On the size of coset unions

Benjamin Sambale* and Marius Tărnăuceanu†

June 9, 2021

Abstract

Let $g_1H_1, \ldots, g_nH_n$ be cosets of subgroups $H_1, \ldots, H_n$ of a finite group G such that $g_1H_1 \cup \ldots \cup g_nH_n \neq G$. We prove that $|g_1H_1 \cup \ldots \cup g_nH_n| \leq \gamma_n|G|$ where $\gamma_n < 1$ is a constant depending only on n. In special cases we show that $\gamma_n = (2^n - 1)/2^n$ is the best possible constant with this property and we conjecture that this is generally true.

Keywords: union of cosets, subgroup covering
AMS classification: 20D30, 20D60, 05E16

1 Introduction

It is well-known that a finite group G cannot be the union of two proper subgroups. In fact, an easy counting argument shows that such a union covers at most three quarters of the elements of G. It is equally well-known that G is never a union of conjugates of a proper subgroup H. Cameron–Cohen [4] have shown more precisely that there are at least $|H|$ elements outside such a union. On the other hand, it may happen that G is covered by $n \geq 3$ arbitrary proper subgroups $H_1, \ldots, H_n \leq G$. While many authors classified such groups for a given n (the interested reader is referred to the survey [2]), we are interested in the situation where $H_1 \cup \ldots \cup H_n \neq G$. We show that a portion of elements, depending only on n, lies outside this union. In fact, this holds more generally for union of cosets of subgroups. To the authors' knowledge, this has apparently not been observed in the literature. In the first part of the paper we prove more precisely that $|G|/(2n!)$ elements lie outside such a coset union. In the second part we investigate our conjecture that even $|G|/2^n$ elements lie outside the union. For elementary abelian groups we obtain in Theorem 6 the best possible bound of that kind by using a linear algebra approach due to Alon–Füredi [1]. We like to mention that there are other open conjectures on union of cosets such as the long-standing Herzog–Schönheim Conjecture [5].

2 Main result

Theorem 1. *For every positive integer n there exists a constant $\gamma_n < 1$ with the following property: For every finite group G and every n subgroups $H_1, \ldots, H_n \leq G$ and $g_1, \ldots, g_n \in G$ either $g_1H_1 \cup \ldots \cup g_nH_n = G$ or $|g_1H_1 \cup \ldots \cup g_nH_n| \leq \gamma_n|G|$.*

*Institut für Algebra, Zahlentheorie und Diskrete Mathematik, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany, sambale@math.uni-hannover.de
†Faculty of Mathematics, "Al.I. Cuza" University, Iaşi, Romania, tarnauc@uaic.ro

Proof. We argue by induction on n. For $n = 1$ the claim holds with $\gamma_1 = \frac{1}{2}$ by Lagrange's Theorem. Now let $n \geq 2$, $H_1, \ldots, H_n \leq G$ and $g_1, \ldots, g_n \in G$ such that $g_1H_1 \cup \ldots \cup g_nH_n \neq G$. Let $s_i := |G : H_i|$ for $i = 1, \ldots, n$. We may assume that $s_1 \leq \ldots \leq s_n$. Let α_n be the smallest positive integer such that $\gamma_{n-1} + \frac{1}{\alpha_n} < 1$. If $s_n \geq \alpha_n$, then induction yields

$$|g_1H_1 \cup \ldots \cup g_nH_n| \leq |g_1H_1 \cup \ldots \cup g_{n-1}H_{n-1}| + |g_nH_n| \leq \Big(\gamma_{n-1} + \frac{1}{s_n}\Big)|G| \leq \Big(\gamma_{n-1} + \frac{1}{\alpha_n}\Big)|G|.$$

