Block Theory of Finite Groups – Research Report

Benjamin Sambale

February 21, 2012

1 Introduction

Following R. Brauer, the group algebra of a finite group G over a field of characteristic p (or a complete discrete valuation ring of residue characteristic p) splits into blocks. This leads to a distribution of the irreducible (ordinary and Brauer) characters of G into blocks. For a block B, $k(B)$ denotes the number of irreducible ordinary characters of G associated with B, and $l(B)$ denotes the number of irreducible Brauer characters of G associated with B. Many of the central open problems in representation theory are concerned with these numbers. For example, Alperin's Weight Conjecture [1] relates $l(B)$ to the number of B-weights. The number $k(B)$ appears in Brauer's $k(B)$-Conjecture [2] which predicts $k(B) \le |D|$ where D is a defect group of B.

It is therefore an interesting task to determine the block invariants $k(B)$ and $l(B)$ with respect to a fixed defect group. Here it is often useful to study the heights of the irreducible characters. For an irreducible character χ of a block B with defect group D the height of χ is the largest integer $h(\chi) \ge 0$ such that $p^{h(\chi)}|G : D|_p$ divides $\chi(1)$. The number of characters of height i is denoted by $k_i(B)$.

2 Block invariants

In my PhD thesis 2010, I determined the block invariants of 2-blocks with metacyclic defect groups [16]. It turned out that these numbers only depend on the fusion system of the block (this was independently obtained by Craven-Glesser [4]). The following result relies on preliminary work of Puig-Usami [12].

Theorem 1. *Let B be a 2-block of a finite group G with a metacyclic defect group D. Then one of the following holds:*

(i) B is nilpotent. Then $k_i(B)$ is the number of ordinary characters of D of degree 2^i. In particular $k(B)$ is the number of conjugacy classes of D and $k_0(B) = |D : D'|$. Moreover, $l(B) = 1$.

(ii) D has maximal class. Then Theorem 3 below applies.

(iii) D is a direct product of two isomorphic cyclic groups. Then $k(B) = k_0(B) = \frac{|D|+8}{3}$ and $l(B) = 3$.

It follows easily that the major counting conjecture are satisfied in this case.

Later in collaboration with Charles Eaton and Burkhard Külshammer, I obtained the block invariants of 2-blocks with minimal nonabelian defect groups [17, 5]. Here minimal nonabelian means that all proper subgroups are abelian, but the whole group is not. Rédei gave a classification of the minimal nonabelian p-groups [13]. We use the notation $[x, y] := xyx^{-1}y^{-1}$ and $[x, x, y] := [x, [x, y]]$.

Theorem 2. *Let B be a 2-block of a finite group G with a minimal nonabelian defect group D. Then one of the following holds:*

(i) B is nilpotent. Then $k(B) = \frac{5}{8}|D|$, $k_0(B) = \frac{1}{2}|D|$, $k_1(B) = \frac{1}{8}|D|$ and $l(B) = 1$.

(ii) $|D| = 8$. Then Theorem 3 applies.

(iii) $D \cong \langle x, y \mid x^{2^r} = y^2 = [x, y]^2 = [x, x, y] = [y, x, y] = 1\rangle$ for some $r \ge 2$. Then $k(B) = 5 \cdot 2^{r-1}$, $k_0(B) = 2^{r+1}$, $k_1(B) = 2^{r-1}$ and $l(B) = 2$.

(iv) $D \cong \langle x, y \mid x^{2^r} = y^{2^r} = [x, y]^2 = [x, x, y] = [y, x, y] = 1\rangle$ for some $r \ge 2$. Then B is Morita equivalent to the group algebra of $D \rtimes E$ where E is a subgroup of $\mathrm{Aut}(D)$ of order 3. In particular, $k(B) = \frac{5 \cdot 2^{2r-2}+16}{3}$, $k_0(B) = \frac{2^{2r}+8}{3}$, $k_1(B) = \frac{2^{2r-2}+8}{3}$ and $l(B) = 3$.

The last possibility in this theorem gives an example of Donovan's Conjecture.

