Fusion systems of groups and blocks

Young researchers seminar
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1 Motivation

Let G be a finite group and $H \leq G$. Elements $x, y \in H$ are called *fused* if they are conjugate in G, but not in H.

Aim: Find "small" subgroup $K \supseteq H$ *controlling* fusion in H, i. e. $x, y \in H$ are fused in G iff x, y are fused in K.

Main interest: $H \in \mathrm{Syl}_p(G)$.

In the following let $P \in \mathrm{Syl}_p(G)$.

Theorem 1.1 (Burnside)**.** *$\mathrm{N}_G(P)$ controls fusion in $\mathrm{Z}(P)$.*

Theorem 1.2 (Frobenius)**.** *If P controls fusion in P ("no fusion"), then G is p-*nilpotent*, i. e. $G = N \rtimes P$.*

Theorem 1.3 ((Hyper)focal subgroup theorem)**.**

$$\langle xy^{-1} : x, y \in P \text{ are conjugate in } G\rangle = G' \cap P \qquad \text{(focal subgroup)},$$
$$\langle xy^{-1} : x, y \in P \text{ are conjugate by a } p'\text{-element}\rangle = \mathrm{O}^p(G) \cap P \qquad \text{(hyperfocal subgroup)}$$

where $G' = [G, G]$ and $\mathrm{O}^p(G) = \langle p'\text{-elements}\rangle$.

Theorem 1.4 (Z^*-theorem)**.** *If $x \in \mathrm{Z}(P)$ is not fused to any other element of P, then $x\,\mathrm{O}_{p'}(G) \in \mathrm{Z}(G/\mathrm{O}_{p'}(G))$ where $\mathrm{O}_{p'}(G)$ is the largest normal p'-subgroup of G.*

Theorem 1.5 (ZJ-theorem)**.** *Suppose that $p > 2$ and G does not involve $Qd(p) := C_p^2 \rtimes \mathrm{SL}_2(p)$. Then $\mathrm{N}_G(\mathrm{Z}(\mathrm{J}(P)))$ controls fusion in P where $\mathrm{J}(P)$ is the Thompson subgroup of P.*

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2 Fusion systems

Definition 2.1 (Puig)**.** A (saturated) *fusion system* on a finite p-group P is a category $\mathcal{F}$ with

- objects = subgroups of P
- morphisms = injective group homomorphisms such that
  - $\mathrm{Hom}_P(S,T) := \{\varphi : S \to T : \exists g \in P : \varphi(s) = s^g = g^{-1}sg\ \forall s \in S\} \subseteq \mathrm{Hom}_{\mathcal{F}}(S,T)$ for $S,T \le P$,
  - $\varphi \in \mathrm{Hom}_{\mathcal{F}}(S,T) \Longrightarrow \varphi \in \mathrm{Hom}_{\mathcal{F}}(S,\varphi(S))$, $\varphi^{-1} \in \mathrm{Hom}_{\mathcal{F}}(\varphi(S),S)$,
  - for every $S \le P$ there exists an isomorphism $S \to T$ in $\mathcal{F}$ such that $\mathrm{Aut}_P(T) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(T))$ and every isomorphism $\varphi : R \to T$ in $\mathcal{F}$ extends to $\{x \in \mathrm{N}_P(R) : \exists y \in \mathrm{N}_P(T) : \varphi(r^x) = \varphi(r)^y\ \forall r \in R\}$.

Example 2.2. Every finite group G induces a fusion system $\mathcal{F}_P(G)$ on $P \in \mathrm{Syl}_p(G)$ via $\mathrm{Hom}_{\mathcal{F}}(S,T) := \mathrm{Hom}_G(S,T)$ for $S,T \le P$ (Exercise). In particular, there is always the *trivial* fusion system $\mathcal{F}_P(P)$. There are *exotic* fusion systems not arising from finite groups. For example on the non-abelian group $P = 7_+^{1+2}$ of order 7^3 and exponent 7.

