1})/2$. Setting $\alpha := (p^{d_2 - d_1} - 1)/2$ we have

$$\overline{C_u} = p^{d_1} \begin{pmatrix} \alpha + 1 & \alpha \\ \alpha & \alpha + 1 \end{pmatrix}.$$

Hence, $p^{-d_1}\overline{C}_u$ is the Cartan matrix of the principal block of $Z_{p^{d_2-d_1}} \rtimes Z_2$. So we are basically in the situation of Lemma 3.4 and Theorem 2.1 implies

$$\begin{aligned} k(B) &\leq p^{d_1} \left(\frac{|\langle u \rangle| - 1}{r} \frac{p^{d_2-d_1} - 1}{2} + 2 \frac{|\langle u \rangle| - 1}{r} + r \frac{p^{d_2-d_1} - 1}{2} + \frac{r}{2} \right) \\ &= p^{d_1} \left(\frac{|\langle u \rangle| - 1}{r} \frac{p^{d_2-d_1} + 3}{2} + \frac{rp^{d_2-d_1}}{2} \right) < \left(\frac{|\langle u \rangle| - 1}{r} + r \right) (c_{11} + c_{22} - c_{12}) \end{aligned} \quad (5.1)$$

(cf. Lemma 3.5, we omit the details). Note that r must be even. Now it remains to show that the right hand side of (5.1) does not get smaller if we change the basic set. For

$$U := \begin{pmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{pmatrix} \begin{cases} \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{cases} \begin{matrix} c_{11} - c_{12} \\ \\ c_{12} \\ \\ c_{22} - c_{12} \end{matrix}$$

we have $U^T U = \overline{C}_u$ and U has just $c_{11} + c_{22} - c_{12}$ rows. Suppose that $S \in \text{GL}(2, \mathbb{Z})$, and let $S^T \overline{C}_u S = (c'_{ij})$. Then the matrix US still has $c_{11} + c_{22} - c_{12}$ (non-zero) rows and satisfies $(US)^T (US) = (c'_{ij})$. An application of the quadratic form corresponding to the Dynkin diagram of type A_2 (as in Theorem 2.6) shows that $c_{11} + c_{22} - c_{12} \leq c'_{11} + c'_{22} - c'_{12}$ and we are done. \square

In practice, if \overline{C}_u and \mathcal{N} are known and the action on $\text{IBr}(b_u)$ is unknown, one can consider all possible actions, apply Theorem 2.1 in each case and take the bound from the worst case.

We finish the paper with some more remarks:

- (i) The idea of Theorem 3.1 goes back to Kiyota [14]. On page 40 he considers the case $p = 3$ and $D \rtimes I(B) \cong \mathfrak{S}_3^2$ where b_u has two irreducible Brauer characters which are interchanged by $N_G(\langle u \rangle, b_u)$. Using the integral basis $1, e^{2\pi i/3}$ he obtains the matrix $\begin{pmatrix} 5 & 1 \\ 1 & 2 \end{pmatrix}$. In comparison, we use the integral basis $e^{2\pi i/3}, e^{-2\pi i/3}$ and get the equivalent matrix $\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$ in Lemma 3.5. This is, by the way, the Cartan matrix of the generalized dihedral group $Z_3^2 \rtimes Z_2$. It is a bit harder to find such non-trivial examples for $p = 2$. Gluck [10, Question 1 and Section 2] gave an example with $|D| = 32$.
- (ii) Whenever we have a major B -subsection (u, b_u) such that \overline{b}_u has cyclic defect groups, it is obvious that the defect group D of B is abelian of rank at most 2. Hence, Theorem 3.1 covers a large portion of these cases. However, in principle it is possible to do similar things for non-major subsections. The details are necessarily more complicated. In particular, one needs to combine Lemmas 2.5 and 3.4. Since we do not know any interesting applications, we chose to stick to major subsections.
- (iii) Brauer's permutation lemma (see [8, Lemma IV.6.10]) asserts in most cases that the number of integral columns of the generalized decomposition matrix of a block B coincides with the number of p -rational irreducible characters of B . Thus, if we have some knowledge about the fusion system of B , we can obtain lower bounds for the number of integral decomposition numbers $d_{\chi\varphi}^u$ where (u, b_u) is a fixed subsection. This has consequences for the possible factorizations $M = Q^T Q$ in Theorem 2.1. Additional information can be taken into account by using the Broué-Puig $*$ -construction in order to compare the contributions of distinct subsections. We have successfully applied these arguments in [29, Lemma 1].
- (iv) Judging from the examples we have considered, it seems that the non-zero elementary divisors of the matrix M in Theorem 2.1 are always p -powers. We do not know if these numbers have interesting interpretations. At least, M does not seem to be related to the Cartan matrix of $B_u = b_u^{N_G(\langle u \rangle, b_u)}$ (see Example 2.2). Moreover, none of the inequalities

$$k(B) \leq k(b_u), \quad k(B) \leq |\langle u \rangle| k(\overline{b}_u), \quad k(B) \leq k(B_u)$$

is true in general. Counterexamples are given by the principal 3-blocks of the groups `SmallGroup(108, 17)`, `SmallGroup(324, 117)` and \mathfrak{S}_3^2 respectively (see [9]).

- (v) The Cartan matrix of (the principal block of) $G = \mathfrak{A}_4^2$ is given by $C = (c_{ij}) = (1 + \delta_{ij})_{i,j=1}^3 \otimes (1 + \delta_{ij})_{i,j=1}^3$. We have shown in [26, end of Section 4.1] that there is no positive definite, integral quadratic form $q(x) = \sum_{1 \leq i \leq j \leq 9} q_{ij} x_i x_j$ such that

$$\sum_{1 \leq i \leq j \leq 9} q_{ij} c_{ij} = 16 = k(G). \quad (5.2)$$

Now let q be the tensor product of the quadratic form corresponding to the Dynkin diagram of type A_3 with itself. Then q is not integral anymore, but by Lemma 2.3, q can be used to bound $k(B)$ and it turns out that (5.2) holds. Therefore, we really benefit from giving up integrity. In fact, we do not know any Cartan matrix where this method fails to give Brauer's $k(B)$ -Conjecture $k(B) \leq |D|$.

- (vi) Thinking more general, one can try to replace subsections by subpairs (Q, b_Q) where $Q \leq D$ and b_Q is a Brauer correspondent of B in $C_G(Q)Q$. This seems a bit out of reach for the following reason. The block b_D is nilpotent and has Cartan matrix $(|D|)$. Hence, a general version of Proposition 1.1 would already imply Brauer's $k(B)$ -Conjecture for all blocks.

Acknowledgment

This work is supported by the German Research Foundation (project SA 2864/1-1) and the Daimler and Benz Foundation (project 32-08/13). The author thanks Gabriel Navarro for the hospitality at the University of Valencia and for showing him his conjecture. The author is also grateful to Britta Späth and Burkhard Külshammer for answering some questions concerning Proposition 3.2.

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