Determination of block invariants

Morita equivalence problems for blocks of finite groups
CIB Lausanne

Benjamin Sambale*

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1 Local and global invariants

Let G be a finite group, and let p be a prime. We consider a *p-modular system* $(K, \mathcal{O}, F)$ with the following properties:

- $\mathcal{O}$ is a complete discrete valuation ring with valuation ν and field of fractions K,
- K has characteristic 0 and contains a primitive $|G|$-th root of unity,
- $F = \mathcal{O}/J(\mathcal{O})$ is an algebraically closed field of characteristic p.

The group algebra $\mathcal{O}G$ decomposes into a direct sum of indecomposable (twosided) ideals

$$\mathcal{O}G = B_1 \oplus \ldots \oplus B_n.$$

The summands B_i are called the *(p-)blocks* of $\mathcal{O}G$. The natural homomorphism $\mathcal{O} \to F$, $\alpha \mapsto \alpha + J(\mathcal{O})$ induces a bijection between the blocks of $\mathcal{O}G$ and the blocks of FG. In the following we assume that B is a block of RG where $R \in \{\mathcal{O}, F\}$ (whatever is appropriate). Then B is a subalgebra of RG and the unity element 1_B is a primitive idempotent of the center $\mathrm{Z}(RG)$.

Definition 1.1 (Global numerical invariants)**.**

(i) Let $\mathrm{Irr}(B)$ be the set of irreducible characters of G over K belonging to B. We set $k(B) := |\mathrm{Irr}(B)|$. Then the *defect* $d(B) \geq 0$ of B is defined by

$$p^{d(B)} \min\{\chi(1)_p : \chi \in \mathrm{Irr}(B)\} = |G|_p.$$

The *height* $h(\chi)$ of $\chi \in \mathrm{Irr}(B)$ is determined via

$$p^{d(B)-h(\chi)}\chi(1)_p = |G|_p.$$

We set $\mathrm{Irr}_i(B) := \{\chi \in \mathrm{Irr}(B) : h(\chi) = i\}$ and $k_i(B) := |\mathrm{Irr}_i(B)|$ for $i \geq 0$.

(ii) The sets $\mathrm{Irr}_i(B)$ can be partitioned further into families of *p-conjugate* characters. These are the orbits of the Galois group $\mathcal{G}$ of the cyclotomic field extension $\mathbb{Q}_{|G|} \supseteq \mathbb{Q}_{|G|_{p'}}$. The *p-rational* characters are the fixed points under this action. Note that $\mathcal{G} \cong (\mathbb{Z}/|G|_p\mathbb{Z})^\times$.

*Fachbereich Mathematik, TU Kaiserslautern, 67653 Kaiserslautern, Germany, sambale@mathematik.uni-kl.de

(iii) Similarly, one may count the *real* characters in $\mathrm{Irr}(B)$ and determine their *Frobenius-Schur-Indicators*.

(iv) The F-representations of G determine *Brauer characters*. The irreducible Brauer characters $\mathrm{IBr}(G)$ of G can be distributed into the blocks. Accordingly we define $l(B) := |\mathrm{IBr}(B)|$. This is also the number of simple B-modules.

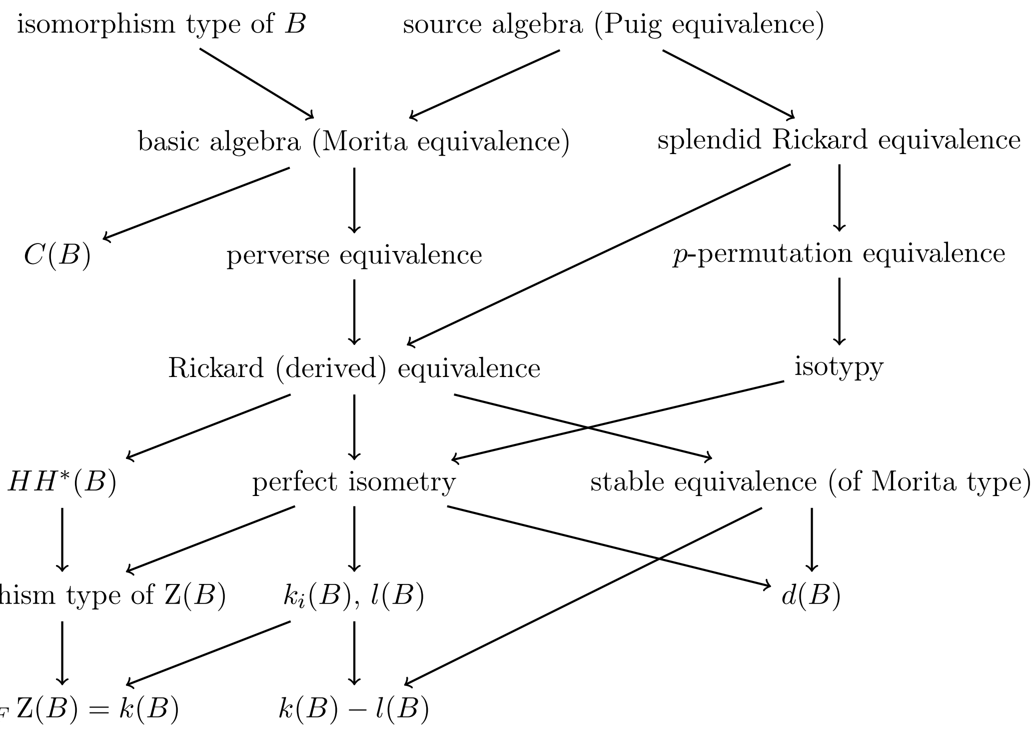
(v) There exist non-negative integers $d_{\chi\varphi}$ such that