In recent papers [19, 15, 14], I was also able to handle 2-blocks with defect group $M \times C_{2^m}$ or $M * C_{2^m}$. Here M is a 2-group of maximal class, C_{2^m} is a cyclic group of order 2^m and $M * C_{2^m}$ denotes the central product. Moreover, D_{2^n} (resp. Q_{2^n}, SD_{2^n}) is the dihedral (resp. quaternion, semidihedral) group of order 2^n. The following result generalizes work by Brauer [3] and Olsson [10].

Theorem 3. *Let B be a nonnilpotent 2-block of a finite group G with defect group D, and let $m \geq 0$.*

(i) If $D \cong D_{2^n} \times C_{2^m}$ for some $n \geq 3$, then $k(B) = 2^m(2^{n-2}+3)$, $k_0(B) = 2^{m+2}$ and $k_1(B) = 2^m(2^{n-2}-1)$. According to two different fusion systems, $l(B)$ is 2 or 3.

(ii) If $D \cong Q_8 \times C_{2^m}$ or $D \cong Q_8 * C_{2^{m+1}}$, then $k(B) = 2^m \cdot 7$, $k_0(B) = 2^{m+2}$ and $k_1(B) = 2^m \cdot 3$ and $l(B) = 3$.

(iii) If $D \cong Q_{2^n} \times C_{2^m}$ or $D \cong Q_{2^n} * C_{2^{m+1}}$ for some $n \geq 4$, then $k_0(B) = 2^{m+2}$ and $k_1(B) = 2^m(2^{n-2}-1)$. According to two different fusion systems, one of the following holds

(a) $k(B) = 2^m(2^{n-2}+4)$, $k_{n-2}(B) = 2^m$ and $l(B) = 2$.

(b) $k(B) = 2^m(2^{n-2}+5)$, $k_{n-2}(B) = 2^{m+1}$ and $l(B) = 3$.

(iv) If $D \cong SD_{2^n} \times C_{2^m}$ for some $n \geq 4$, then $k_0(B) = 2^{m+2}$ and $k_1(B) = 2^m(2^{n-2}-1)$. According to three different fusion systems, one of the following holds

(a) $k(B) = 2^m(2^{n-2}+3)$ and $l(B) = 2$.

(b) $k(B) = 2^m(2^{n-2}+4)$, $k_{n-2}(B) = 2^m$ and $l(B) = 2$.

(c) $k(B) = 2^m(2^{n-2}+4)$, $k_{n-2}(B) = 2^m$ and $l(B) = 3$.

Notice that $Q_{2^n} * C_{2^m} \cong D_{2^n} * C_{2^m} \cong SD_{2^n} * C_{2^m}$ for $m \geq 2$. It should be pointed out that also the invariants for the defect group $D_4 \times C_{2^m}$ and $D_4 * C_{2^m}$ are known by work of Puig-Usami [12] and Kessar-Koshitani-Linckelmann [7].

These theorems together with one half of Brauer's Height Zero Conjecture (which was proved recently by Kessar-Malle [8]) imply that the invariants of 2-blocks with defect at most 4 are known in almost all cases. Here, only for a block with elementary abelian defect group of order 16 and inertial index 15 it is not clear to my knowledge if Alperin's Weight Conjecture holds (see [9]).

3 Conjectures

In the last two years I also made progress on some of the open conjectures in representation theory.

Theorem 4. *Brauer's $k(B)$-Conjecture holds for defect groups which contain a central, cyclic subgroup of index at most 9.*

Theorem 5. *Let B be a block with a defect group which is a central extension of a group Q of order 16 by a cyclic group. If Q is not elementary abelian or if 9 does not divide the inertial index of B, then Brauer's $k(B)$-conjecture holds for B.*

As a corollary one gets Brauer's $k(B)$-Conjecture for the 3-blocks of defect at most 3 and most 2-blocks of defect at most 5 (see [18]).

Another related conjecture was proposed by Olsson [11]: For a block B with defect group D it holds that $k_0(B) \leq |D : D'|$ where D' is the commutator subgroup of D. In a joint work with Lászlo Héthelyi and Burkhard Külshammer, I verified Olsson's Conjecture under certain hypotheses [6].

