On Loewy lengths of blocks (joint work with S. Koshitani and B. Külshammer)

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Notation

- G – finite group
- p – prime number
- F – algebraically closed field of characteristic p
- B – block of FG
- $J(B)$ – Jacobson radical of B (as an algebra)
- Let $LL(B) := \min\{n \geq 0 : J(B)^n = 0\}$ be the Loewy length of B
- Let D be a defect group of B. This is p-subgroup of G, unique up to conjugation.

Question

What can be said about the structure of D if $LL(B)$ is given?

Theorem (Okuyama)

Let δ be the defect of B. Then

1. $LL(B) = 1$ *iff* $\delta = 0$.
2. $LL(B) = 2$ *iff* $\delta = 1$ *and* $p = 2$.
3. $LL(B) = 3$ *iff one of the following holds:*
   - (a) $p = \delta = 2$ *and* B *is Morita equivalent to* $F[C_2 \times C_2]$ *or to* FA_4.
   - (b) $p > 2$, $\delta = 1$, *the inertial index of* B *is* $e(B) \in \{p - 1, (p-1)/2\}$, *and the Brauer tree of* B *is a straight line with exceptional vertex at the end (if it exists).*

Theorem (Koshitani-Külshammer-S.)

If B has defect δ and $LL(B) > 1$, then

$$\delta \leq \binom{LL(B)}{2}(2\lfloor \log_p(LL(B) - 1)\rfloor + 1).$$

Sketch of the proof.

- Let D be a defect group of B and set $p^\epsilon = \exp D$.
- Moreover, let ρ be the rank of D.
- A result of Oppermann shows $\rho \leq LL(B) - 1$.
- A result of Külshammer implies $\epsilon \leq 1 + \lfloor \log_p(LL(B) - 1)\rfloor$.
- By elementary group theory we have $\delta \leq \binom{\rho+1}{2}(2\epsilon - 1)$.
- Combine these equations. □

Remarks

Brauer's Problem 21

Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that $\lim_{n\to\infty} f(n) = \infty$ and $f(\delta) \leq \dim_F \mathrm{Z}(B)$.

Proposition

Let B be a block with cyclic defect group D and inertial index $e(B)$. Then

$$LL(B) \geq \frac{|D| - 1}{e(B)} + 1.$$

Blocks with $LL(B) = 4$

Proposition

Let B be a p-block with defect δ, defect group D and $LL(B) = 4$. Then

$$\delta \leq \begin{cases} 18 & \text{if } p \leq 3, \\ 5 & \text{if } p = 5, \\ 6 & \text{if } p \geq 7. \end{cases}$$

In case $p = 5$ (resp. $p = 7$) there are at most 10 (resp. 12) isomorphism types for D. These can be given by generators and relations. All these groups have exponent p and rank at most 3.

Blocks with $LL(B) = 4$

Proposition

If G is p-solvable and $LL(B) = 4$, then $p = 2$ and one of the following holds

- $D \cong C_4$,
- $D \cong C_2 \times C_2 \times C_2$,
- $D \cong D_8$.

Theorem

Let $G = S_n$ and $LL(B) = 4$. Then $n = 4$ and B is the principal 2-block.

Principal blocks

We denote the principal block of G by $B_0(G)$.

Theorem

Suppose $p \geq 5$ and $LL(B_0(G)) = 4$. Then $H := \mathrm{O}^{p'}(G/\mathrm{O}_{p'}(G))$ is simple and $LL(B_0(H)) = 4$.

Theorem (Koshitani)

If $p = 2$ and $LL(B_0(G)) = 4$, then $\mathrm{O}^{2'}(G/\mathrm{O}_{2'}(G))$ is one of the following groups:

- C_4,
- $C_2 \times C_2 \times C_2$,
- $C_2 \times \mathrm{PSL}(2, q)$ *for* $q \equiv 3 \pmod 8$,
- $\mathrm{PGL}(2, q)$ *for* $q \equiv 3 \pmod 8$.

Simple groups

Proposition

If G is simple of Lie type in defining characteristic $p > 2$, then $LL(B_0(G)) \neq 4$.

Proposition

If G is sporadic, $p > 2$ and $LL(B_0(G)) = 4$, then $G = M$ and $p = 11$.

We do not know if $LL(B_0(M)) = 4$ for $p = 11$ (probably not).

Examples

- Let $p \equiv 1 \pmod 3$, $n := (p-1)/3$ and $G := \mathrm{PSL}(n, q)$ where q has order n modulo p, but not modulo p^2 (q always exists). Then $LL(B_0(G)) = 4$.
- However, all these blocks have defect 1.
- There are similar examples for other groups of Lie type.
- There are (not necessarily principal) blocks of Loewy length 4 of the following groups:
    - $G = 12.M_{22}$ for $p \in \{5, 7, 11\}$,
    - $G = 6.A_7$ for $p \in \{5, 7\}$,
    - $G = 3.O'N$ for $p = 5$,
    - $G = Ru$ and $G = 2.Ru$ for $p = 7$.
- We do not have any examples for $p = 3$.