Theorem 2.3 (Frobenius)**.** *$\mathcal{F}_P(G) = \mathcal{F}_P(P) \Longrightarrow G$ p-nilpotent.*

In the following let $\mathcal{F}$ be a fusion system on P. We call $x,y \in P$ *$\mathcal{F}$-conjugate* if there exists a morphism in $\mathcal{F}$ sending x to y.

Definition 2.4. $Q < P$ is called *essential* if

- for every isomorphism $Q \to S$ in $\mathcal{F}$ we have $|\mathrm{N}_P(Q)| \ge |\mathrm{N}_P(S)|$ and $\mathrm{C}_P(S) \le S$,
- there exists a *strongly p-embedded* subgroup $H < \mathrm{Out}_{\mathcal{F}}(Q) := \mathrm{Aut}_{\mathcal{F}}(Q)/\mathrm{Inn}(Q)$, i.e. $|H|_p \ne 1$ and $|H \cap H^x|_p = 1$ for every $x \in \mathrm{Out}_{\mathcal{F}}(Q) \setminus H$ (cf. Frobenius complement).

Remark 2.5. Essential subgroups Q are self-centralizing ($\mathrm{C}_P(Q) \le Q$) and *radical*, i.e. $\mathrm{O}_p(\mathrm{Aut}_{\mathcal{F}}(Q)) = \mathrm{Inn}(Q)$ (Exercise).

Theorem 2.6 (Alperin's fusion theorem)**.** *Every isomorphism in $\mathcal{F}$ is a composition of restrictions from* $\mathrm{Aut}_{\mathcal{F}}(P) \cup \bigcup_{Q \text{ essential}} \mathrm{Aut}_{\mathcal{F}}(Q)$.

Theorem 2.7. *A group G contains a strongly p-embedded subgroup iff one of the following holds:*

(1) $\mathrm{O}_p(G) = 1$ *and the Sylow p-subgroups of G are cyclic or quaternion.*

(2) $\mathrm{O}^{p'}(G/\mathrm{O}_{p'}(G))$ *is one of the following:*

- $\mathrm{PSL}(2,p^n)$ *for* $n \ge 2$,
- $\mathrm{PSU}(3,p^n)$ *for* $n \ge 1$,
- $\mathrm{Sz}(2^{2n+1})$ *for* $p = 2$ *and* $n \ge 1$,
- ${}^2G_2(3^{2n-1})$ *for* $p = 3$ *and* $n \ge 1$,
- A_{2p} *for* $p \ge 5$,
- $\mathrm{PSL}_3(4)$, M_{11} *for* $p = 3$,
- $\mathrm{Aut}(\mathrm{Sz}(32))$, ${}^2F_4(2)'$, McL, Fi_{22} *for* $p = 5$,

- J_4 *for* $p = 11$.

Consequence: Most fusion systems are *controlled*, i. e. there are no essential subgroups and $\mathcal{F} = \mathcal{F}_P(P \rtimes \mathrm{Out}_{\mathcal{F}}(P))$. In fact "most" fusion systems are trivial.

Theorem 2.8 (Burnside)**.** *P abelian $\Longrightarrow$ $\mathcal{F}$ controlled.*

Example 2.9. P cyclic 2-group $\Longrightarrow$ $\mathcal{F}$ trivial.

Definition 2.10.

(1) Let $\mathrm{O}_p(\mathcal{F})$ be the largest subgroup $Q \le \bigcap\limits_{E \text{ essential}} E$ such that $f(Q) = Q$ $\forall f \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$ (Exercise: Show well-defined).

(2) $\mathcal{F}$ is called *constrained*, if $\mathrm{C}_P(\mathrm{O}_p(\mathcal{F})) \le \mathrm{O}_p(\mathcal{F})$.

Theorem 2.11 (Model theorem)**.** *Every constrained fusion system $\mathcal{F}$ has a unique* model *G, i. e. $P \in \mathrm{Syl}_p(G)$, $\mathrm{O}_p(\mathcal{F}) = \mathrm{O}_p(G)$, $\mathrm{C}_G(\mathrm{O}_p(G)) \le \mathrm{O}_p(G)$ and $\mathcal{F} = \mathcal{F}_P(G)$. In particular, $\mathcal{F}$ is non-exotic.*

Example 2.12.

