Groups with supersolvable automorphism group

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Abstract

We call a finite group G ultrasolvable if it has a characteristic subgroup series whose factors are cyclic. It was shown by Durbin–McDonald that the automorphism group of an ultrasolvable group is supersolvable. The converse statement was established by Baartmans–Woeppel under the hypothesis that G has no direct factor isomorphic to the Klein four-group. We extend this result by proving that $\mathrm{Aut}(G)$ is supersolvable if and only if G is ultrasolvable or $G = H \times C_2 \times C_2$ where H is ultrasolvable of odd order. This corrects an erroneous claim by Corsi Tani. Our proof is more elementary than Baartmans–Woeppel's and uses some ideas of Corsi Tani and Laue.

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1 Introduction

A finite group G is solvable if and only if it has a subnormal series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n = G$$

such that the factors G_i/G_{i-1} are cyclic for $i = 1, \ldots, n$. Moreover, G is called supersolvable if there is a analogous *normal* series (i. e. $G_i \trianglelefteq G$ for $i = 1, \ldots, n$) with cyclic factors. It is therefore natural to investigate groups G with a characteristic series (i. e. $\alpha(G_i) = G_i$ for all $\alpha \in \mathrm{Aut}(G)$ and $i = 1, \ldots, n$) with cyclic factors. Durbin–McDonald [4] have called these groups c.c.s. groups (*characteristic cyclic series*), but we like to call them *ultrasolvable*. It was shown in [4, Theorem 1] that the automorphism group of an ultrasolvable group is supersolvable. We provide an elementary proof in the next section for the convenience of the reader (Theorem 3).

The characteristically simple Klein four-group $V \cong C_2 \times C_2$ with supersolvable automorphism group isomorphic to the symmetric group S_3 shows that the converse does not hold in general. Nevertheless, Durbin–McDonald have conjectured that the converse holds under the hypothesis that G has no direct factor isomorphic to V. This was eventually proven by Baartmans–Woeppel [1, Theorem C] relying on deep theorems of Baer [2]. Corsi Tani [3, Theorema 3] showed that a p-group $P \not\cong V$ is ultrasolvable if and only if $\mathrm{Aut}(P)$ is supersolvable. (A more precise description of the automorphism group in this case has been obtained by Lakatos [8].) It is claim in [3, footnote (**) on p. 106] (and in its MathSciNet

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review MR0670814) that the same result holds for arbitrary finite groups not isomorphic of V. But this is plainly false with $G = C_6 \times C_2$ being a counterexample ($\mathrm{Aut}(G) \cong S_3 \times C_2$).

The aim of this paper is the following characterization of groups with supersolvable automorphism group.

Theorem 1. *For every finite group G, the following statements are equivalent:*

(1) $\mathrm{Aut}(G)$ *is supersolvable.*

(2) G is ultrasolvable or $G \cong H \times C_2 \times C_2$ where H is ultrasolvable of odd order.

The proof is mostly self-contained and makes use of ideas of Corsi Tani and Laue [9].

2 A-solvable groups

Our notation is standard and follows Kurzweil–Stellmacher's book [7]. From now on, G will always be a finite group.

Definition 2. A group A acts on G via a homomorphism $A \to \mathrm{Aut}(G)$ (in most cases we consider $A \le \mathrm{Aut}(G)$ with the embedding homomorphism). A subnormal series $1 = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n = G$ is called an *A-series* if each G_i is A-invariant, i. e. $\alpha(G_i) = G_i$ for all $\alpha \in A$. If G has an A-series with cyclic factors G_i/G_{i-1} for $i = 1, \ldots, n$, then G is called *A-solvable*.

Note that G is 1-solvable, $\mathrm{Inn}(G)$-solvable or $\mathrm{Aut}(G)$-solvable if and only if G is solvable, supersolvable or ultrasolvable respectively ($\mathrm{Inn}(G)$ denotes the inner automorphism group of G). An A-series of G with cyclic factors can be refined to an A-series with factors of prime order, because subgroups of cyclic groups are characteristic. This will often be used in the following. In the usual manner, one verifies that A-invariant subgroups of A-solvable groups are A-solvable. The same holds for quotients by A-invariant normal subgroups.

Our first theorem is slightly more general than [4, Theorem 1], while its proof is more economic.

