The Alperin-McKay Conjecture for a special class of defect groups

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Introduction

- Let G be a finite group and p be a prime.
- Let B be a p-block of G, i. e. an ideal of $\mathcal{O}G$ where $\mathcal{O}$ is a complete discrete valuation ring of characteristic 0.
- Let $k_i(B)$ be the number of irreducible characters of height $i \geq 0$ in B. Then $k(B) := \sum k_i(B) = |\mathrm{Irr}(B)|$.
- Let $\mathrm{Irr}_0(B)$ be the subset of $\mathrm{Irr}(B)$ of characters of height 0.
- Let $l(B)$ be the number of irreducible Brauer characters of B.
- Suppose that B has metacyclic defect group D, i. e. D has a cyclic normal subgroup such that the corresponding quotient is also cyclic.

What is known?

The case $p = 2$:

- D is dihedral, semidihedral or quaternion (tame case):
    - $k(B)$, $k_i(B)$, $l(B)$ computed by Brauer and Olsson
    - perfect isometries constructed by Cabanes-Picaronny
    - Dade's Invariant Conjecture verified by Uno
    - Donovan's Conjecture almost settled by Erdmann and Holm (up to certain scalars)
- $D \cong C_{2^n} \times C_{2^n}$ is homocyclic for some $n \geq 1$:
    - case $n = 1$ known to Brauer (also tame case)
    - perfect isometries constructed by Usami-Puig
    - Donovan's Conjecture and Broué's Conjecture recently checked as follows:

What is known?

Theorem (Eaton-Kessar-Külshammer-S., 2013)

Suppose that B is a 2-block with homocyclic defect group D. Then one of the following holds:

1. *B is nilpotent and thus Morita equivalent to $\mathcal{O}D$.*
2. *B is Morita equivalent to $\mathcal{O}[D \rtimes C_3]$.*
3. *$D \cong C_2 \times C_2$ and B is Morita equivalent to $B_0(\mathcal{O}A_5)$.*

- remaining metacyclic 2-groups:
  - B is nilpotent (Craven-Glesser, Robinson, S. independently)
  - algebra structure of B known by a result of Puig

Conclusion: The case $p = 2$ is well-understood.

What is known?

The case $p > 2$:

- Brauer's $k(B)$-Conjecture is true (Gao)
- Olsson's Conjecture is true (Yang)
- Brauer's Height Zero Conjecture is true (S.)
- Fusion system $\mathcal{F}$ on the subpairs of B is controlled (Stancu)
- D is cyclic:
    - case $|D| = p$ known to Brauer
    - in general, $k(B)$, $k_i(B)$, $l(B)$ computed by Dade
    - Donovan's Conjecture verified (Brauer trees)
    - Broué's Conjecture verified by Rickard and Linckelmann

What is known?

- D is abelian but non-cyclic:
  - smallest case $D \cong C_3 \times C_3$ still open! Partial results by Kiyota and Watanabe.
  - Broué's Conjecture and Donovan's Conjecture checked for principal blocks in case $D \cong C_3 \times C_3$ by Koshitani, Kunugi and Miyachi.
  - perfect isometries known for $D \cong C_{3^m} \times C_{3^n}$ if $n \neq m$ (Usami-Puig)
  - More partial results for $D \cong C_p \times C_p$ by Kessar-Linckelmann
- D is non-abelian and non-split:
  - $\mathrm{Aut}(D)$ is a p-group (Dietz)
  - all blocks are nilpotent

What is known?

- D is non-abelian and split:
  - $k(B)$, $k_i(B)$, $l(B)$ known if B has maximal defect (Gao)
  - perfect isometries constructed if B is principal by Horimoto and Watanabe
  - the p-solvable case follows from a result by Külshammer
  - G is not quasisimple by work of An
  - $D \cong C_{p^m} \rtimes C_p$:
    - $k(B) - l(B)$ known by Gao-Zeng
    - Holloway, Koshitani and Kunugi determined $k(B)$, $k_i(B)$, $l(B)$ under additional assumptions on G
    - Partial results in case $|D| = p^3$ by Hendren
    - $k(B)$, $k_i(B)$, $l(B)$ determined for $|D| = 3^3$ by S.