(1) controlled $\Longrightarrow$ constrained ($\mathrm{O}_p(\mathcal{F}) = P$).

(2) $\mathcal{F}_{D_8}(S_4)$ is constrained ($\mathrm{O}_p(\mathcal{F}) = V_4$), but not controlled.

(3) $\mathcal{F}_{D_8}(\mathrm{GL}_3(2))$ is not constrained (Exercise).

Definition 2.13. A group G is called *metacyclic* if there exists $N \trianglelefteq G$ such that N and G/N are cyclic.

Theorem 2.14. *If P is metacyclic, then one of the following holds:*

(1) $\mathcal{F}$ is trivial.

(2) P is abelian and $\mathrm{Aut}_{\mathcal{F}}(P)$ is a p'-subgroup of $\mathrm{GL}_2(p)$.

(3) $p > 2$, $P = C_{2^n} \rtimes C_{2^m}$, $\mathcal{F}$ is controlled and $\mathrm{Out}_{\mathcal{F}}(P) \le C_{p-1}$.

(4) $p = 2$, D is dihedral, semidihedral or quaternion (≤ 7 non-trivial fusion systems per order, all coming from "decorated" simple groups).

Definition 2.15.

$$\begin{aligned} \mathrm{Z}(\mathcal{F}) &:= \{x \in P : f(x) = x \ \forall f \in \mathrm{Hom}_{\mathcal{F}}(\langle x \rangle, P)\} && (\textit{center}), \\ \mathfrak{hyp}(\mathcal{F}) &:= \langle f(x)x^{-1} : x \in Q \le P,\ f \in \mathrm{O}^p(\mathrm{Aut}_{\mathcal{F}}(Q)) \rangle && (\textit{hyperfocal subgroup}). \end{aligned}$$

Proposition 2.16.

(1) $\mathcal{F}$ trivial $\Longleftrightarrow$ $\mathfrak{hyp}(\mathcal{F}) = 1$ (Exercise).

(2) P abelian $\Longrightarrow$ $P = \mathfrak{hyp}(\mathcal{F}) \times \mathrm{Z}(\mathcal{F})$ (Exercise).

(3) $\mathfrak{hyp}(\mathcal{F})$ cyclic $\Longrightarrow$ $\mathcal{F}$ controlled and $\mathrm{Out}_{\mathcal{F}}(P) \le C_{p-1}$.

3 Blocks

Let F be an algebraically closed field of characteristic p, and let B be a *block* of FG, i. e. an indecomposable direct summand of the group algebra FG. As usual, the irreducible ordinary and modular characters can be distributed into blocks.

Definition 3.1. A *defect group* of B is a maximal p-subgroup $D \le G$ such that there exists $\psi \in \mathrm{Irr}(\mathrm{N}_G(D))$ with

$$\Big(\sum_{\chi\in\mathrm{Irr}(B)} \chi(1)(\chi, \psi^G)\Big)_p = \psi^G(1)_p.$$

Definition 3.2 (Alperin-Broué). B determines a fusion system $\mathcal{F}_D(B)$ on D such that $\mathrm{Hom}_{\mathcal{F}}(S,T) \subseteq \mathrm{Hom}_G(S,T)$ for $S,T \le D$ (makes use of *Brauer pairs*).

In the following let $\mathcal{F} = \mathcal{F}_D(B)$.

Example 3.3. If $B = B_0(G)$ is the principal block ($1 \in \mathrm{Irr}(B)$), then $D \in \mathrm{Syl}_p(G)$ and $\mathcal{F} = \mathcal{F}_D(G)$.

Open: Is $\mathcal{F} = \mathcal{F}_D(H)$ for some finite group H?

Definition 3.4. B is called *nilpotent* if $\mathcal{F}$ is trivial.

Theorem 3.5 (Puig)**.** *If B is nilpotent, then $B \cong (FD)^{n\times n}$ for some $n \ge 1$. In particular, B and FD are* Morita equivalent*, i. e. they have equivalent module categories.*

Example 3.6. G p-nilpotent iff $B_0(G)$ nilpotent.