$$\chi(g) = \sum_{\varphi \in \mathrm{IBr}(B)} d_{\chi\varphi}\varphi(g)$$

for every $\chi \in \mathrm{Irr}(B)$ and $g \in G_{p'}$. Let $Q := (d_{\chi\varphi}) \in \mathbb{Z}^{k(B)\times l(B)}$ be the *decomposition matrix* of B. Then $C(B) := Q^{\mathrm{T}}Q$ is the *Cartan matrix* of B.

(vi) The *Loewy length* $LL(B)$ of B is the smallest positive integer l such that $J(B)^l = 0$. Also $LL(\mathrm{Z}(B))$ is of interest.

Global structural invariants:



Definition 1.2 (Local data)**.**

(i) Recall that a *defect group* $D = D(B)$ of B is a p-subgroup of G which is unique up to conjugation. Moreover, $|D| = p^{d(B)}$. Let b_D be a *Brauer correspondent* of B in $\mathrm{C}_G(D)$. Then

$$I(B) := \mathrm{N}_G(D, b_D)/D\mathrm{C}_G(D)$$

is the *inertial quotient* of B. A result by Külshammer shows that $B_D := b_D^{\mathrm{N}_G(D)}$ is Morita equivalent to a twisted group algebra of the form $R_\alpha[D \rtimes I(B)]$. The 2-cocycle α is called the *Külshammer-Puig class* of B_D.

(ii) The *fusion system* $\mathcal{F} = \mathcal{F}(B)$ of B is a category with

- objects: subgroups of D,
- morphisms: certain conjugation maps induced by elements in G.

We have $I(B) \cong \mathrm{Out}_{\mathcal{F}}(D)$ and this is p'-group.

It is conjectured that every fusion system of a block is represented by a finite group H with $D \in \mathrm{Syl}_p(H)$.

Philosophy: Local data determine global invariants

Conjectures:

- $k_0(B) = k_0(B_D)$ (Alperin-Mckay)
- $k(B) = k_0(B)$ if and only if D abelian (Brauer)
- $\inf\{i \geq 1 : k_i(B) > 0\} = \inf\{i \geq 1 : k_i(D) > 0\}$ (Eaton-Moretó)
- $l(B)$ is determined by $\mathcal{F}$ and Külshammer-Puig classes of certain Brauer correspondents (Alperin)
- $k_i(B)$ is determined in a similar fashion (Dade, Robinson)
- If D is abelian, then B is (splendid) Rickard equivalent to B_D (Broué)
- There are only finitely many Morita (Puig) equivalence classes of p-blocks with a given defect (Donovan, Puig)

  ⋮

Results:

Theorem 1.3. *If G is p-solvable, then the source algebra of B can be described locally.*

Theorem 1.4.

(i) B is a simple algebra if and only if $D = 1$. In this case $B \cong R^{n\times n}$ where n is the degree of the unique irreducible character of B.

(ii) B has finite representation type *if and only if D is cyclic. In this case the source algebra of B is determined by an endo-permutation module for D and the planar embedding of a Brauer tree.*

(iii) B has tame representation type *if and only if $p = 2$ and D is dihedral, semidihedral or (generalized) quaternion. Here, the Morita equivalence classes are determined by Auslander-Reiten quivers (up to scalars).*

(iv) In all other cases B has wild representation type.

Aim: Say more in the wild case!

2 Methods

Given: D
Wanted: Global invariants of B

2.1 Fusion systems

Theorem 2.1 (Alperin's Fusion Theorem)**.** *The morphisms of $\mathcal{F}$ are compositions of restrictions from* $\mathrm{Aut}(S)$ *where $S = D$ or S is* essential*, i. e.* $\mathrm{Out}_{\mathcal{F}}(S)$ *contains a* strongly p-embedded *subgroup.*

Groups with strongly embedded p-subgroups are classified. Moreover, the automorphism group of a p-group is "almost always" a p-group. In this way one can classify all (saturated) fusion systems on D.