Conclusion: Many things are open in case $p > 2$.

New results

Let $p > 2$ and

$$D = \langle x, y \mid x^{p^m} = y^{p^n} = 1,\ yxy^{-1} = x^{1+p^{m-1}} \rangle \cong C_{p^m} \rtimes C_{p^n}$$

where $m \geq 2$ and $n \geq 1$.

- These are precisely the metacyclic defect groups D such that $|D'| = p$.
- These are precisely the metacyclic, minimal non-abelian groups, i. e. all proper subgroups of D are abelian.
- The family includes the groups $D \cong C_{p^m} \rtimes C_p$ mentioned above.

New results

Theorem (S., 2014)

The Alperin-McKay Conjecture holds for all blocks with metacyclic, minimal non-abelian defect groups.

Sketch of proof

Idea: Compute $k_0(B)$ in terms of the fusion system $\mathcal{F}$ only.

- By a result of Stancu, $\mathcal{F}$ is controlled, i. e. any conjugation on subpairs is induced from $\mathrm{Aut}(D)$.
- Moreover, $\mathrm{Out}_{\mathcal{F}}(D)$ is cyclic of order dividing $p-1$ by a result of Sasaki.
- In particular, $\mathcal{F}$ only depends on the inertial index $e(B) := |\mathrm{Out}_{\mathcal{F}}(D)|$.
- Let
$$\mathfrak{foc}(B) := \langle f(a)a^{-1} : a \in Q \leq D,\ f \in \mathrm{Aut}_{\mathcal{F}}(Q) \rangle$$
be the focal subgroup of B.
- It follows that $\mathfrak{foc}(B)$ lies in the cyclic normal subgroup $\langle x \rangle$. In particular $p^n \mid |D : \mathfrak{foc}(B)|$.

Sketch of proof

- By Broué-Puig and Robinson, $D/\mathfrak{foc}(B)$ acts freely on $\mathrm{Irr}_0(B)$ via the $*$-construction.
- In particular $p^n \mid k_0(B)$.
- On the other hand, we have upper bounds for $k_0(B)$ and $\sum p^{2i} k_i(B)$ from Héthelyi-Külshammer-S. (using properties of decomposition numbers)
- Finally, a formula by Brauer gives a lower bound for $k(B)$.
- The claim follows by a combination of these estimates. $\square$

Remarks

- The proof does not rely on the classification.
- As a corollary, one gets $k_1(B) = k(B) - k_0(B)$.
- This confirms less-known conjectures by Eaton, Eaton-Moretó, Robinson and Malle-Navarro, for B.

Remarks

Isaacs and Navarro proposed a refinement of the Alperin-McKay-Conjecture:

Conjecture (Isaacs-Navarro, 2002)

Let b_D be the Brauer correspondent of B in $\mathrm{N}_G(D)$. Then for every p-automorphism $\gamma \in \mathrm{Gal}(\mathbb{Q}_{|G|}|\mathbb{Q}_{|G|_{p'}})$ we have

$$|\{\chi \in \mathrm{Irr}_0(B) : {}^{\gamma}\chi = \chi\}| = |\{\chi \in \mathrm{Irr}_0(b_D) : {}^{\gamma}\chi = \chi\}|.$$

Remarks

Proposition (S.)

The Isaacs-Navarro Conjecture holds for all blocks with defect group $C_{p^2} \rtimes C_p$.

In fact every p-automorphism $\gamma \in \mathrm{Gal}(\mathbb{Q}_{|G|}|\mathbb{Q}_{|G|_{p'}})$ acts trivially on $\mathrm{Irr}_0(B)$.

The case $p = 3$

Theorem (S., 2014)

Let B be a non-nilpotent 3-block with metacyclic, minimal non-abelian defect groups. Then

$$k_0(B) = \frac{3^{m-2}+1}{2}3^{n+1}, \qquad k_1(B) = 3^{m+n-3},$$

$$k(B) = \frac{11 \cdot 3^{m-2}+9}{2}3^{n-1}, \qquad l(B) = 2$$

where m and n are the parameters in the presentation of D.

