Cartan matrices and Brauer's $k(B)$-Conjecture

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Notation

- G – finite group
- p – prime
- B – p-block of G
- D – defect group of B
- $\mathrm{Irr}(B)$ – irreducible ordinary characters in B
- $\mathrm{IBr}(B)$ – irreducible Brauer characters in B
- $k(B) := |\mathrm{Irr}(B)|$
- $l(B) := |\mathrm{IBr}(B)|$

Notation

- For $\chi \in \mathrm{Irr}(B)$ there exist non-negative integers $d_{\chi\psi}$ such that

$$\chi(x) = \sum_{\varphi \in \mathrm{IBr}(B)} d_{\chi\varphi}\varphi(x)$$

  for all p'-elements $x \in G$.
- $Q = (d_{\chi\varphi}) \in \mathbb{Z}^{k(B)\times l(B)}$ – decomposition matrix of B
- Let c_{ij} be the multiplicity of the i-th simple B-module as a composition factor of the j-th indecomposable projective B-module
- $C = (c_{ij}) \in \mathbb{Z}^{l(B)\times l(B)}$ – Cartan matrix of B

Facts

- all of the $k(B)$ rows of Q are non-zero
- $C = Q^{\mathsf{T}}Q$ is symmetric and positive definite
- $|D|$ is the unique largest elementary divisor of C

Observation: There should be a relation between $k(B)$, C and $|D|$.

Facts

- Obvious: $l(B) \leq k(B) \leq \mathrm{tr}(C)$.
- Brandt: $k(B) \leq \mathrm{tr}(C) - l(B) + 1$.
- Külshammer-Wada: $k(B) \leq \mathrm{tr}(C) - \sum c_{i,i+1}$ where $C = (c_{ij})$.
- Wada: $k(B) \leq \rho(C)l(B)$ where $\rho(C)$ is the Perron-Frobenius eigenvalue of C.
- Brauer-Feit: $k(B) \leq |D|^2$.
- Brauer's $k(B)$-Conjecture: $k(B) \leq |D|$.

Indecomposable matrices

Example (naive)

$$Q = \begin{pmatrix} 1 & . \\ . & 1 \\ . & 1 \end{pmatrix} \Longrightarrow C = \begin{pmatrix} 1 & . \\ . & 2 \end{pmatrix} \Longrightarrow k(B) = 3 > 2 = |D|?!$$

Definition

A matrix $A \in \mathbb{Z}^{k \times l}$ is indecomposable (as a direct sum) if there is no $S \in \mathrm{GL}(l, \mathbb{Z})$ such that $AS = \begin{pmatrix} * & . \\ . & * \end{pmatrix}$.

Indecomposable matrices

Proposition

The decomposition matrix Q is indecomposable.

- This has been known for $S = 1$ in the definition above.
- The proof of the general result makes use the contribution matrix $M = |D|QC^{-1}Q^{\mathsf{T}} \in \mathbb{Z}^{k(B)\times k(B)}$.
- The proposition remains true if the irreducible Brauer characters are replaced by an arbitrary basic set, i. e. a basis for the $\mathbb{Z}$-module of generalized Brauer characters spanned by $\mathrm{IBr}(B)$.
- Open: Is C also indecomposable in the sense above?

A result

Lemma

Let $A \in \mathbb{Z}^{k\times l}$ be indecomposable of rank l without vanishing rows. Then

$$\det(A^{\mathsf{T}}A) \geq l(k-l)+1.$$

Main Theorem I

With the notation above we have

$$k(B) \leq \frac{\det(C)-1}{l(B)} + l(B) \leq \det(C).$$

Remarks

- $\det(C)$ is locally determined by the theory of lower defect groups.
- Fujii gave a sufficient criterion for $\det(C) = |D|$.
- The Brauer-Feit bound is often stronger.
- What about equality?

Equality?

Proposition

Suppose that

$$k(B) = \frac{\det(C) - 1}{l(B)} + l(B).$$

Then the following holds:

- $\det(C) = |D|$.
- $C = (m + \delta_{ij})_{i,j}$ *up to basic sets where* $m := \frac{|D|-1}{l(B)}$.
- *All irreducible characters of* B *have height* 0.