Corollary 2.2. *"Most" blocks are* nilpotent*, i. e. the morphisms of $\mathcal{F}$ are restrictions from* $\mathrm{Inn}(D)$*.*

Theorem 2.3 (Puig)**.** *The block B is nilpotent if and only if $B \cong (\mathcal{O}D)^{n\times n}$ for some $n \geq 1$.*

Example 2.4.

(i) If G has a normal p-complement, then B is nilpotent. The converse holds for the principal block.

(ii) If D is abelian and $I(B) = 1$, then B is nilpotent.

(iii) If D is a cyclic 2-group, then B is nilpotent.

If $p > 2$ and $\mathcal{F}$ is not nilpotent, then there exists a finite group H with $D \in \mathrm{Syl}_p(H)$ and without normal p-complement (not necessarily with the same fusion system).

2.2 Subsections

A *(B)-subsection* is a pair (u, b) where $u \in G_p$ and b is a Brauer correspondent of B in $\mathrm{C}_G(u)$.

Proposition 2.5. *Choose a set $\mathcal{R}$ of representatives for the $\mathcal{F}$-orbits of D such that $|\mathrm{C}_D(u)|$ is as large as possible for every $u \in \mathcal{R}$. Then there exist blocks b_u $(u \in \mathcal{R})$ with the following properties:*

(i) every subsection is G-conjugate to exactly one (u, b_u) with $u \in \mathcal{R}$,

(ii) $D(b_u) = \mathrm{C}_D(u)$ and $\mathcal{F}(b_u) = \mathrm{C}_{\mathcal{F}}(u)$, in particular $I(b_u) \cong \mathrm{C}_{\mathrm{Out}_{\mathcal{F}}(\mathrm{C}_D(u))}(u)$,

(iii) b_u dominates a unique block $\overline{b_u}$ of $\mathrm{C}_G(u)/\langle u\rangle$ with $D(\overline{b_u}) = \mathrm{C}_D(u)/\langle u\rangle$ and $\mathcal{F}(\overline{b_u}) = \mathrm{C}_{\mathcal{F}}(u)/\langle u\rangle$, in particular $I(\overline{b_u}) \cong I(b_u)$,

(iv) $C(b_u) = |\langle u\rangle| C(\overline{b_u})$, in particular $l(b_u) = l(\overline{b_u})$,

(v)

$$\boxed{k(B) = \sum_{u\in\mathcal{R}} l(b_u) = \sum_{u\in\mathcal{R}} l(\overline{b_u}).}$$

Corollary 2.6. *The difference $k(B) - l(B)$ is determined locally.*

2.3 Generalized decomposition numbers

Proposition 2.7. *Let $u \in G_p$ and let $\chi \in \mathrm{Irr}(B)$. Then there are uniquely determined algebraic integers $d^u_{\chi\varphi}$ in the cyclotomic field $\mathbb{Q}_{|\langle u\rangle|}$ such that*

$$\chi(uv) = \sum_{\varphi\in\mathrm{IBr}(\mathrm{C}_G(u))} d^u_{\chi\varphi}\varphi(v) \qquad \forall v \in \mathrm{C}_G(u)_{p'}.$$

The numbers $d^u_{\chi\varphi}$ are called generalized decomposition numbers*.*

Theorem 2.8 (Brauer's Second Main Theorem)**.** *Let $\chi \in \mathrm{Irr}(B)$, $u \in G_p$ and $\varphi \in \mathrm{IBr}(b)$ for some block b of $\mathrm{C}_G(u)$. Then $d^u_{\chi\varphi} = 0$ unless $b^G = B$.*

For $u \in \mathcal{R}$ we write

$$Q_u := (d^u_{\chi\varphi})_{\chi\in\mathrm{Irr}(B),\,\varphi\in\mathrm{IBr}(b_u)} \in \mathcal{O}^{k(B)\times l(b_u)}.$$

Note that $Q_1 = Q$. By choosing an integral basis of $\mathbb{Q}_{|\langle u\rangle|}$, we may replace Q_u by its integral coefficient matrix.

Theorem 2.9 (Orthogonality relations)**.** *For $u, v \in \mathcal{R}$ we have*

$$\boxed{Q_u^{\mathrm{T}}\overline{Q_v} = \delta_{uv}C(b_u).}$$

For $u \in \mathcal{R}$ let

$$M_u := (m^u_{\chi\psi})_{\chi,\psi\in\mathrm{Irr}(B)} = \overline{Q_u}C(b_u)^{-1}Q_u^{\mathrm{T}} = \overline{Q_u}(Q_u^{\mathrm{T}}\overline{Q_u})^{-1}Q_u^{\mathrm{T}} \in \mathbb{C}^{k(B)\times k(B)}$$

be the *contribution matrix* of B with respect to (u, b_u).