Now let $s_n \leq \alpha_n$ and $H := H_1 \cap \ldots \cap H_n$. Using Poincaré's formula $|G : H_i \cap H_j| \leq |G : H_i||G : H_j|$ repeatedly, we get $|G : H| \leq s_1 \ldots s_n \leq \alpha_n^n$. Since $g_1H_1 \cup \ldots \cup g_nH_n$ is a union of H-cosets, it follows that

$$|g_1H_1 \cup \ldots \cup g_nH_n| \leq |G| - |H| \leq \Big(1 - \frac{1}{\alpha_n^n}\Big)|G|.$$

Hence, the claim holds with

$$\gamma_n := \max\Big\{\gamma_{n-1} + \frac{1}{\alpha_n}, 1 - \frac{1}{\alpha_n^n}\Big\} < 1. \qquad \square$$

The proof of Theorem 1 yields only a very crude bound on γ_n. With some more effort we can prove an effective bound as follows.

Proposition 2. *Theorem 1 holds with* $\gamma_n = \frac{2n!-1}{2n!}$.

Proof. We reuse the notation from the proof of Theorem 1. We already know that the claim holds for $n = 1$. Thus, let $n \geq 2$. If $n + 1 \leq s_1 \leq \ldots \leq s_n$, then

$$|g_1H_1 \cup \ldots \cup g_nH_n| \leq |H_1| + \ldots + |H_n| \leq \frac{n}{n+1}|G| \leq \frac{2n! - 1}{2n!}|G|$$

as desired.

Now let $s_1 \leq n$. Since G is the union of all cosets of H_1, there exists a coset gH_1 such that $gH_1 \nsubseteq g_1H_1 \cup \ldots \cup g_nH_n$. Since $|g_1H_1 \cup \ldots \cup g_nH_n| = |g^{-1}(g_1H_1 \cup \ldots \cup g_nH_n)|$, we may replace g_i by $g^{-1}g_i$ for $i = 1, \ldots, n$. Then $H_1 \nsubseteq g_1H_1 \cup \ldots \cup g_nH_n$ and $H_1 \cap g_1H_1 = \varnothing$. It follows that

$$g_1H_1 \cup \ldots \cup g_nH_n \subseteq (G \setminus H_1) \,\dot\cup\, \bigcup_{i=2}^{n}(g_iH_i \cap H_1).$$

If $g_iH_i \cap H_1 \neq \varnothing$, then $g_iH_i \cap H_1 = h_i(H_i \cap H_1)$ for some $h_i \in H_1$. By induction on n, we conclude that

$$\Big|(G \setminus H_1) \,\dot\cup\, \bigcup_{i=2}^{n}(g_iH_i \cap H_1)\Big| \leq \frac{s_1 - 1}{s_1}|G| + \gamma_{n-1}|H_1| \leq \frac{s_1 + \gamma_{n-1} - 1}{s_1}|G|.$$

Since $\gamma_{n-1} - 1 < 0$, it follows that

$$\frac{s_1 + \gamma_{n-1} - 1}{s_1} \leq \frac{n + \gamma_{n-1} - 1}{n} = \frac{2n! + 2(n-1)! - 1 - 2(n-1)!}{2n!} = \gamma_n$$

as desired. $\square$

In most cases cosets can cover more elements than subgroups. For instance, if G is a p-group, then two distinct cosets of a maximal subgroup cover $\frac{2}{p}|G|$ elements while two distinct maximal subgroups only cover $\frac{2p-1}{p^2}|G|$ elements (see also Theorem 6 below).

In order to compute a lower bound on γ_n, let us consider an elementary abelian 2-group $G = \langle x_1, \ldots, x_n \rangle$ rank n. Let $H_i := \langle x_j : j \neq i \rangle$. Then $H_1 \cup \ldots \cup H_n = G \setminus \{x_1 \ldots x_n\}$ and therefore $\gamma_n \geq (2^n - 1)/2^n$ for all $n \geq 1$. If we restrict ourselves to union of subgroups, we can show that this bound is indeed optimal for small n.