Examples

- Let $d \geq 1$, $t \mid p^d - 1$ and $T \leq \mathbb{F}_{p^d}^\times$ such that $|T| = t$. Then the principal block of $\mathbb{F}_{p^d} \rtimes T$ satisfies the proposition with $l(B) = t$.
- If D is cyclic, then the proposition applies by Dade's Theorem.
- In view of Brauer's Height Zero Conjecture, one expects that the defect groups are abelian.
- The stronger condition $k(B) = \det(C)$ implies $k(B) = |D|$ and $l(B) \in \{1, |D| - 1\}$. In both cases D is abelian by results of Okuyama-Tsushima and Héthelyi-Külshammer-Kessar-S.
- The classification of the blocks with $k(B) = |D|$ is open even in the local case where $D \trianglelefteq G$ (Schmid).

Some consequences

- Brandt's result $k(B) \leq \operatorname{tr}(C) - l(B) + 1$ holds for any basic set. This makes it possible to apply the LLL reduction.
- $l(B) \leq 3 \Longrightarrow k(B) \leq |D|$. This improves a result by Olsson. The proof makes use of the reduction theory of quadratic forms.
- If D is abelian and B has Frobenius inertial quotient, then

$$k(B) \leq \frac{|D| - 1}{l(B)} + l(B).$$

This relates to work by Kessar-Linckelmann. If the inertial quotient is also abelian, then Alperin's Conjecture predicts equality.

Major subsections

- Many of the previous results remain true if C is replaced by a "local" Cartan matrix.
- Let $u \in \mathrm{Z}(D)$, and let b_u be a Brauer correspondent of B in $\mathrm{C}_G(u)$ with Cartan matrix C_u.
- The pair (u, b_u) is called major subsection.
- It is known that b_u has defect group D.
- Moreover, $C_u = Q_u^\mathsf{T}\overline{Q_u}$ where $Q_u \in \mathbb{C}^{k(B)\times l(b_u)}$ is the generalized decomposition matrix of B with respect to (u, b_u).

Problems

- In general, Q_u is not integral, but consists of algebraic integers of a cyclotomic field. Take coefficients with respect to an integral basis instead.
- $\det(C_u) > |D|$ unless $u = 1$ or $l(b_u) = 1$.
- Nevertheless, b_u dominates a block $\overline{b_u}$ of $\mathrm{C}_G(u)/\langle u\rangle$ with Cartan matrix $\overline{C_u} = |\langle u\rangle|^{-1}C_u$.
- It is not clear if there is a corresponding factorization $\overline{C_u} = R^\mathsf{T} R$ where R has at most $|\langle u\rangle|^{-1}k(B)$ non-zero rows (but there is a factorization where R has $k(\overline{b_u})$ rows).

Some local results

- (S.) $l(b_u) \leq 2 \implies k(B) \leq |D|$.
- (Héthelyi-Külshammer-S.)

$$k(B) \leq \sum_{1 \leq i \leq j \leq l(b_u)} q_{ij} c_{ij}$$

where

$$q = \sum_{1 \leq i \leq j \leq l(b_u)} q_{ij} X_i X_j$$

is a positive definite, integral quadratic form and $C_u = (c_{ij})$. This generalizes Külshammer-Wada.

Example

The last formula often implies $k(B) \leq |D|$, but not always:

Example

Let B be the principal 2-block of $A_4 \times A_4$. Then $l(B) = 9$ and

$$C = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \otimes \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \quad \text{(Kronecker product).}$$

There is no quadratic form q such that

$$\sum_{1 \leq i \leq j \leq l(b_u)} q_{ij} c_{ij} \leq 16 = |D|.$$

A different approach

- C_u determines a positive definite, integral quadratic form

$$q(x) := |D| x C_u^{-1} x^{\mathsf{T}} \qquad (x \in \mathbb{Z}^{l(b_u)}).$$

- The equivalence class of q does not depend on the basic set for b_u.
- $\mu(b_u) := \min\{q(x) : 0 \neq x \in \mathbb{Z}^{l(b_u)}\}$.
- Behaves nicely: $\mu(b_u) = \mu(\overline{b_u})$.

Lemma

- (*Brauer*) $\mu(b_u) \geq l(b_u) \Longrightarrow k(B) \leq |D|$.
- (*Robinson*) $\mu(b_u) = 1 \Longrightarrow k(B) \leq |D|$.