Proposition 2.10 (Divisibility relations)**.** *For $u \in \mathcal{R}$ the following holds:*

(i) $\nu\big(p^{d(b_u)}m^u_{\chi\psi}\big) \ge 0$. Equality holds if and only if $h(\chi) = h(\psi) = 0$. In particular, for every $\chi \in \mathrm{Irr}_0(B)$ there exists a $\varphi \in \mathrm{IBr}(b_u)$ such that $d^u_{\chi\varphi} \ne 0$.

(ii) $\nu\big(p^{d(B)}m^u_{\chi\psi}\big) \ge h(\chi)$. Equality holds if and only if $u \in \mathrm{Z}(D)$ and $h(\psi) = 0$. In particular, for every $u \in \mathrm{Z}(D)$ and $\chi \in \mathrm{Irr}(B)$ there exists a $\varphi \in \mathrm{IBr}(b_u)$ such that $d^u_{\chi\varphi} \ne 0$.

Corollary 2.11. *The numbers $k_i(B)$ can be read off from Q_u whenever $u \in \mathrm{Z}(D)$.*

Proof. Pick a $\psi \in \mathrm{Irr}(B)$ with $p^{d(B)}m^u_{\psi\psi} \in \mathcal{O}^\times$. Then $h(\psi) = 0$ and $h(\chi) = \nu\big(p^{d(B)}m^u_{\chi\psi}\big)$ for every $\chi \in \mathrm{Irr}(B)$ by Proposition 2.10. □

Proposition 2.12 (Surjectivity of decomposition)**.** *Let $\widetilde{Q}$ be a matrix whose columns form a basis of the $\mathbb{Z}$-module*

$$\big\{v \in \mathbb{Z}^{k(B)} : Q_u^{\mathrm{T}}v = 0\ \forall u \in \mathcal{R}\setminus\{1\}\big\}.$$

Then there exists $S \in \mathrm{GL}(l(B), \mathbb{Z})$ such that $Q = \widetilde{Q}S$.

Remark 2.13. Arguing by induction on $|D|$, we may assume that $C(\overline{b_u})$ and therefore $C(b_u)$ is known for $1 \ne u \in \mathcal{R}$. By the "integrity" of Q_u (and the Brauer-Feit bound), there are only finitely many solutions of the matrix equation $Q_u^{\mathrm{T}}\overline{Q_u} = C(b_u)$. The solutions can be determined with an algorithm by Plesken (implemented as `OrthogonalEmbeddings` in GAP). Now Proposition 2.12 implies that Q can be computed up to *basic sets* from the Q_u ($u \ne 1$). Here, a basic set is a basis for $\mathbb{Z}\,\mathrm{IBr}(B)$. Note that $C(B) = Q^{\mathrm{T}}Q = S^{\mathrm{T}}\widetilde{Q}^{\mathrm{T}}\widetilde{Q}S$. In particular, the elementary divisors and the determinant of $C(B)$ are encoded in $\widetilde{Q}$. This can be stated more explicitly in terms of *lower defect groups*. We will see later that not all elements $u \in \mathcal{R}\setminus\{1\}$ in Proposition 2.12 are needed. Observe also that the contribution matrices M_u do not depend on the basic sets of b_u (but on the order of $\mathrm{Irr}(B)$).

2.4 Galois actions

Since the matrix factorization $X^{\mathrm{T}}\overline{X} = C(b_u)$ has usually many solutions X, it is of interest to investigate relations between the Q_u's.

The *generalized decomposition matrix* of B is defined by

$$Q_* := \bigl(d^u_{\chi\varphi} : \chi \in \mathrm{Irr}(B),\ u \in \mathcal{R},\ \varphi \in \mathrm{IBr}(b_u)\bigr) = (Q_u : u \in \mathcal{R}) \in \mathcal{O}^{k(B)\times k(B)}.$$

Example 2.14. If $G = D$, then Q_* is just the character table of G.

Proposition 2.15. *The Galois group $\mathcal{G}$ introduced in Definition 1.1(ii) acts on the rows and on the columns of Q_* such that*

$$\boxed{\gamma(d^u_{\chi\varphi}) = d^{u^\gamma}_{\chi\varphi} = d^u_{\chi^\gamma,\varphi}}$$

for $\gamma \in \mathcal{G}$. The number of orbits on both sets is the same. If $p > 2$, then the number of p-rational characters in $\mathrm{Irr}(B)$ coincides with the number of integral columns of Q_*.