Proposition 3. *For every finite group G and every set of subgroups $H_1, \ldots, H_n \leq G$ with $n \leq 5$ either $H_1 \cup \ldots \cup H_n = G$ or $|H_1 \cup \ldots \cup H_n| \leq \frac{2^n-1}{2^n}|G|$. Equality can only hold if $|G : H_1 \cap \ldots \cap H_n| = 2^n$.*

Proof. We may assume that $n \geq 2$, $H_1 \cup \ldots \cup H_n \neq G$ and $H_i \nsubseteq \bigcup_{j \neq i} H_j$ for $i = 1, \ldots, n$. Let $N := \{1, \ldots, n\}$ and $H_I := \bigcap_{i \in I} H_i$ for $I \subseteq N$. Suppose first that $L := H_{N \setminus \{i\}} \nsubseteq H_i$ for some i, say $i = 1$. Let $U := G \setminus \bigcup_{i=2}^n H_i$. By induction on n we have $|U| \geq |G|/2^{n-1}$. Moreover, U is a union of L-cosets. If $g \in G$ and $x \in gL \cap H_1$, then

$$|gL \cap H_1| = |x(L \cap H_1)| = |L \cap H_1| = |H_N|.$$

Hence, $|U \cap H_1| \leq \sum_{gL \subseteq U} |H_N| = \frac{|U|}{|L|}|H_N|$. It follows that

$$\Big|G \setminus \bigcup_{i=1}^n H_i\Big| = |U \setminus H_1| = |U| - |U \cap H_1| \geq |U|\Big(1 - \frac{1}{|L : H_N|}\Big) \geq \frac{|G|}{2^n}$$

as desired. Equality can only hold if $|U| = |G|/2^{n-1}$ and $|L : H_N| = 2$. In this case, induction yields $|G : L| = 2^{n-1}$ and $|G : H_N| = 2^n$.

Hence, in the following we will assume that $H_{N \setminus \{i\}} \subseteq H_i$ for $i = 1, \ldots, n$. In particular, $n \geq 3$. Since $H_1 \cup \ldots \cup H_n$ is a union of H_N-cosets, we may also assume that $|G : H_N| > 2^n$ as in the proof of Theorem 1. We need to show the strict inequality $|H_1 \cup \ldots \cup H_n| < \frac{2^n-1}{2^n}|G|$. Using

$$|H_1 \cup \ldots \cup H_n| \leq |H_1 \cup \ldots \cup H_{n-1}| + |H_n| - |H_1 \cap H_n| \leq \Big(\gamma_{n-1} + \frac{1}{s_n} - \frac{1}{s_n^2}\Big)|G|$$

and induction, the indices $s_i := |G : H_i|$ can be bounded. In particular, there are only finitely many choices. By using

$$|G : H_I| \;\Big|\; |G : H_{I \cup J}| \leq |H_{I \cap J} : H_I||G : H_J| = \frac{|G : H_I||G : H_J|}{|G : H_{I \cap J}|}$$

for $I, J \subseteq N$, we can enumerate all possible indices $|G : H_I|$ for $I \subseteq N$ by computer. The claim can then be checked with the exclusion-inclusion principle. Note that for $n = 3$ this becomes

$$|H_1 \cup H_2 \cup H_3| = \Big(\frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3} - \frac{2}{|G : H_N|}\Big)|G|$$

where $9 \leq |G : H_N| = |G : H_1 \cap H_2| \leq s_1 s_2$ and $s_1 \leq s_2 \leq s_3$. It is easy to see that this implies $|H_1 \cup H_2 \cup H_3| \leq \frac{7}{9}|G|$ with equality if and only if $s_1 = s_2 = s_3 = 3$. For $n = 4$ we obtain similarly

$$|H_1 \cup \ldots \cup H_4| \leq \Big(\frac{1}{s_1} + \ldots + \frac{1}{s_4} - \frac{1}{s_1 s_2} - \frac{1}{s_1 s_3} - \ldots - \frac{1}{s_3 s_4} + \frac{3}{|G : H_N|}\Big)|G|$$

where $|G : H_N| \geq 17$ (in fact, $|G : H_N| \geq 18$ since $|G : H_N|$ cannot be a prime). It can be seen that this is again strictly less than $\frac{15}{16}|G|$.

