New inequalities concerning Olsson's Conjecture

Benjamin Sambale

(joint work with L. Héthelyi and B. Külshammer)

11. 11. 2011

Notations

- G is a finite group
- p is a prime number
- B is a p-block of G
- D is a defect group of B
- $\mathrm{Irr}(B)$ is the set of irreducible ordinary characters of B
- $k(B) := |\mathrm{Irr}(B)|$
- $\mathrm{IBr}(B)$ is the set of irreducible Brauer characters of B
- $l(B) := |\mathrm{IBr}(B)|$

Heights

- For $\chi \in \mathrm{Irr}(B)$ define the height $h(\chi) \in \mathbb{N}_0$ by

$$\chi(1)_p = p^{h(\chi)}|G : D|_p.$$

- $k_i(B) := |\{\chi \in \mathrm{Irr}(B) : h(\chi) = i\}|$ for $i \geq 0$.
- D' is the commutator subgroup of D

Olsson's Conjecture (1975)

For every block B with defect group D we have $k_0(B) \leq |D : D'|$.

Known results

- In general Olsson's Conjecture for a block B would follow from the Alperin-McKay Conjecture for B which asserts $k_0(B) = k_0(b)$ for the Brauer correspondent b of B in $\mathrm{N}_G(D)$.
- In particular, Olsson's Conjecture holds for p-solvable, symmetric or alternating groups G.
- If D is abelian, Olsson's Conjecture for B would follow from Brauer's $k(B)$-Conjecture which asserts $k(B) \leq |D|$.
- Olsson's Conjecture is satisfied if D is metacyclic.
- If D is extraspecial of order p^3, Olsson's Conjecture was proved by Hendren in some, but not all cases. These cases concern the inertial group of B.

Subsections

- Let $u \in D$, and let b_u be a block of $C_G(u)$ with Brauer correspondent B.
- Then the pair (u, b_u) is called subsection for B.

Proposition (Robinson)

If b_u has defect d, then we have $k_0(B) \leq p^d \sqrt{l(b_u)}$.

- The conjugation of subsections takes place in the fusion system $\mathcal{F}$ of B.
- The block B is controlled if $\mathcal{F}$ is controlled by the inertial group of B.

Subsections

- A given subsection (u, b_u) can be replaced by a conjugate such that $\langle u \rangle$ is fully $\mathcal{F}$-normalized in D.
- This means that $|\mathrm{N}_D(\langle u \rangle)|$ is as large as possible among all $\mathcal{F}$-conjugates of u.
- In this case $\mathrm{C}_D(u)$ is a defect group of b_u.
- If B is controlled, then all subgroups of D are fully $\mathcal{F}$-normalized.

The case $p = 2$

Theorem

Let $p = 2$, and let (u, b_u) be a subsection such that $\langle u \rangle$ is fully $\mathcal{F}$-normalized and u is conjugate to u^{-5^n} for some $n \in \mathbb{Z}$ in D. If $l(b_u) \le 2$, then

$$k_0(B) \le 2|\mathrm{N}_D(\langle u \rangle)/\langle u \rangle|.$$

- The idea of the proof goes back to Brauer and uses the generalized decomposition numbers $d^u_{\chi\varphi}$ for $\chi \in \mathrm{Irr}(B)$ and $\varphi \in \mathrm{IBr}(b_u)$.
- Here the following result by Broué is important.

Sketch of the proof

Proposition (Broué)

If $\chi \in \mathrm{Irr}(B)$ has height 0, then $d^u_{\chi\varphi} \neq 0$ for some $\varphi \in \mathrm{IBr}(b_u)$.

- It is known that $d^{u^{-5^n}}_{\chi\varphi} = d^u_{\chi\varphi'}$ for some $\varphi' \in \mathrm{IBr}(b_u)$, since u and u^{-5^n} are $\mathcal{F}$-conjugate.
- On the other hand $d^{u^\gamma}_{\chi\varphi} = \gamma(d^u_{\chi\varphi})$ for an automorphism γ in the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta)|\mathbb{Q}) \cong \mathrm{Aut}(\langle u \rangle)$ where ζ is a $|\langle u \rangle|$-th root of unity.
- A comparison of these numbers implies the result.

Example

- Let D be a modular 2-group and $x \in D$ such that $|D : \langle x\rangle| = 2$.
- Since $\langle x\rangle \trianglelefteq D$, the subgroup $\langle x\rangle$ is fully $\mathcal{F}$-normalized.
- Moreover, $l(b_x) = 1$, because b_x has cyclic defect group $\mathrm{C}_D(x) = \langle x\rangle$.
- However, x and x^{-5^n} are not conjugate in D for all $n \geq 0$.
- It is known that B is nilpotent and thus

$$k_0(B) = |D : D'| = |D|/2.$$

- This example shows that the conjugation condition is necessary.

Application

Corollary

Let D be a 2-group and $x \in D$ such that $|D : \langle x \rangle| \leq 4$, and suppose that one of the following holds:

- *x is conjugate to x^{-5^n} in D for some $n \in \mathbb{Z}$,*
- *$\langle x \rangle \trianglelefteq D$.*

Then Olsson's Conjecture holds for all blocks with defect group D.

This includes the 2-groups of maximal class for which Olsson's Conjecture was already proved by Brauer and Olsson.

The case $p > 2$

We call a B-subsection (u, b_u) major if b_u and B have the same defect.