Sketch of proof. The equation is a direct consequence of Proposition 2.7. By Brauer's Permutation Lemma, every $\gamma \in \mathcal{G}$ has the same number of fixed points on the rows as on the columns of Q_*. Hence, Burnside's Lemma implies that the number of orbits coincides. Finally, if $p > 2$, then $\mathcal{G}$ is cyclic and the last claim follows. □

If u and u^γ ($\gamma \in \mathcal{G}$) lie in the same $\mathcal{F}$-orbit, then there exists a $g \in \mathrm{N}_G(\langle u\rangle, b_u)$ such that $d^{u^\gamma}_{\chi\varphi} = d^{u^g}_{\chi\varphi} = d^u_{\chi,\varphi^g}$. Thus, γ permutes the columns of Q_u in this case. Here, at least one column is fixed if $p = 2$. Also note that for the computation of Q in Proposition 2.12 we only need Q_u with $u \in \mathcal{R}'$ where $\mathcal{R}'$ is a set of representatives of $\mathcal{R} \setminus \{1\}$ under $\mathcal{G}$.

2.5 Broué-Puig's $*$-construction

Let $\mathbb{Z}\,\mathrm{Irr}(D)^{\mathcal{F}}$ be the $\mathbb{Z}$-module of $\mathcal{F}$-stable generalized characters of D. Then

$$\mathrm{rk}\bigl(\mathbb{Z}\,\mathrm{Irr}(D)^{\mathcal{F}}\bigr) = |D/\mathcal{F}| = |\mathcal{R}|.$$

For $\chi \in \mathrm{Irr}(B)$ and $\lambda \in \mathbb{Z}\,\mathrm{Irr}(D)^{\mathcal{F}}$ there exists a character $\lambda * \chi \in \mathbb{Z}\,\mathrm{Irr}(B)$. It follows that

$$\sum_{u\in\mathcal{R}} \lambda(u) M_u = \bigl((\lambda * \chi, \psi)_G\bigr)_{\chi,\psi\in\mathrm{Irr}(B)} \in \mathbb{Z}^{k(B)\times k(B)}.$$

If $\lambda = 1$, this simplifies to

$$\sum_{u\in\mathcal{R}} M_u = 1$$

(which is also a consequence of Theorem 2.9). If λ is the regular character of D and $\chi \in \mathrm{Irr}_0(B)$, then every $\psi \in \mathrm{Irr}(B)$ is a constituent of $\lambda * \chi$ with multiplicity $p^{d(B)} m^1_{\chi\psi} \ne 0$.

The *hyperfocal subgroup* of B is defined by

$$\mathfrak{hyp}(B) := \langle x^{-1}x^f : x \in S \le D,\ f \in \mathrm{O}^p(\mathrm{Aut}_{\mathcal{F}}(S))\rangle.$$

Moreover, $\mathfrak{foc}(B) := D'\mathfrak{hyp}(B)$ is the *focal subgroup* of B. If $\mathcal{F}$ is realized by a finite group H, then we may use the focal subgroup theorem $\mathfrak{foc}(B) = D \cap H'$. Observe that

$$\overline{D} := D/\mathfrak{foc}(B) \cong \mathrm{Irr}(D/\mathfrak{foc}(B)) \subseteq \mathrm{Irr}(D)^{\mathcal{F}}.$$

Proposition 2.16. *If $\chi \in \operatorname{Irr}_i(B)$ and $\lambda \in \operatorname{Irr}(\overline{D})$, then $\lambda * \chi \in \operatorname{Irr}_i(B)$ with*

$$\boxed{d^u_{\lambda*\chi,\varphi} = \lambda(u) d^u_{\chi\varphi}}$$

for $u \in \mathcal{R}$. This induces an action on the rows of Q_* with the following properties:

(i) $\overline{D}$ acts semiregularly on $\operatorname{Irr}_0(B)$. In particular, $k_0(B) \equiv 0 \pmod{|\overline{D}|}$. The action is regular if and only if B is nilpotent.

(ii) $\overline{\mathrm{Z}(D)}$ acts semiregularly on $\operatorname{Irr}_i(B)$. In particular, $k_i(B) \equiv 0 \pmod{|\overline{\mathrm{Z}(D)}|}$ and $C(B) \equiv 0 \pmod{|\overline{\mathrm{Z}(D)}|}$ for $i \geq 0$.

(iii) $\mathcal{G}$ acts on the set of $\overline{D}$-orbits of $\operatorname{Irr}(B)$.

