Local determination of Frobenius–Schur indicators

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It is a difficult problem to decide whether a real-valued character χ of a finite group G can be afforded by a real representation. Although this information is encoded by the *Frobenius–Schur indicator* (F-S indicator)

$$\epsilon(\chi) := \frac{1}{|G|}\sum_{g\in G}\chi(g^2)$$

(being 1 if and only if χ comes from a real representation), it cannot, for instance, be read off from the character table of G (as D_8 and Q_8 witness). On the other hand, the F-S indicators of characters in a given block B are, as always, influenced by a defect group of B. Motivated by John Murray's results on cyclic defect groups, I obtained the number of real characters in nilpotent blocks (this number is not invariant under Morita equivalence).

Theorem 1. *Let B be a real, nilpotent p-block of a finite group G with defect group D. Let b_D be a Brauer correspondent of B in $D\mathrm{C}_G(D)$. Then the number of real characters in $\mathrm{Irr}(B)$ of height h coincides with the number of characters $\lambda \in \mathrm{Irr}(D)$ of degree p^h such that $\lambda^t = \overline{\lambda}$ where*

$$\mathrm{N}_G(D, b_D)^*/D\mathrm{C}_G(D) = \langle tD\mathrm{C}_G(D)\rangle.$$

If $p > 2$, then all real characters in $\mathrm{Irr}(B)$ have the same F-S indicator.

Here $\mathrm{N}_G(D, b_D)^* := \{g \in \mathrm{N}_G(D) : b_D^g \in \{b_D, \overline{b_D}\}\}$ is the *extended* stabilizer of b_D. For $p = 2$, Rod Gow and John Murray have introduced the *extended* defect group E of B such that $|E : D| = 2$ unless B is the principal block. It seems that the pair (D, E) fully determines the F-S indicators in nilpotent blocks as follows:

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Conjecture 2. *Let B be a real, nilpotent, non-principal 2-block of a finite group G with defect pair (D,E). Then there exists a height preserving bijection $\Gamma : \mathrm{Irr}(D) \to \mathrm{Irr}(B)$ such that*

$$\boxed{\epsilon(\Gamma(\lambda)) = \frac{1}{|D|}\sum_{e\in E\setminus D}\lambda(e^2)}$$

for all $\lambda \in \mathrm{Irr}(D)$.

Theorem 3. *Conjecture 2 holds in each of the following cases:*

(i) D is abelian or a dihedral group.

(ii) G is solvable or quasisimple.

In general, Conjecture 2 is implied by the following local conjecture:

Conjecture 4. *Let B be a real, non-principal 2-block with defect pair (D,E). Let (x,b) be a B-subsection with defect pair $(\mathrm{C}_D(x), \mathrm{C}_E(x))$ such that b has a unique projective indecomposable character Φ. Then for every $y \in G$ with $y^2 = x$, we have*

$$[\Phi_{\mathrm{C}_G(y)}, 1_{\mathrm{C}_G(y)}] = |y^{\mathrm{C}_G(x)} \cap E \setminus D|,$$

where $y^{\mathrm{C}_G(x)}$ denotes the conjugacy class of y in $\mathrm{C}_G(x)$.

A consequence of Conjecture 4 is an interesting formula involving the generalized decomposition numbers $d^x_{\chi\varphi}$ where $\mathrm{IBr}(b) = \{\varphi\}$:

$$\sum_{\chi\in\mathrm{Irr}(B)} \epsilon(\chi) d^x_{\chi\varphi} = |\{y \in E \setminus D : y^2 = x\}|.$$

Murray has also investigated blocks with dihedral defect groups. Using similar idea as above, I excluded two families appearing in his classification.

Theorem 5. *Let B be real block of a finite group G with dihedral defect group D of order $2^d \geq 8$ and extended defect group E. Let $\epsilon_1, \ldots, \epsilon_4$ be the F-S indicators of the four irreducible characters of height 0 in B. There is a unique family of 2-conjugate characters of height 1 in $\mathrm{Irr}(B)$ of size 2^{d-3}. Let μ be the common F-S indicator of those characters. The possible values for $\epsilon_1, \ldots, \epsilon_4, \mu$ are given as follows, while the remaining $2^{d-3} - 1$ characters (of height 1) all have F-S indicator 1:*

Morita equivalence class	$l(B)$	E	$\epsilon_1,\ldots,\epsilon_4;\mu$
D (*nilpotent*)	1	$D,\ D\times C_2$	$1,1,1,1;1$
		$D*C_4$	$1,1,1,1;-1$
		$D_{2^{d+1}}$	$0,0,1,1;1$
		$SD_{2^{d+1}}$	$0,0,1,1;-1$
		$C_{2^{d-1}}\rtimes C_2^2,\ d\ge 4$	$1,1,1,1;0$
$\mathrm{PGL}(2,q),\ \lvert q-1\rvert_2=2^{d-1}$	2	$D,\ D\times C_2$	$1,1,1,1;1$
		$C_{2^{d-1}}\rtimes C_2^2,\ d\ge 4$	$1,1,1,1;0$
$\mathrm{PGL}(2,q),\ \lvert q+1\rvert_2=2^{d-1}$	2	$D,\ D\times C_2$	$1,1,1,1;1$
$\mathrm{PSL}(2,q),\ \lvert q-1\rvert_2=2^{d}$	3	$D,\ D\times C_2$	$1,1,1,1;1$
		$D_{2^{d+1}}$	$0,0,1,1;1$
		$SD_{2^{d+1}}$	$0,0,1,1;-1$
		$C_{2^{d-1}}\rtimes C_2^2,\ d\ge 4$	$1,1,1,1;0$
$\mathrm{PSL}(2,q),\ \lvert q+1\rvert_2=2^{d}$	3	$D,\ D\times C_2$	$0,0,1,1;1$
		$D_{2^{d+1}}$	$1,1,1,1;1$
$A_7,\ d=3$	3	$D,\ D\times C_2$	$1,1,1,1;1$

All cases occur for all d as indicated.

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References

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