Finally, let $n = 5$. Here we first estimate the union of four out of five subgroups. This leaves us with a short list of exceptional cases. In all those cases there exist subgroups $A, B, C, D \in \{H_1, \ldots, H_5\}$ with the following indices

$$|G : A \cap B| = 4, \qquad |G : C \cap D| = |G : C||G : D|,$$
$$|G : A \cap B \cap C| = 2|G : C|, \qquad |G : A \cap B \cap D| = 2|G : D|.$$

From $|G : A \cap B| = 4$ we obtain $|G : A| = |G : B| = 2$ and $A \cap B \trianglelefteq G$ (the proof of Proposition 2 now already yields $\gamma_5 = 31/32$, but we need a strict inequality here). In particular $(A \cap B)C \le G$ with $|(A \cap B)C : A \cap B| = |C : A \cap B \cap C| = 2$. Thus, $A, B, (A \cap B)C$ are the maximal subgroups of G containing $A \cap B$ and therefore, $G = A \cup B \cup (A \cap B)C$. From $|G : A \cap B \cap D| = 2|G : D| = |G : D \cap A| = |G : D \cap B|$ we obtain $D \cap A = D \cap B$. It follows that

$$D = (D \cap A) \cup (D \cap B) \cup (D \cap (A \cap B)C) = (D \cap A) \cup (D \cap (A \cap B)C).$$

Since D is not the union of two proper subgroups, we conclude that $D \subseteq (A \cap B)C$. But also $C \subseteq (A \cap B)C$. Now $G = CD \subseteq (A \cap B)C$, because $|G : C \cap D| = |G : C||G : D|$. Contradiction. □

We remark that the following alternative procedure applies more generally to union of cosets. Note that G acts on $\bigcup_{i=1}^n G/H_i$ by left multiplication. The kernel N of this action is contained in $H_1 \cap \ldots \cap H_n$. Since $g_1H_1 \cup \ldots \cup g_nH_n$ is the union of the cosets in $g_1(H_1/N) \cup \ldots \cup g_n(H_n/N)$, we may replace G by G/N. Then G is isomorphic to a subgroup of a direct product of symmetric groups $\prod_{i=1}^n S_{|G:H_i|}$. In principle we can enumerate those subgroups by computer, but doing so becomes impractical when n is large.

3 Nilpotent groups

In order to extend Proposition 3 to other cases, we provide a reduction theorem for nilpotent groups. Let $\delta_n(G)$ be the largest constant such that $|G \setminus (g_1H_1 \cup \ldots \cup g_nH_n)| \ge \delta_n(G)|G|$ whenever $g_1H_1 \cup \ldots \cup g_nH_n \ne G$. We wish to show that $\delta_n(G) \ge 1/2^n$.

Lemma 4. *Let $n \ge 1$. Suppose that for every p-group P and every $m \le n$ we have $\delta_m(P) \ge 1/2^m$. Then $\delta_n(G) \ge 1/2^n$ for every nilpotent group G.*

Proof. Let G be a nilpotent group. Let $p_1, \ldots, p_k$ be the distinct prime divisors of $|G|$. Let $P_i := \mathrm{O}_{p_i}(G)$ the Sylow p_i-subgroup and $Q_i := \mathrm{O}_{p_i'}(G)$ such that $G = P_1 \times \ldots \times P_k$. Let $g_1H_1, \ldots, g_nH_n$ be cosets of subgroups of G such that $g_1H_1 \cup \ldots \cup g_nH_n \ne G$. Suppose that $|G : H_1|$ is divisible by p_i and p_j with $i \ne j$. Let $K := H_1\mathrm{N}_{P_i}(H_1)$ and $L := H_1\mathrm{N}_{P_j}(H_1)$ Then $g_1H_1 = g_1(K \cap L) = g_1K \cap g_1L$ and $g_1K \cup g_2H_2 \cup \ldots \cup g_nH_n \ne G$ or $g_1L \cup g_2H_2 \cup \ldots \cup g_nH_n \ne G$. Thus, we may replace H_1 by K or L respectively. Since every subgroup of G is subnormal, we may continue in this way until $|G : H_1|$ is a prime power. We repeat this process with H_i for $i = 2, \ldots, k$. Then every H_i contains a unique Q_j.