(iv) The number of $\overline{D}$-orbits is

$$\boxed{|\operatorname{Irr}(B)/\overline{D}| = \sum_{u \in \mathcal{R} \cap \mathfrak{foc}(B)} l(b_u).}$$

Sketch of proof. Once $d^u_{\lambda*\chi,\varphi} = \lambda(u) d^u_{\chi\varphi}$ is proven, the claims about semiregularity follow from Proposition 2.10. A result by Kessar-Linckelmann-Navarro provides the characterization of nilpotent blocks. Part (iii) is easy, since $\mathcal{G}$ acts naturally on $\operatorname{Irr}(\overline{D})$. The last claim is explained in Remark 2.17 below. □

In general $\overline{D}$ does not act on the columns of Q_*.

Remark 2.17. Now we combine the actions of $\mathcal{G}$ and $\overline{D}$. Let $\widehat{\mathcal{G}} := \overline{D} \rtimes \mathcal{G}$, and let $\mathcal{S}$ be a set of representatives for the $\widehat{\mathcal{G}}$-orbits of $\operatorname{Irr}(B)$. For $x \in \mathbb{Q}_{|G|_p}$ let

$$\operatorname{tr}(x) := \frac{1}{|\mathcal{G}|} \sum_{\gamma \in \mathcal{G}} \gamma(x) \in \mathbb{Q}.$$

Define

$$\widehat{Q}_u := \big(|\widehat{\mathcal{G}} : \widehat{\mathcal{G}}_\chi| \operatorname{tr}(d^u_{\chi\varphi}) : \chi \in \mathcal{S},\ \varphi \in \operatorname{IBr}(b_u)\big)$$

for $u \in \mathcal{R}'$. Since Q is constant on the $\widehat{\mathcal{G}}$-orbits, we can recover Q from $\widehat{Q} := (d_{\chi\varphi} : \chi \in \mathcal{S},\ \varphi \in \operatorname{IBr}(B))$. The columns of $\widehat{Q}$ form a basis of the $\mathbb{Z}$-module

$$\big\{v \in \mathbb{Z}^{|\mathcal{S}|} : \widehat{Q}_u^{\mathrm{T}} v = 0\ \forall u \in \mathcal{R}' \cap \mathfrak{foc}(B)\big\}.$$

Example 2.18 (Robinson)**.** If $p \geq 5$, $D \neq 1$ and $I(B) = 1$, then $\overline{D} \neq 1$.

Note that B is nilpotent if and only if $\mathfrak{hyp}(B) = 1$.

Theorem 2.19 (Watanabe)**.** *If $\mathfrak{hyp}(B)$ is cyclic, then $l(B) = |I(B)|$ and $k(B) = k(B_D) = k(D \rtimes I(B))$.*

Let

$$\mathrm{Z}(\mathcal{F}) := \{x \in D : x^f = x \text{ for every morphism } f \text{ in } \mathcal{F}\} \leq D.$$

Proposition 2.20. *For $u \in \mathrm{Z}(\mathcal{F})$ we have $k(B) \geq k(b_u)$ and $l(B) \geq l(b_u)$. If (in addition) D is abelian, then equality holds and $\mathrm{Z}(B) \cong \mathrm{Z}(b_u)$.*

It is conjectured that B and b_u are Morita equivalent whenever $u \in \mathrm{Z}(\mathcal{F})$. This can be regarded as a Z^*-*Theorem* for blocks.

2.6 Quadratic forms

By construction, the Cartan matrix $C(b_u)$ $(u \in \mathcal{R})$ is symmetric and positive definite. Hence, it gives rise to an integral quadratic form

$$q_u : \mathbb{Z}^{l(b_u)} \to \mathbb{Z}, \qquad x \mapsto xC(b_u)x^{\mathrm{T}}.$$

If we change the basic set of b_u, $C(b_u)$ becomes $SC(b_u)S^{\mathrm{T}}$ for some $S \in \mathrm{GL}(l(b_u), \mathbb{Z})$. This yields an *equivalent* quadratic form.

Theorem 2.21 (Reduction of quadratic form)**.** *There are only finitely many equivalence classes of integral, positive definite quadratic forms with given dimension and determinant (discriminant).*

Theorem 2.22. *There are only finitely many isotypy classes of p-blocks with a given defect.*

Sketch of proof. For a p-block B with a given defect there are only finitely many possible defect groups D. Let $u \in \mathcal{R}$. By Brauer-Feit, $l(b_u) \le k(B) \le p^{2d(B)}$. Moreover, the elementary divisors of $C(b_u)$ divide $p^{d(b_u)} \le p^{d(B)}$. In particular, $\det(C(b_u)) \le p^{d(B)l(b_u)}$. Hence, by Theorem 2.21 there are only finitely many possibilities for $C(b_u)$ up to basic sets. If a basic set for b_u is fixed, then there are only finitely many choices for Q_u. Since $|\mathcal{R}| \le p^{d(B)}$, there are only finitely many possibilities for Q_* up to basic sets. This allows only finitely many perfect isometry classes. The refinement to isotypy classes can be achieved inductively. □

Corollary 2.23. *There are only finitely many isomorphism types of centers of p-blocks with given defect.*

This corollary can be shown more directly by observing that $\mathrm{Z}(B)$ has an F-basis of the form

$$L_1^+ 1_B, \ldots, L_k^+ 1_B$$

where $L_1^+, \ldots, L_k^+$ are class sums of G. Then the structure constants lie in $\mathbb{F}_p$ and there are only finitely many multiplication tables.

