Character counting conjectures for π-separable groups

Abstract

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Many of the open conjectures in modular representation theory of finite groups are known to be true for p-solvable groups where p is the relevant prime. Richard Brauer and others have tried to replace p by a set of prime π. A convincing theory of π-blocks was eventually developed by Slattery for the family of π-separable groups. Here a finite group G is called π-separable if every composition factor of G is a π-group or a π'-group. Moreover, a π-block of G is a minimal non-empty subset $B \subseteq \mathrm{Irr}(G)$ such that B is a union of p-blocks for every $p \in \pi$. Note that $\{p\}$-separable is p-solvable and a $\{p\}$-block is a p-block. As in the original theory, let $k(B) := |B|$. Using a variant of the Fong-Reynolds Theorem, Slattery defined defect groups D of B by induction on $|G|$. In this framework it is natural to ask which of the open conjectures still hold for π-blocks. For instance, Brauer's Height Zero Conjecture and the Alperin-McKay Conjecture were proved for B above by Manz–Staszewski and Wolf respectively. In 2017, I verified Brauer's $k(B)$-Conjecture for B (stating that $k(B) \le |D|$) which was put forward previously by Y. Liu. The proof is reduced to a non-abelian $k(GV)$-Theorem which answers a question by Pálfy and Pyber. In a second paper, I proved Brauer Problem 21 for π-blocks which states that there exists a function $f : \mathbb{N} \to \mathbb{N}$ (independent of π, B or D) such that $|D| \le f(k(B))$. This extends the corresponding result for p-solvable groups by Külshammer. It also generalizes the classical theorem of Landau that there are only finitely many finite groups with a given class number. Finally, in a joint paper with Gabriel Navarro, we proved in 2018 a version of Alperin's Weight Conjecture for π-solvable groups. Here G is called π-solvable if the composition factors of G are π'-groups or solvable π-groups. Moreover, a π-weight of G is a pair (P, ψ) where P is a nilpotent π-subgroup of G and $\psi \in \mathrm{Irr}(\mathrm{N}_G(P)/P)$ satisfies $\psi(1)_\pi = |\mathrm{N}_G(P)/P|_\pi$. We showed that the number of conjugacy classes of π-weights of G equals the number of π-regular conjugacy classes of G. As an interesting and perhaps surprising special case one recovers Carter's theorem that every solvable group has exactly one conjugacy class of selfnormalizing nilpotent subgroups. In my talk I also proposed a (groupwise) version of Dade's conjecture which seems to hold for any π-separable group.

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