Let $\mathcal{H}_i := \{H_j : Q_i \subseteq H_j\}$ for $i = 1, \ldots, k$. Then $\{H_1, \ldots, H_n\}$ is the disjoint union of $\mathcal{H}_1, \ldots, \mathcal{H}_k$. In particular, $n = |\mathcal{H}_1| + \ldots + |\mathcal{H}_k|$. Moreover, an element $(x_1, \ldots, x_k) \in P_1 \times \ldots \times P_k$ does not lie in $g_1H_1 \cup g_2H_2 \cup \ldots \cup g_nH_n$ if and only if x_iQ_i does not lie in $\bigcup_{H_j \in \mathcal{H}_i} g_j(H_j/Q_i)$ for $i = 1, \ldots, k$. If we regard H_j/Q_i as subgroups of $P_i \cong G/Q_i$, it follows that

$$|G \setminus (g_1H_1 \cup \ldots \cup g_nH_n)| \ge \prod_{i=1}^k \delta_{|\mathcal{H}_i|}(P_i)|P_i| \ge \prod_{i=1}^k \frac{|P_i|}{2^{|\mathcal{H}_i|}} = \frac{1}{2^n}|G|. \qquad \square$$

Unfortunately, we are unable to prove $\delta_n(P) \geq 1/2^n$ for p-groups in general. Nevertheless, we provide an optimal bound for elementary abelian p-groups by making use of combinatorial theorems of Alon–Füredi [1] (see also [6]). The following variant of the Schwartz–Zippel Lemma is an explicit version of [1, Theorem 5].

Lemma 5. *Let p be a prime and let $\alpha \in \mathbb{F}_p[X_1, \ldots, X_k]$ be a polynomial of total degree $d = a + b(p-1)$ where $0 \leq a \leq p-2$. If α does not vanish identically on $\mathbb{F}_p^k$, then α is non-zero on at least $p^{k-b-1}(p-a)$ points of $\mathbb{F}_p^k$. This bound is best possible for $d \leq k(p-1)$.*

Proof. We argue by induction on k. Without loss of generality we may assume that $k > b$. If $k = 1$, then α has at most $d = a$ roots in $\mathbb{F}_p$, so it is non-zero on at least $p - a$ points. Now let $k \geq 2$. By Fermat's little theorem, $x^p = x$ for all $x \in \mathbb{F}_p$. Hence, we can reduce all powers of X_1 such that the degree of α in X_1 is at most $p - 1$. This might decrease d, so the bound will be even stronger. For $x \in \mathbb{F}_p$ let

$$\gamma_x := \alpha(x, X_2, \ldots, X_k) \in \mathbb{F}_p[X_2, \ldots, X_k].$$

Let $C \subseteq \mathbb{F}_p$ be the set of $x \in \mathbb{F}_p$ such that γ_x does not vanish identically on $\mathbb{F}_p^{k-1}$. By hypothesis, $C \neq \varnothing$. Let $p' := p - |C|$. Let