Theorem 3 (Durbin–McDonald)**.** *Let G be A-solvable for some $A \le \mathrm{Aut}(G)$. Then A is supersolvable.*

Proof. Let $1 = G_0 \trianglelefteq \ldots \trianglelefteq G_n = G$ be an A-series with cyclic factors. Let $B \trianglelefteq A$ be the kernel of the action of A on G_{n-1}. Let $C \trianglelefteq A$ be the kernel of the action of A on G/G_1. By induction on $|G|$, we may assume that $A/B \le \mathrm{Aut}(G_{n-1})$ and $A/C \le \mathrm{Aut}(G/G_1)$ are supersolvable. Then $A/(B \cap C) \le A/B \times A/C$ is supersolvable. It suffices to show that $D := B \cap C$ is cyclic. Let $G_1 = \langle x \rangle$ and $G/G_{n-1} = \langle y \rangle G_{n-1}$. We choose $\alpha \in D \setminus \{1\}$ with $\alpha(x) = xy^a$ such that $a \ge 0$ is as small as possible. Since α is uniquely determined by $\alpha(x)$, we must have $a \ge 1$. Let $\beta \in D$ be arbitrary with $\beta(x) = xy^b$. By Euclidean division, there exist $q, r \in \mathbb{Z}$ such that $b = qa + r$ and $0 \le r < a$. Since

$$(\beta^{-1}\alpha^q)(x) = \beta^{-1}(x)y^{qa} = xy^{qa-b} = xy^r,$$

it follows that $r = 0$ and $\beta = \alpha^q$. This shows that $D = \langle \alpha \rangle$. □

The next lemma is well-known, but we include a proof for sake of completeness. As usual, $\mathrm{O}_\pi(G)$ denotes the largest normal π-subgroup of G.

Lemma 4 (Sylow tower property)**.** *Let G be supersolvable and $n \in \mathbb{N}$. Let π be the set of primes $p \geq n$. Then G has a normal Hall π-subgroup. In particular, G is 2-nilpotent, i. e. G has a normal 2-complement.*

Proof. We argue by induction on $|G|$. Let $N \trianglelefteq G$ be a minimal normal subgroup. Then N is a chief factor and therefore $q := |N|$ is a prime. By induction G/N has a normal Hall π-subgroup $K/N \leq G/N$. If $q \geq n$, then K is a normal Hall π-subgroup of G. Thus, let $q < n$. Then N has a complement L in K by the Schur–Zassenhaus theorem. Since $|\mathrm{Aut}(N)| = q - 1$ has no prime divisor in π, we obtain $N \leq \mathrm{Z}(K)$ and $K = N \times L$. It follows that $L = \mathrm{O}_\pi(K) = \mathrm{O}_\pi(G)$ is a normal Hall π-subgroup of G. The second claim follows with $n = 3$. □

The following result generalizes [3, Lemma 1] and is related to the uniqueness in the Krull–Remak–Schmidt theorem (see [6, I.12.6]).

Lemma 5. *Let $G = A \times B$ with $\gcd(|A/A'|, |\mathrm{Z}(B)|) \neq 1$. Then A is not characteristic in G.*

Proof. By hypothesis, there exists a non-trivial homomorphism $\varphi : A \to A/A' \to \mathrm{Z}(B)$. It is easy to check that $\psi : G \to G$, $(a, b) \mapsto (a, \varphi(a)b)$ is an automorphism with $\psi(A) \neq A$. □

Corollary 6. *Let $P = A \times B$ be a p-group such that A is characteristic in P. Then $A = 1$ or $B = 1$.*

Now we prove [3, Lemma 2], which is a consequence of Laue [9, Satz 3].

Lemma 7. *For every non-abelian p-group P, we have $\mathrm{O}_{p'}(\mathrm{Aut}(P)) = 1$.*

Proof. Let $A := \mathrm{O}_{p'}(\mathrm{Aut}(P))$. Then the normal p-subgroup $P/\mathrm{Z}(P) \cong \mathrm{Inn}(P) \trianglelefteq \mathrm{Aut}(P)$ is centralized by A. It follow that $[P, A] \leq \mathrm{Z}(P) < P$, since P is non-abelian. From the theory of coprime actions (see [7, 8.2.7]), we obtain $P = [P, A]\mathrm{C}_P(A)$ and $[P, A] = [P, A, A] \leq [\mathrm{Z}(P), A]$. Since $\mathrm{Z}(P)$ is abelian, we further know that

$$[P, A] \cap \mathrm{C}_P(A) = [\mathrm{Z}(P), A] \cap \mathrm{C}_{\mathrm{Z}(P)}(A) = 1$$

by [7, 8.4.2]. Therefore, $P = [P, A] \times \mathrm{C}_P(A)$ is a decomposition into characteristic subgroups, because $A \trianglelefteq \mathrm{Aut}(P)$. Corollary 6 implies $P = \mathrm{C}_P(A)$ and $A = 1$. □

3 Strictly p-closed groups

The following definition goes back to Baer [2].