Example

The inequality $\mu(B) \geq l(B)$ is often true, but not always:

Example

Let B be the principal 2-block of $Z_2^3 \rtimes (Z_7 \rtimes Z_3)$. Then

$$8C^{-1} = \begin{pmatrix} 4 & 2 & 2 & 2 & 2 \\ 2 & 5 & 1 & 1 & 1 \\ 2 & 1 & 5 & 1 & 1 \\ 2 & 1 & 1 & 5 & 1 \\ 2 & 1 & 1 & 1 & 5 \end{pmatrix}$$

and $\mu(B) = 4 < 5 = l(B)$. Nevertheless, there is no factorization $C = R^{\mathsf{T}} R$ where R has more than 8 non-zero rows.

A result

Proposition (S.)

Let (u, b_u) be a major subsection such that u has order p^r. If $\det(\overline{C_u}) = |D|p^{-r}$, then $k(B) \leq |D|$.

The proof uses the following observation to show $\mu(b_u) \geq l(b_u)$.

Lemma

Let $A \in \mathbb{Z}^{k \times l}$ be indecomposable of rank l without vanishing rows. Let $\widetilde{A} = A^{\mathsf{T}} A$. Then

$$\min\{\det(\widetilde{A}) x \widetilde{A}^{-1} x^{\mathsf{T}} : 0 \neq x \in \mathbb{Z}^l\} \geq l.$$

Some consequences

Corollary

1. *(Brauer) If D is abelian of rank ≤ 2, then $k(B) \le |D|$.*
2. *If D is non-abelian of order p^3, then $k(B) \le |D|$.*
3. *If $D/\langle u \rangle$ is metacyclic and $p \le 5$, then $k(B) \le |D|$.*

Brauer's original proof of (1) uses of Dade's theory of cyclic defect groups. The new proof is quite elementary.
Part (3) relies on the following two results:

Tools

Theorem (Watanabe)

If D is non-abelian and metacyclic of odd order, then $l(B) \mid p-1$ and C has only two elementary divisors up to multiplicity.

Theorem (Mordell)

Let $S \in \mathbb{Z}^{l\times l}$ be symmetric and positive semidefinite with $l \le 5$. Then there exists $R \in \mathbb{Z}^{k\times l}$ such that $S = R^{\mathsf{T}}R$.

Unfortunately, Mordell's Theorem fails for $l \ge 6$ as one can see by the Gram matrix of the E_6 lattice.

Inertial indices

In the following we assume that the defect group D of B is abelian.

- Let b_D be a Brauer correspondent of B in $\mathrm{C}_G(D)$. Then

$$I(B) := \mathrm{N}_G(D, b_D)/\,\mathrm{C}_G(D)$$

  is the inertial quotient of B.
- $I(B) \le \mathrm{Aut}(D)$ is a p'-group.
- $D = [D, I(B)] \times \mathrm{C}_D(I(B))$.
- B is nilpotent iff $I(B) = 1$. In this case $k(B) \le |D|$.

Some results

- (Kessar-Malle) all irreducible characters in B have height 0 (uses CFSG)
- (Brauer, Kessar-Malle) $k(B) \leq \sqrt{l(b_u)}|D|$.
- (Robinson) If $I(B)$ is abelian, then $k(B) \leq |D|$.
- (S.) $k(B) \leq |D|^{\frac{3}{2}}$.
- (S.) $|I(B)| \leq 255 \Longrightarrow k(B) \leq |D|$.

The proofs rely on the existence of regular orbits.

A blockwise Z^*-theorem

Theorem (Watanabe)

For $u \in \mathrm{C}_D(I(B))$ we have $k(B) = k(b_u)$ and $l(B) = l(b_u)$. Moreover, C and C_u have the same elementary divisors counting multiplicities.

- Even more, the centers $\mathrm{Z}(B)$ and $\mathrm{Z}(b_u)$ are isomorphic algebras over an algebraically closed field of characteristic p.
- Open: Is $C = C_u$ up to basic sets?

Another local result

Theorem (S.)

Suppose there exists $u \in D$ such that $\mathrm{C}_{I(B)}(u)$ acts freely on $[D, \mathrm{C}_{I(B)}(u)]$. Then $k(B) \leq |D|$. This applies in particular, if $\mathrm{C}_{I(B)}(u)$ has prime order or if $[D, \mathrm{C}_{I(B)}(u)]$ is cyclic.

- The proof uses the Broué-Puig $*$-construction to show that $\overline{C_u} = R^\mathsf{T} R$ where R has $|\langle u\rangle|^{-1} k(B)$ rows.
- By a result of Halasi-Podoski there is always some $u \in D$ such that $\mathrm{C}_{I(B)}(u)$ has a regular orbit on $[D, \mathrm{C}_{I(B)}(u)]$.