$$\alpha = \alpha_1 X_1^{p-1} + \alpha_2 X_1^{p-2} + \ldots + \alpha_p$$

with $\alpha_i \in \mathbb{F}_p[X_2, \ldots, X_k]$ and $\deg(\alpha_i) \leq d - p + i$ for $i = 1, \ldots, p$. We arrange the elements of $\mathbb{F}_p^{k-1}$ in some fixed order, say $\mathbb{F}_p^{k-1} = \{v_1, v_2, \ldots, v_{p^{k-1}}\}$, and define $\overline{\alpha_i} := (\alpha_i(v_1), \alpha_i(v_2), \ldots, \alpha_i(v_{p^{k-1}}))^{\mathrm{t}} \in \mathbb{F}_p^{p^{k-1} \times 1}$. For $x \in \mathbb{F}_p \setminus C$ we obtain a linear equation $x^{p-1}\overline{\alpha_1} + x^{p-2}\overline{\alpha_2} + \ldots + \overline{\alpha_p} = 0$. The Vandermonde matrix $A := (x^i : i = 0, \ldots, p'-1, x \in \mathbb{F}_p \setminus C)$ is invertible and

$$(\overline{\alpha_p}, \overline{\alpha_{p-1}}, \ldots, \overline{\alpha_{|C|+1}})A = -(x^{p-1}\overline{\alpha_1} + \ldots + x^{p'}\overline{\alpha_{|C|}} : x \in \mathbb{F}_p \setminus C).$$

Therefore, we can express the vectors $\overline{\alpha_{|C|+1}}, \ldots, \overline{\alpha_p}$ as linear combinations of $\{x^{p-1}\overline{\alpha_1} + \ldots + x^{p'}\overline{\alpha_{|C|}} : x \in \mathbb{F}_p \setminus C\}$. Hence, we may replace each α_i with $|C| < i \leq p$ by a linear combination of $\alpha_1, \ldots, \alpha_{|C|}$ without changing the values on $\mathbb{F}_p^{k-1}$. Eventually, $\deg(\alpha_i) \leq d - p'$ for all i and $\deg(\gamma_x) \leq d - p'$ for $x \in C$. By induction, γ_x is non-zero on at least $p^{k-b'-2}(p-a')$ points of $\mathbb{F}_p^{k-1}$ where $d - p' = a' + b'(p-1)$ with $0 \leq a' \leq p-2$. Consequently, α is non-zero on at least

$$|C|p^{k-b'-2}(p-a') = p^{k-b'-2}(p-a')(p-p')$$

points of $\mathbb{F}_p^k$.

Suppose first that $p' \leq a$. Then $a' = a - p'$ and $b' = b$. It follows that

$$p^{k-b-2}(p-a+p')(p-p') \geq p^{k-b-2}(p-a+p')(p-a) \geq p^{k-b-1}(p-a)$$

and we are done. Now let $a < p' \leq p-1$. Then $a' = a - p' + p - 1$ and $b' = b - 1$. Since $(p'-a)(p-p') \geq p' - a$, we obtain $(p'-a+1)(p-p') \geq p - a$. This yields $p^{k-b'-2}(p-a')(p-p') \geq p^{k-b-1}(p-a)$ as desired.

To see that the bound is best possible, just consider

$$\alpha = \prod_{i=1}^{b}(X_i^{p-1} - 1)\prod_{j=1}^{a}(X_{b+1} - j)$$

where j is interpreted as $1 + \ldots + 1 \in \mathbb{F}_p$ (j summands). □

Theorem 6. *Let G be an elementary abelian p-group, $H_1, \ldots, H_n \leq G$ and $g_1, \ldots, g_n \in G$ such that $g_1H_1 \cup \ldots \cup g_nH_n \neq G$. Let $n = a + b(p-1)$ where $0 \leq a \leq p-2$. Then*

$$|g_1H_1 \cup \ldots \cup g_nH_n| \leq \frac{p^{b+1} - p + a}{p^{b+1}}|G| \leq \frac{2^n - 1}{2^n}|G|$$

and the first inequality is best possible.