Theorem 6. *Let $p > 3$. Then Olsson's Conjecture holds for all p-blocks with defect groups of p-rank 2 and for all p-blocks with minimal non-abelian defect groups.*

More detailed information is available if one involves the notion of subsections. A subsection for the block B is a pair (u, b_u) where u is p-element of G and b_u is a Brauer correspondent of B in $\mathrm{C}_G(u)$. If b_u and B have the same defect, the subsection is called major.

Theorem 7. *Let B be a p-block of a finite group G where p is an odd prime, and let (u, b_u) be a B-subsection such that $l(b_u) = 1$ and b_u has defect d. Moreover, let $\mathcal{F}$ be the fusion system of B and $|\mathrm{Aut}_{\mathcal{F}}(\langle u\rangle)| = p^s r$, where $p \nmid r$ and $s \geq 0$. Then we have*

$$k_0(B) \leq \frac{|\langle u\rangle| + p^s(r^2-1)}{|\langle u\rangle| \cdot r} p^d. \tag{1}$$

If (in addition) (u, b_u) is major, we can replace $k_0(B)$ by $\sum_{i=0}^{\infty} p^{2i} k_i(B)$ in (1).

References

[1] J. L. Alperin, *Weights for finite groups*, in The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986), 369–379, Proc. Sympos. Pure Math., Vol. 47.1, Amer. Math. Soc., Providence, RI, 1987.

[2] R. Brauer, *Representations of finite groups*, in Lectures on Modern Mathematics, Vol. I, 133–175, Wiley, New York, 1963.

[3] R. Brauer, *On* 2*-blocks with dihedral defect groups*, in Symposia Mathematica, Vol. XIII (Convegno di Gruppi e loro Rappresentazioni, INDAM, Rome, 1972), 367–393, Academic Press, London, 1974.

[4] D. A. Craven and A. Glesser, *Fusion systems on small p-groups*, arXiv:1007.1639v1.

[5] C. W. Eaton, B. Külshammer and B. Sambale, 2*-Blocks with minimal nonabelian defect groups II*, J. Group Theory (to appear), DOI: 10.1515/JGT.2011.110.

[6] L. Héthelyi, B. Külshammer and B. Sambale, *A note on Olsson's Conjecture*, submitted.

[7] R. Kessar, S. Koshitani and M. Linckelmann, *Conjectures of Alperin and Broué for* 2*-blocks with elementary abelian defect groups of order* 8, J. Reine Angew. Math. (to appear), DOI: 10.1515/CRELLE.2011.

[8] R. Kessar and G. Malle, *Quasi-isolated blocks and Brauer's height zero conjecture*, arXiv:1112.2642v1.

[9] B. Külshammer and B. Sambale, *The* 2*-blocks of defect* 4, submitted.

[10] J. B. Olsson, *On* 2*-blocks with quaternion and quasidihedral defect groups*, J. Algebra **36** (1975), 212–241.

[11] J. B. Olsson, *On the number of characters in blocks of finite general linear, unitary and symmetric groups*, Math. Z. **186** (1984), 41–47.

[12] L. Puig and Y. Usami, *Perfect isometries for blocks with abelian defect groups and cyclic inertial quotients of order* 4, J. Algebra **172** (1995), 205–213.

[13] L. Rédei, *Das „schiefe Produkt“ in der Gruppentheorie*, Comment. Math. Helv. **20** (1947), 225–264.

[14] B. Sambale, *Blocks with central product defect group $D_{2^n} * C_{2^m}$*, Proc. Amer. Math. Soc. (to appear), arXiv:1105.4977v1.

[15] B. Sambale, *Blocks with defect group $Q_{2^n} \times C_{2^m}$ and $SD_{2^m} \times C_{2^m}$*, submitted.

[16] B. Sambale, *Fusion systems on metacyclic* 2*-groups*, Osaka J. Math. (to appear), arXiv:0908.0783v3.

[17] B. Sambale, 2*-Blocks with minimal nonabelian defect groups*, J. Algebra **337** (2011), 261–284.

[18] B. Sambale, *Cartan matrices and Brauer's $k(B)$-conjecture II*, J. Algebra **337** (2011), 345–362.

[19] B. Sambale, *Blocks with defect group $D_{2^n} \times C_{2^m}$*, J. Pure Appl. Algebra **216** (2012), 119–125.