Sketch of proof

- Since B is non-nilpotent and $e(B) \mid p-1$, we have $e(B) = 2$.
- The theory of lower defect groups implies $l(B) \in \{2, 3\}$.
- Use induction on n. In case $n = 1$ and $l(B) = 3$, decomposition numbers are in exceptional configuration $\rightarrow$ contradiction.
- Let $n \geq 2$. Then induction gives $k(B) - l(B)$.
- By a result of Robinson, $\mathrm{Z}(D)\mathfrak{foc}(B)/\mathfrak{foc}(B)$ acts freely on $\mathrm{Irr}(B)$ by the $*$-construction.
- In particular $3^{n-1} \mid k(B)$, and the result follows. $\square$

Remarks

- The proof is still classification-free.
- The induction argument works for any prime $p > 2$.
- Therefore, it suffices to handle the defect groups $D \cong C_{p^m} \rtimes C_p$ for $m \geq 2$.

Remarks

- Since $\mathcal{F}$ is controlled and $\mathrm{Out}_{\mathcal{F}}(D)$ is cyclic, Alperin's Weight Conjecture asserts $l(B) = e(B)$.
- The Ordinary Weight Conjecture is equivalent Dade's Projective Conjecture and predicts $k_i(B)$ in terms of $\mathcal{F}$.

Corollary

Alperin's Weight Conjecture and the Ordinary Weight Conjecture are satisfied for every 3-block with metacyclic, minimal non-abelian defect groups.

The case $p = 5$

Theorem (S., 2014)

Let B be a 5-block of a finite group with non-abelian defect group $C_{25} \rtimes C_{5^n}$ where $n \geq 1$. Then

$$k_0(B) = \left(\frac{4}{e(B)} + e(B)\right)5^n, \qquad k_1(B) = \frac{4}{e(B)}5^{n-1},$$

$$k(B) = \left(\frac{24}{e(B)} + 5e(B)\right)5^{n-1}, \qquad l(B) = e(B).$$

Again Alperin's Weight Conjecture and the Ordinary Weight Conjecture are satisfied in this special case.

Partial results

Proposition

Let $p \in \{7, 11, 13, 17, 23, 29\}$ *and let* B *be a* p*-block with defect group* $C_{p^2} \rtimes C_{p^n}$ *where* $n \geq 1$*. If* $e(B) = 2$*, then*

$$k_0(B) = \frac{p+3}{2} p^n, \qquad k_1(B) = \frac{p-1}{2} p^{n-1},$$

$$k(B) = \frac{p^2+4p-1}{2} p^{n-1}, \qquad l(B) = 2.$$

Final remarks

- Let B be a block with defect group D and fusion system $\mathcal{F}$.
- Then the hyperfocal subgroup of B is defined by

$$\mathfrak{hyp}(B) := \langle f(a)a^{-1} : a \in Q \leq D,\ f \in \mathrm{O}^p(\mathrm{Aut}_{\mathcal{F}}(Q))\rangle$$

- By a result of Puig the source algebra iBi of B can be expressed as a crossed product:

$$iBi = \bigoplus_{x \in D/\mathfrak{hyp}(B)} \mathcal{H}x$$

where $\mathcal{H}$ is the hyperfocal subalgebra of iBi.

- $\mathcal{H}$ is unique up to $(iBi^D)^\times$-conjugation as D-stable unitary subalgebra of iBi.

Final remarks

- Moreover, $\mathcal{H} \cap Di = \mathfrak{hyp}(B)i$.
- If D is non-abelian, metacyclic for an odd prime p, then $\mathfrak{hyp}(B) \subseteq \mathfrak{foc}(B)$ are cyclic.
- Assume that $F = \mathcal{O}/\operatorname{Rad}(\mathcal{O})$ is an algebraically closed field of characteristic p.
- It follows from Watanabe that $\mathcal{H}$, considered as an algebra over F, has finite representation type.