Proof. We regard G as the $\mathbb{F}_p$-vector space $\mathbb{F}_p^k$. Each coset g_iH_i is the set of solutions of a linear system $A_ix = b_i$. By hypothesis, there exists $x \in G \setminus (g_1H_1 \cup \ldots \cup g_nH_n)$. For each i we choose a row a_i of A_i such that $a_ix \neq \beta_i$ where $\beta_i \in \mathbb{F}_p$ is the corresponding entry of b_i. Then the polynomial

$$\alpha(X_1, \ldots, X_k) := \prod_{i=1}^{n}(a_i(X_1, \ldots, X_k)^{\mathrm{t}} - \beta_i) \in \mathbb{F}_p[X_1, \ldots, X_k]$$

of degree n does not vanish on x. By Lemma 5, α is non-zero on at least $p^{k-b-1}(p-a) = \frac{p-a}{p^{b+1}}|G|$ points of G. All these points lie outside of $g_1H_1 \cup \ldots \cup g_nH_n$. This implies the first inequality. For the second, we may assume that $a = p-2$ and $b+1 = \frac{n-a}{p-1} + 1 = \frac{n+1}{p-1}$. It suffices to show that $2^{n+1} \geq p^{b+1}$, i.e. $(n+1)\log_p(2) \geq \frac{n+1}{p-1}$. This is true since $2^{p-1} \geq p$.

In order to show that the first inequality is optimal, we choose $H_1 = \ldots = H_{p-1}$ as a maximal subgroup of G and $g_1, \ldots, g_{p-1} \in G$ such that $G \setminus H_1 = g_1H_1 \cup \ldots \cup g_{p-1}H_1$. Similarly, choose $H_p = \ldots = H_{2p-1}$ as a maximal subgroup of H_1 such that $H_1 \setminus H_p = g_pH_p \cup \ldots \cup g_{2p-1}H_p$ and so on. This will certainly yield the exact bound. □

We remark that Theorem 6 extends to arbitrary finite p-groups as long as $n \leq 2p-2$. To see this, consider $g_1H_1 \cup \ldots \cup g_nH_n \neq G$ where G is a finite p-group. If all $H_1, \ldots, H_n$ are maximal subgroups of G, then, by the remark at the end of Section 2, we can go over to the elementary abelian group $G/\Phi(G)$ where $\Phi(G)$ is the Frattini subgroup of G. In this case the claim follows from Theorem 6. Otherwise we may assume that H_n is not maximal. Then the claim follows by induction on n, because

$$|g_1H_1 \cup \ldots \cup g_nH_n| \leq |g_1H_1 \cup \ldots \cup g_{n-1}H_{n-1}| + |H_n| \leq |g_1H_1 \cup \ldots \cup g_{n-1}H_{n-1}| + \frac{1}{p^2}|G|.$$

On a different note we mention that the subgroup lattice of some (but not all) p-groups can be embedded into the subgroup lattice of an elementary abelian p-group (see [3]). On this basis we suspect that Theorem 6 holds for all p-groups. Moreover, the following general conjecture seems reasonable.

Conjecture 7. *The best possible bound in Theorem 1 is $\gamma_n = (2^n - 1)/2^n$ for all n.*

Acknowledgment

We thank an anonymous reviewer for his/her valuable comments. The first author is supported by the German Research Foundation (SA 2864/1-2 and SA 2864/3-1).

References

[1] N. Alon and Z. Füredi, *Covering the cube by affine hyperplanes*, European J. Combin. **14** (1993), 79–83.

[2] M. Bhargava, *Groups as unions of proper subgroups*, Amer. Math. Monthly **116** (2009), 413–422.

[3] L. M. Butler and A. W. Hales, *Generalized flags in p-groups*, Comput. Math. Appl. **39** (2000), 67–76.

[4] P. J. Cameron and A. M. Cohen, *On the number of fixed point free elements in a permutation group*, Discrete Math. **106/107** (1992), 135–138.

[5] M. Herzog and J. Schönheim, *Research Problem No. 9*, Canad. Math. Bull. **17** (1974), 150.

[6] B. Szegedy, *Coverings of abelian groups and vector spaces*, J. Combin. Theory Ser. A **114** (2007), 20–34.