Definition 8. Let p be a prime. Then G is called *strictly p-closed* if G has a normal Sylow p-subgroup P such that G/P is abelian of exponent dividing $p - 1$, i. e. $x^{p-1} \in P$ for all $x \in G$.

It is a routine exercise to show that subgroups and quotients of strictly p-closed groups are strictly p-closed. Moreover, G is strictly p-closed if and only if $G/\mathrm{O}_p(G)$ is strictly p-closed (see also Lemma 14 below). In particular, every p-group is strictly p-closed, and for 2-groups the converse is also true.

Lemma 9. *Let G be strictly p-closed. If G acts irreducibly on an elementary abelian p-group V, then $|V| = p$.*

Proof. We may assume that G acts faithfully on V. Let P be the unique Sylow p-subgroup of G. By orbit counting, we have $U := \mathrm{C}_V(P) \neq 1$. Since $P \trianglelefteq G$, U is G-invariant. Since V is irreducible, $U = V$ and $P = 1$. Hence, G is abelian of exponent dividing $p-1$. By Schur's lemma, G is cyclic, say $G = \langle x \rangle$ (see [7, 8.3.3 or 8.6.1]). Let $f \in \mathrm{GL}(V)$ be the linear map induced by x. Since $x^{p-1} = 1$, the minimal polynomial of f divides $X^{p-1}-1$. It follows that f has an eigenvalue in $\mathbb{F}_p^\times$. A corresponding eigenvector generates a G-invariant subspace of dimension 1. Therefore, $|V| = p$ (this also follows directly from [7, 8.6.1(b)]). □

Now we are in a position to prove an extension of [3, Teorema 8(b)], which is related to [2, Theorem 2.1]. As usual, we denote the Frattini subgroup of G by $\Phi(G)$.

Theorem 10. *Let P be a p-group and $A \leq \mathrm{Aut}(P)$. Then the following statements are equivalent:*

(i) P is A-solvable.

(ii) $P/\Phi(P)$ is A-solvable.

(iii) A is strictly p-closed.

Proof. Suppose first that P is A-solvable. Then the normal series $\Phi(P) \leq P$ can be refined to an A-series $\Phi(P) = P_0 \trianglelefteq P_1 \trianglelefteq \ldots \trianglelefteq P_n = P$ such that $|P_i/P_{i-1}| = p$ for $i = 1, \ldots, n$ using the Jordan–Hölder theorem for operator groups (see [7, 1.8.1]). Hence, $P/\Phi(P)$ is A-solvable. Now given the A-series as above, A acts on $\bigtimes_{i=1}^n P_i/P_{i-1}$ fixing each factor. The kernel $B \trianglelefteq A$ of this action is a p-group by a theorem of Burnside (see [7, 8.2.9]). Since

$$A/B \leq \bigtimes_{i=1}^{n} \mathrm{Aut}(P_i/P_{i-1}) \cong C_{p-1}^n$$

is abelian of exponent dividing $p-1$, we conclude that A is strictly p-closed.

Suppose conversely that A is strictly p-closed. By standard group theory,

$$\Omega := \Omega(\mathrm{Z}(P)) := \{z \in \mathrm{Z}(P) : z^p = 1\}$$

is a characteristic non-trivial elementary abelian p-subgroup of P. By Lemma 9, there exists an A-invariant subgroup $Q \leq \Omega$ of order p. By induction on $|P|$, P/Q is A-solvable and so is P. □

Since "most" p-groups have no non-trivial p'-automorphisms, there is no hope to classify ultrasolvable (p-)groups. Concrete examples are the p-groups of maximal nilpotency class and order $\geq p^4$ (see [6, Hilfssätze III.14.2, III.14.4]). The following corollary is not needed in the sequel, but interesting on its own.

Corollary 11. *A 2-group P is ultrasolvable if and only if $\mathrm{Aut}(P)$ is 2-group.*

Corollary 12 (Baer)**.** *Every strictly p-closed group is supersolvable.*

Proof. Let G be strictly p-closed with normal Sylow p-subgroup P. Then P is G-solvable by Theorem 10. Hence, G is supersolvable. □

We obtain a partial converse of Corollary 12.