If $l(B)$ is small, one can determine a set of representatives for $C = C(B)$ up to basic sets (reductions by Gauß, Minkowski, Hermite, …).

Example 2.24. If $l(B) = 2$, then there exists a basic set for B such that

$$C = \frac{\det(C)}{p^{d(B)}} \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$$

with $0 \le 2\beta \le \alpha \le \gamma$. It follows that

$$\frac{3}{4}\alpha^2 \le \alpha\gamma - \beta^2 = \frac{p^{2d(B)}}{\det(C)} \le p^{d(B)}.$$

It is conjectured that $\beta > 0$ (or more generally that q_u is *indecomposable*).

If $l(B)$ is large, heuristics can be used to make the entries of C "small" (LLL algorithm).

We are also interested in the *dual* quadratic form

$$q_u^* : \mathbb{Z}^{l(b_u)} \to \mathbb{Z}, \qquad x \mapsto p^{d(b_u)} xC(b_u)^{-1}x^{\mathrm{T}}.$$

Let $\min q_u^* := \min\{q_u^*(x) : x \in \mathbb{Z}^{l(b_u)} \setminus \{0\}\} > 0$.

Proposition 2.25 (Brauer)**.** *If* $u \in \mathrm{Z}(D)$*, then* $k(B)\min q_u^* \le l(b_u)p^{d(B)}$*.*

Sketch of proof. By construction, $M_u^2 = M_u$. It follows that the eigenvalues of M_u are 0 and 1. Therefore,

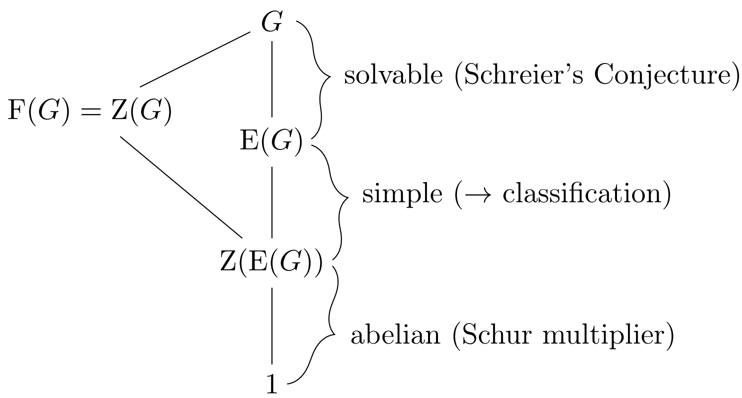
$$p^{d(B)}l(b_u) = p^{d(b_u)} \operatorname{rk} M_u = \operatorname{tr}\bigl(p^{d(b_u)}M_u\bigr) = \sum_{\chi\in\mathrm{Irr}(B)} q_u^*(d^u_{\chi.}) \ge k(B)\min q_u^*. \qquad \square$$

Quadratic forms with small minimum have been studied. For example, if q_u is indecomposable, then $\min q_u^* > 1$ or $l(b_u) = 1$. In general there are only finitely many vectors $x \in \mathbb{Z}^{l(b_u)}$ with $q_u^*(x) = \min q_u^*$.

2.7 Reduction to quasisimple groups

The methods we have covered so far are in general not sufficient to determine e. g. $k(B)$ from local data. In the following we present a rather different approach which often helps to overcome difficulties encountered otherwise.