Theorem 3.7 (Külshammer)**.** *If $D \trianglelefteq G$, then $\mathcal{F}$ is controlled and B is Morita equivalent to a twisted group algebra $F_\alpha[D \rtimes \mathrm{Out}_{\mathcal{F}}(P)]$ where $\alpha \in \mathrm{H}^2(\mathrm{Out}_{\mathcal{F}}(P), F^\times)$.*

Theorem 3.8 (Külshammer)**.** *If G is p-solvable, then $\mathcal{F}$ is constrained and B is Morita equivalent to $F_\alpha H$ where H is a model for $\mathcal{F}$ and $\alpha \in \mathrm{H}^2(H, F^\times)$.*

Theorem 3.9. *If D is a metacyclic 2-group, then one of the following holds:*

(1) B is nilpotent.

(2) D is dihedral, semidihedral or quaternion and B has tame *representation type (Morita equivalence classes classified up to scalars).*

(3) $D \cong C_{2^n}^2$ and B is Morita equivalent to $F[D \rtimes C_3]$.

(4) $D \cong C_2^2$ and B is Morita equivalent to $B_0(A_5)$.

Remark 3.10. Puig's theorem classifies blocks with "minimal" fusion. The following is the other extreme.

Theorem 3.11. *If every two non-trivial elements of D are $\mathcal{F}$-conjugate, then one of the following holds:*

(1) D is elementary abelian and the possible $\mathrm{Aut}_{\mathcal{F}}(D)$ are classified by Hering (transitive linear groups).

(2) $D = 3_+^{1+2}$ and $\mathcal{F} = \mathcal{F}_D(H)$ where $H \in \{{}^2F_4(2)', J_4\}$.

(3) $D = 5_+^{1+2}$, $\mathcal{F} = \mathcal{F}_D(Th)$ and B is Morita equivalent to $B_0(Th)$.

Conjecture 3.12 (Blockwise Z*-conjecture). *B is Morita equivalent to its Brauer correspondent B_Z in $\mathrm{C}_G(\mathrm{Z}(\mathcal{F}))$.*

Remark 3.13. Let $B = B_0(G)$. Since $B_0(G) \cong B_0(G/\operatorname{O}_{p'}(G))$, we may assume that $\operatorname{O}_{p'}(G) = 1$. Then the Z*-theorem implies $\mathrm{Z}(\mathcal{F}) = \mathrm{Z}(G)$ and $B = B_Z$.

Theorem 3.14 (Külshammer-Okuyama, Watanabe). *$|\mathrm{Irr}(B)| \geq |\mathrm{Irr}(B_Z)|$ and $|\mathrm{IBr}(B)| \geq |\mathrm{IBr}(B_Z)|$ with equality in both cases if D is abelian.*

Conjecture 3.15 (Rouquier). *If $\mathfrak{hyp}(\mathcal{F})$ is abelian, then B is derived equivalent to its Brauer correspondent B_H in $\mathrm{N}_G(\mathfrak{hyp}(\mathcal{F}))$.*

Remark 3.16. Suppose that D is abelian. In view of Conjecture 3.12, lets assume that $\mathrm{Z}(\mathcal{F}) \leq \mathrm{Z}(G)$. Then $\mathrm{N}_G(\mathfrak{hyp}(\mathcal{F})) = \mathrm{N}_G(D)$ (since $D = \mathfrak{hyp}(\mathcal{F}) \times \mathrm{Z}(\mathcal{F})$) and Rouquier's conjecture becomes *Broué's conjecture.*

Theorem 3.17 (Watanabe). *If $\mathfrak{hyp}(\mathcal{F})$ is cyclic, then*

$$|\mathrm{Irr}(B)| = |\mathrm{Irr}(B_H)| = |\mathrm{Irr}(D \rtimes \mathrm{Out}_{\mathcal{F}}(D))|,$$
$$|\mathrm{IBr}(B)| = |\mathrm{IBr}(B_H)| = |\mathrm{Out}_F(D)|.$$

Remark 3.18. If $p > 2$ and D non-abelian metacyclic, then Theorem 3.17 applies.