Lemma 13. *Let G be a supersolvable group such that $\mathrm{O}_{p'}(G) = 1$ for some prime p. Then G is strictly p-closed.*

Proof. By Lemma 4, G has a normal Sylow p-subgroup P (and no other normal Sylow subgroups). By the Schur–Zassenhaus theorem (or Hall's theorem for solvable groups), P has a complement K in G. Since $\mathrm{O}_{p'}(G) = 1$, K acts faithfully on P. Moreover, P is a K-solvable group, because G is supersolvable. By Theorem 10, K is strictly p-closed and so must be G. □

It is easy to see that the condition $\mathrm{O}_{p'}(G) = 1$ is not fulfilled by strictly p-closed groups in general. In order to obtain an equivalent characterization, we recall the notation $\mathrm{O}_{pp'}(G)/\mathrm{O}_p(G) := \mathrm{O}_{p'}(G/\mathrm{O}_p(G))$.

Lemma 14. *For a supersolvable group G the following assertions are equivalent:*

(1) G is strictly p-closed.

(2) $\mathrm{O}_{pp'}(G)$ is strictly p-closed.

(3) $\mathrm{O}_{pp'}(G)/\mathrm{O}_p(G)$ is abelian of exponent dividing $p-1$.

Proof. It is clear that (1) implies (2) and (2) implies (3). Now assume (3). If p does not divide $|G|$, then $G = \mathrm{O}_{p'}(G) = \mathrm{O}_{pp'}(G)$ is strictly p-closed by hypothesis. Hence, we may assume that p divides $|G|$. By Lemma 4, G has a normal Sylow q-subgroup Q, where q is the largest prime divisor of $|G|$. Moreover, $Q \le \mathrm{O}_{pp'}(G)$ and therefore $q = p$. Since $G/\mathrm{O}_p(G) = G/Q$ is a p'-group, it follows again that $G = \mathrm{O}_{pp'}(G)$ is strictly p-closed. □

Lemma 15 (Durbin–McDonald). *Let $P = C_{p^{a_1}} \times \ldots \times C_{p^{a_n}}$ be an abelian p-group with $1 \le a_1 \le \ldots \le a_n$. Then $\mathrm{Aut}(P)$ is supersolvable if and only if $a_1 < \ldots < a_n$ or $P = C_2 \times C_2$.*

Proof. If $P \cong C_2 \times C_2$, then $\mathrm{Aut}(P) \cong \mathrm{GL}(2,2) \cong S_3$ is supersolvable. Suppose next that $a_1 < \ldots < a_n$. Let $A := \mathrm{Aut}(P)$. By Theorem 10 and Corollary 12, it suffices to show that $P/\Phi(P) \cong C_p^n$ is A-solvable. Define characteristic subgroups

$$P_i := \{x \in P : x^{p^{a_i}} = 1\}\Phi(P) \le P$$

for $i = 1, \ldots, n$. Then $P_0 := \Phi(P) < P_1 < \ldots < P_n = P$ and $|P_i : P_{i-1}| = p$ for $i = 1, \ldots, n$. So we are done.

Conversely, let $a_k = a_{k+1}$ for some k. Since $\mathrm{Aut}(C_p^{a_k} \times C_p^{a_k})$ is a subgroup of A, we may assume that $P \cong C_{p^a} \times C_{p^a}$ in order to show that A is not supersolvable. Let $P = \langle x, y\rangle$ and $x', y' \in P$ such that $\{x'\Phi(P), y'\Phi(P)\}$ is a basis of the elementary abelian group $P/\Phi(P) \cong C_p \times C_p$. Recall that Burnside's basis theorem implies $P = \langle x', y'\rangle$. It is easy to check that there exists an automorphism $\gamma \in A$ such that $\gamma(x) = x'$ and $\gamma(y) = y'$. This shows that the restriction map

$$\Gamma : A \to \mathrm{Aut}(P/\Phi(P)) \cong \mathrm{GL}(2, p)$$

is surjective. If $p \ge 5$, then $\mathrm{GL}(2, p)$ (and A in turn) is not even solvable. For $p = 3$, $\mathrm{GL}(2, 3) \cong Q_8 \rtimes C_3$ is not supersolvable by Lemma 4, for instance. Finally, let $p = 2$ and $a \ge 2$. By Burnside's theorem mentioned before, the kernel of Γ is a 2-group. Thus, $|A| = 2^s 3$ for some $s \geq 1$. It is easy to check that the maps $\alpha, \beta : P \to P$ with

$$\begin{aligned} \alpha(x) &= y, & \alpha(y) &= (xy)^{-1}, \\ \beta(x) &= y^{-1}, & \beta(y) &= xy^{-1} \end{aligned}$$

are automorphisms of order 3 and $\langle \alpha \rangle \neq \langle \beta \rangle$. In particular, A does not have a normal Sylow 3-subgroup. By Lemma 4, A is not supersolvable. □

We end this section by proving the converse of Theorem 3 for p-groups.