- In order to determine the basic algebra of B we may change G in accordance with Fong's reductions. After the first reduction, B is *quasiprimitive*, i. e. for every $N \trianglelefteq G$, B covers a unique block of N. By the second reduction, we may assume that $\mathrm{O}_{p'}(G)$ is cyclic and central in G. Both reductions preserve D and $\mathcal{F}$.
- The Külshammer-Puig Theorem describes the source algebra of a block covering a nilpotent block. This is often helpful to show that $\mathrm{O}_p(G) = 1$. A recent result by Puig gives information in the opposite case where B is covered by a nilpotent block.
- By the previous steps, the *Fitting subgroup* is given by $\mathrm{F}(G) = \mathrm{Z}(G) = \mathrm{O}_{p'}(G)$. As usual, the *layer* $\mathrm{E}(G)$ of G is a central product of *components* $L_1, \ldots, L_n$ of G. Moreover, B covers a unique block $B_E = B_1 \otimes \ldots \otimes B_n$ of $\mathrm{E}(G)$ with $D(B_E) = D(B_1) \times \ldots \times D(B_n) \le D$. Here, B_E is nilpotent if and only if all B_i are nilpotent.
- In favorable cases we may use the structure of D to prove that $n = 1$, i. e. $\mathrm{E}(G)$ is quasisimple and $S := \mathrm{E}(G)/\mathrm{Z}(\mathrm{E}(G))$ is simple. Moreover, $\mathrm{Z}(G) \le \mathrm{C}_G(\mathrm{E}(G)) \le \mathrm{C}_G(\mathrm{F}^*(G)) \le \mathrm{F}(G) = \mathrm{Z}(G)$ and $G/\mathrm{Z}(G) \le \mathrm{Aut}(\mathrm{E}(G)) \le \mathrm{Aut}(S)$. Thus, we are in a position to apply the classification of the finite simple groups. Clifford theory can be used to minimize $|G/\mathrm{E}(G)|$.



3 Example $D = D_8$

Let $p = 2$ and

$$D = \langle x, y \mid x^4 = y^2 = 1,\ x^y = x^{-1}\rangle \cong D_8.$$

Since a non-trivial p'-automorphism of D must permute the maximal subgroups, we conclude that $\mathrm{Aut}(D)$ is a p-group. In particular, $I(B) = 1$. There are two candidates of essential subgroups: $E_1 := \langle x^2, y\rangle$ and $E_2 := \langle x^2, xy\rangle$. This gives three possible fusion systems represented by the following groups:

(i) D (B is nilpotent),

(ii) S_4 (E_1 is essential),

(iii) $\mathrm{GL}(3,2)$ (E_1 and E_2 are essential).

Let us assume that the second case occurs for B. Then we may choose $\mathcal{R} = \{1, x^2, xy, x\}$. By Proposition 2.5, the blocks b_{x^2}, b_{xy} and b_x are nilpotent and

$$k(B) - l(B) = l(b_{x^2}) + l(b_{xy}) + l(b_x) = 3.$$

Moreover, $C(b_{x^2}) = (8)$ and $C(b_{xy}) = C(b_x) = (4)$. Since $\mathcal{G}$ acts trivially on Q_*, Q_* is integral. Note that $p^{d(b_u)} m^u_{\chi\psi} = d_{\chi\varphi} d_{\psi\varphi}$ for $u \in \mathcal{R} \setminus \{1\}$ and $\mathrm{IBr}(b_u) = \{\varphi\}$. By the orthogonality and divisibility relations for xy, we see that $k_0(B) = 4$. Similarly, if we consider x^2, we get

$$k(B) = k_0(B) + k_1(B) = 4 + 1 = 5, \qquad\qquad l(B) = 2.$$

Moreover, $\mathfrak{foc}(B) = E_1$ and $\overline{D} = D/E_1$ has two orbits of length 2 on $\mathrm{Irr}_0(B)$. It follows that $\widehat{Q}_{x^2} = 2(\epsilon_1, \epsilon_2, \epsilon_3)^{\mathrm{T}}$ where $\epsilon_1, \epsilon_2, \epsilon_3 \in \{\pm 1\}$ (see Remark 2.17). Hence, there exists a basic set for B such that

$$\widehat{Q} = \begin{pmatrix} \epsilon_1 & . \\ . & \epsilon_2 \\ -\epsilon_3 & -\epsilon_3 \end{pmatrix}.$$

We obtain

$$Q_* = \begin{pmatrix} \epsilon_1 & . & \epsilon_1 & \epsilon_1 & \epsilon_1 \\ \epsilon_1 & . & \epsilon_1 & -\epsilon_1 & -\epsilon_1 \\ . & \epsilon_2 & \epsilon_2 & \epsilon_2 & -\epsilon_2 \\ . & \epsilon_2 & \epsilon_2 & -\epsilon_2 & \epsilon_2 \\ -\epsilon_3 & -\epsilon_3 & 2\epsilon_3 & . & . \end{pmatrix}, \qquad\qquad C(B) = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

up to basic sets. Note that the quadratic form q_1 is already reduced by Example 2.24. It is not hard to show that the isotypy class of B is uniquely determined. It is known further that B is Morita equivalent either to the principal block of S_4 or to the principal block of S_5 (recall from Theorem 1.4 that B is tame). Both blocks are Rickard equivalent (confirming a conjecture of Rouquier in this case).