Theorem

Let $p > 2$, and let (u, b_u) be a subsection such that $l(b_u) = 1$ and b_u has defect d. Moreover, let $|\mathrm{Aut}_{\mathcal{F}}(\langle u \rangle)| = p^s r$ where $p \nmid r$ and $s \geq 0$. Then we have

$$k_0(B) \leq \frac{|\langle u \rangle| + p^s(r^2 - 1)}{|\langle u \rangle| \cdot r} p^d.$$

If (in addition) (u, b_u) is major, we can replace $k_0(B)$ by $\sum_{i=0}^{\infty} p^{2i} k_i(B)$.

Example

- Assume that $D = \langle u \rangle$ is cyclic.
- Then $l(b_u) = 1$ and $r := |\mathrm{Aut}_{\mathcal{F}}(\langle u \rangle)|$ is the inertial index of B.
- Thus, the theorem implies

$$k_0(B) \le k(B) \le \sum_{i=0}^{\infty} p^{2i} k_i(B) \le \frac{|D| - 1}{r} + r.$$

- By Dade's Theorem on blocks with cyclic defect groups in fact equality holds.
- This shows that the inequality is sharp.

Remarks

- If $\mathrm{Aut}_{\mathcal{F}}(\langle u\rangle)$ is a p-group or $\mathrm{Aut}_{\mathcal{F}}(\langle u\rangle) = \mathrm{Aut}(\langle u\rangle)$, the theorem implies Robinson's result $k_0(B) \le p^d$ (for $l(b_u) = 1$).
- In all other cases the inequality is even better.
- The claim about major subsections also improves another result by Robinson:

Proposition (Robinson)

If (u, b_u) is a major subsection such that $l(b_u) = 1$, then

$$\sum_{i=0}^{\infty} p^{2i} k_i(B) \le |D|.$$

A related result

The following proposition was obtained by different methods. Here p is arbitrary.

Proposition

Let (u, b_u) be a subsection such that b_u has defect group Q. If $Q/\langle u \rangle$ is cyclic, then

$$k_0(B) \le \left(\frac{|Q/\langle u \rangle| - 1}{l(b_u)} + l(b_u) \right) |\langle u \rangle| \le |Q|.$$

Controlled Blocks

- Assume that B is a controlled block, and the subsection (u, b_u) satisfies $l(b_u) = 1$.
- Then Robinson's result takes the form

$$k_0(B) \leq |\mathrm{C}_D(u)|.$$

- Thus, in order to prove Olsson's Conjecture it suffices to find an element $u \in D$ such that $l(b_u) = 1$ and $|\mathrm{C}_D(u)| \leq |D : D'|$.

Controlled Blocks

Theorem

Let D be a finite p-group, where p is an odd prime, and suppose that one of the following holds:

- *D has maximal class,*
- *D has class 2 and $|D : \Phi(D)| = p^2$,*
- *D' is cyclic and $|D : \Phi(D)| = p^2$,*
- *D has p-rank 2.*

Then Olsson's Conjecture holds for all controlled blocks with defect group D.

Here the *p*-rank denotes the maximal rank of an abelian subgroup.

Sketch of the proof

- Let B be a controlled block with defect group D.
- It is known that the inertial quotient L of B is a p'-subgroup of $\mathrm{Aut}(D)$.
- In all cases except the last one we have $|D : \Phi(D)| = p^2$.
- Hence, we may identify L with a subgroup of $\mathrm{GL}(2, p)$.
- Next we show that the set $S := \{u \in D : |D : \mathrm{C}_D(u)| = |D'|\}$ is nonempty, and L has a regular orbit T on S.
- This implies that the block b_u for some $u \in T$ has inertial index 1.
- Moreover, it is known that b_u is also controlled, and thus nilpotent.

Sketch of the proof

- This shows $l(b_u) = 1$.
- Now assume that D has p-rank 2.
- Then a result of Blackburn implies that we only have to consider two infinite families of p-groups given by generators and relations.
- Here one can use that L acts faithfully on $\Omega(D)/\Phi(\Omega(D))$; again a group of order p^2.
- Recall that $\Omega(D) := \langle x \in D : x^p = 1 \rangle$.

Remarks

- The condition $|D : C_D(u)| = |D'|$ implies that

$$D' = \{[u, v] : v \in D\};$$

  in particular every element of D' is a commutator.
- Hence, our method does not suffice in order to prove Olsson's Conjecture for all controlled blocks.

Applications

- It was shown by Díaz, Ruiz and Viruel that most blocks with a defect group of p-rank 2 are in fact controlled.
- Here for $p > 3$ only an extraspecial defect group D of order p^3 and exponent p is possible for a non-controlled block.
- In this case Hendren showed that there is always a non-major subsection (u, b_u) provided $p > 7$.
- Then b_u has defect group $\mathrm{C}_D(u)$ and $\mathrm{C}_D(u)/\langle u\rangle$ is cyclic.
- Since $|D : D'| = p^2 = |\mathrm{C}_D(u)|$, Olsson's Conjecture follows from one of the previous propositions.

Applications

- Now let $p \in \{5, 7\}$.
- Then by the work of Ruiz and Viruel we only have to consider a few fusion systems for B.
- Kessar and Stancu proved that for $p = 7$ the relevant fusion systems do not occur for blocks.
- For $p = 5$ the only fusion system without non-major subsections is the fusion system of the simple Thompson group.
- Here we have applied the classification of the finite simple groups in order to show Olsson's Conjecture.

Applications

These considerations lead to the following theorem:

Theorem

Let $p > 3$. Then Olsson's Conjecture holds for all p-blocks with defect groups of p-rank 2.

For $p = 3$ there are also non-controlled blocks with defect groups of maximal class and p-rank 2.

Applications

Similar arguments give:

Theorem

Let $p \neq 3$. Then Olsson's Conjecture holds for all p-blocks with minimal nonabelian defect groups.

Here a group D is called minimal nonabelian if all proper subgroups of D are abelian, but D is not.