Theorem 16 (Corsi Tani)**.** *Let $P \ncong C_2 \times C_2$ be a p-group such that $\mathrm{Aut}(P)$ supersolvable. Then P is ultrasolvable.*

Proof. Suppose first that P is abelian. Then $P \cong C_{p^{a_1}} \times \ldots \times C_{p^{a_n}}$ with $a_1 < \ldots < a_n$ by Lemma 15. In fact, we have shown in the proof of Lemma 15 that $P/\Phi(P)$ is A-solvable, where $A := \mathrm{Aut}(P)$. Hence, P is ultrasolvable by Theorem 10. Now assume that P is non-abelian. By Theorem 10, it suffices to show that $\mathrm{Aut}(P)$ is strictly p-closed. But this follows from Lemma 7 and Lemma 13. □

It is straight-forward to deduce a characterization of supersolvable nilpotent groups from Theorem 16, but this will be generalized by our main theorem.

4 Proof of Theorem 1

Let $V = C_2 \times C_2$ be the Klein four-group. If G is ultrasolvable, then $\mathrm{Aut}(G)$ is supersolvable by Theorem 3. If H is ultrasolvable of odd order, then $\mathrm{Aut}(H \times V) = \mathrm{Aut}(H) \times S_3$ is supersolvable. Now assume conversely that $A := \mathrm{Aut}(G)$ is supersolvable. Then $G/\mathrm{Z}(G) \cong \mathrm{Inn}(G) \leq A$ is supersolvable and so is G.

Case 1: $G = H \times V$ for some $H \leq G$.
Suppose first that $|H|$ is even. By Lemma 4, H is 2-nilpotent. In particular, $\gcd(|H/H'|, |V|) \neq 1$. As in the proof of Lemma 5, we construct an automorphism $\alpha : G \to G$, $(h, v) \mapsto (h, \varphi(h)v)$, where $\varphi : H \to V$ is non-trivial. Since V has exponent 2, α is an involution. Let $\beta \in \mathrm{Aut}(V)$ be of order 3. We extend β to G by $\beta(h) = h$ for all $h \in H$. Then $\beta \in \mathrm{O}_{2'}(A)$ and $\gamma := [\alpha, \beta] = \alpha\beta^{-1}\alpha\beta \in \mathrm{O}_{2'}(A)$ by Lemma 4. We compute

$$\gamma(h, v) = (\alpha\beta^{-1}\alpha)(h, \beta(v)) = (\alpha\beta^{-1})(h, \varphi(h)\beta(v)) = \alpha(h, \beta^{-1}(\varphi(h))v) = (h, \varphi(h)\beta^{-1}(\varphi(h))v)$$

for all $(h, v) \in G$. By construction, there exists $h \in H$ such that $w := \varphi(h) \neq 1$. Then $\beta^{-1}(w) \neq w$ and $w\beta^{-1}(w) \neq 1$. This shows that γ has order 2, which contradicts $\gamma \in \mathrm{O}_{2'}(A)$. Therefore, $|H|$ is odd and the claim follows by induction on $|G|$, because $\mathrm{Aut}(H) \leq A$ is supersolvable.

Case 2: V is not a direct factor of G.
Let p be a prime divisor of $|G|$. Let π be the set of primes $q > p$. Then $N := \mathrm{O}_\pi(G)$ is a normal Hall π-subgroup by Lemma 4. For a Sylow p-subgroup P of G we further have that $NP = \mathrm{O}_{\pi \cup \{p\}}(G)$ is characteristic in G. Arguing by induction on $|G|$ (starting with the largest prime divisor p), it suffices to show that $\overline{P} := PN/N \cong P$ is A-solvable. Equivalently, by Theorem 10, we need that the image $\overline{A}$ of A in $\mathrm{Aut}(\overline{P})$ is strictly p-closed. We prove this via Lemma 14. Let $Z := P \cap \mathrm{Z}(G) \leq \mathrm{Z}(P)$. Since A is supersolvable,

$$\overline{P}/\overline{Z} \cong P\mathrm{Z}(G)/\mathrm{Z}(G) \leq G/\mathrm{Z}(G) \cong \mathrm{Inn}(G)$$

is A-solvable. Let $\overline{A}_p := \mathrm{O}_p(\overline{A})$ and $\overline{B} := \mathrm{O}_{pp'}(\overline{A}) = \overline{A}_p \rtimes \overline{K}$ for some complement $\overline{K}$. We need to show that $\overline{K}$ is abelian of exponent dividing $p-1$. We define the auxiliary group

$$\widehat{B} := \overline{P} \rtimes \overline{B} = (\overline{P} \rtimes \overline{A}_p) \rtimes \overline{K}.$$

Since $\overline{B}$ is supersolvable and $\overline{P}/\overline{Z}$ is $\overline{B}$-solvable, there exists a $\widehat{B}$-series from $\overline{Z}$ to $\overline{P}$ to $\overline{P} \rtimes \overline{A}_p$ in $\widehat{B}$ with factors of order p. Let $\overline{C} \trianglelefteq \overline{K}$ be the kernel of the action of $\overline{K}$ on the direct factors of this series. Then $\overline{K}/\overline{C}$ is abelian of exponent dividing $p-1$. Thus, it suffices to prove that $\overline{C} = 1$. Note that $\overline{C}\mathrm{Z}(\overline{A}_p) = \mathrm{C}_{\overline{B}}(\overline{A}_p) \trianglelefteq \overline{A}$ (using [7, 8.2.2]). In particular, $\overline{C} \trianglelefteq \overline{A}$.

By construction, we have $[\overline{P}, \overline{C}] \le \overline{Z}$. As in the proof of Lemma 5, it follows that

$$\overline{P} = [\overline{P}, \overline{C}] \times \mathrm{C}_{\overline{P}}(\overline{C}).$$

The preimage $P_0 \le Z$ of $[\overline{P}, \overline{C}]$ is a direct factor of P. By Gaschütz' theorem, P_0 has a complement K in G (see [7, 3.3.2]). Since $P_0 \le \mathrm{Z}(G)$, we must have $G = P_0 \times K$. Note that Z is A-invariant as the unique Sylow p-subgroup of $\mathrm{Z}(G)$. Moreover, $[\overline{P}, \overline{C}]$ is $\overline{A}$-invariant since $\overline{C} \trianglelefteq \overline{A}$. It follows that $\alpha(P_0) \subseteq NP_0$ for $\alpha \in A$. Hence, $\alpha(P_0) \subseteq Z \cap NP_0 = (Z \cap N)P_0 = P_0$, i. e. P_0 is A-invariant.

If $P_0 = 1$, then $\overline{C}$ acts trivially on $\overline{P}$ and therefore $\overline{C} = 1$. In this case we are done by Lemma 14 Next, let $P_0 \ne 1$. Since $\mathrm{Aut}(P_0) \le A$ is supersolvable and $P_0 \ncong V$, it follows from Theorem 16 that P_0 is ultrasolvable. Since P_0 is A-invariant, it suffices to show that $G/P_0 \cong K$ is ultrasolvable. Every direct factor of K is also a direct factor of G. So by hypothesis, V is not a direct factor of K. Since $\mathrm{Aut}(K) \le A$ is supersolvable, K is ultrasolvable by induction. This completes the proof.

5 Fully solvable groups

We close this paper with an open problem. Recall that a subgroup $H \le G$ is called *fully invariant* if $\alpha(H) \le H$ for every endomorphism $\alpha : G \to G$. Obviously, fully invariant subgroups are characteristic. It is natural to call a group G *fully solvable* if there exists a series of fully invariant subgroups

$$1 = G_0 \trianglelefteq \ldots \trianglelefteq G_n = G$$

such that the factors G_i/G_{i-1} are cyclic for $i = 1, \ldots, n$. Fully solvable groups are certainly ultrasolvable. For abelian groups the converse is also true, because the characteristic subgroups constructed in the proof of Lemma 15 are fully invariant (noticing that $\Phi(P) = \langle x^p : x \in P \rangle$ for every abelian p-group P).

On the other hand, the ultrasolvable dihedral group D_8 is not fully solvable, since none of the three maximal subgroups is fully invariant. Using the computer algebra system GAP [5], one can show that there are 36 ultrasolvable groups and 22 fully solvable groups of order 32.

Problem 17. *Find a "convenient" characterization of fully solvable groups.*

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