Some consequences

Corollary

- *If* $I(B)$ *contains an abelian subgroup of prime index or index* 4*, then* $k(B) \le |D|$*.*
- *If the commutator subgroup* $I(B)'$ *has prime order or order* 4*, then* $k(B) \le |D|$*.*
- *If* $I(B)$ *has prime order or order* 4*, then* $l(B) \le |I(B)|$*.*

Regular orbits

Proposition (S.)

Let P be an abelian p-group, and let $A \leq \mathrm{Aut}(P)$ be a p'-group. If P has no elementary abelian direct summand (i. e. $\Omega(P) \subseteq \Phi(P)$), then A has a regular orbit on P.

Sketch of proof:

- Since A acts faithfully on $\Omega_2(P)$, we may assume that $\exp(P) = p^2$.
- An argument by Hartley-Turull shows that P is A-isomorphic to $\Omega(P) \times \Omega(P)$.
- A theorem by Halasi-Podoski provides a regular orbit on $\Omega(P) \times \Omega(P)$.

Regular orbits

Main Theorem II

Suppose that D has no elementary abelian direct summand of order p^4. Then $k(B) \leq |D|$.

- If p^4 is replaced by p^3, then the previous proposition guarantees an element $u \in D$ such that $[D, \mathrm{C}_{I(B)}(u)]$ is cyclic.
- The general proof goes along the lines of the $k(GV)$-problem which is concerned with the local situation $G = D \rtimes I(B)$.
- One also relies on the existence of perfect isometries for small inertial quotients (Puig-Usami).

Small defects, small primes

Corollary

Brauer's $k(B)$-Conjecture holds for blocks of defect at most 3.

If $p = 2$, then $I(B)$ is solvable by Feit-Thompson. This makes it possible to advance by computing C in small cases explicitly:

Proposition

If $p = 2$ and D has no elementary abelian direct summand of order 2^8, then $k(B) \leq |D|$.

In particular, Brauer's $k(B)$-Conjecture for $p = 2$ holds for abelian defect groups of rank at most 7.

Concluding remarks

- Despite the fact that some of the techniques from the solution of the $k(GV)$-problem carry over, the situation of arbitrary abelian defect groups is significantly harder.
- For instance, there is no reduction to the case where $I(B)$ acts irreducibly on D. This can be seen by the following example.

Example

Let $p = 2$ and $D \rtimes I(B) \cong (Z_2^5 \rtimes (Z_{31} \rtimes Z_5)) \times (Z_2^3 \rtimes (Z_7 \rtimes Z_3))$. Then the largest orbit has length $31 \cdot 7$, i. e. there exists $u \in D$ such that $\mathrm{C}_{I(B)}(u) \cong Z_{15}$. It is currently not known how to deal with this case.

Counterexample?

Suppose that $k(B) > |D|$. How does C look like?

- integral, symmetric, positive definite, permissible elementary divisors
- $l(B) \geq 4$
- $\det(C) > |D|$
- $1 < \mu(B) < l(B)$
- $\sum_{1 \leq i \leq j \leq l(B)} q_{ij} c_{ij} > |D|$ for all positive definite, integral quadratic forms q
- (Brauer) Let $(m_{ij}) = |D| Q C^{-1} Q^{\mathsf{T}}$ be the contribution matrix. Then m_{ii} is either divisible by p^2 or not divisible by p.
- If $m_{ij} = 0$, then $p^2 \mid m_{ii}$ and $p^2 \mid m_{jj}$.

Example (less naive)

$$Q = \begin{pmatrix} 1 & 1 & . & . & . \\ 1 & . & 1 & . & . \\ . & 1 & . & 1 & 1 \\ . & . & 1 & 1 & 1 \\ . & . & . & 1 & -1 \\ . & . & . & 1 & -1 \\ . & . & . & . & 1 \\ . & . & . & . & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 2 & 1 & 1 & . & . \\ 1 & 2 & . & 1 & 1 \\ 1 & . & 2 & 1 & 1 \\ . & 1 & 1 & 4 & . \\ . & 1 & 1 & . & 6 \end{pmatrix}.$$

Then $k(B) = 8$ and C has elementary divisors $1, 1, 2, 2, 4$. However, $m_{88} = 2$. Therefore C does not occur.

- In fact, I do not know any matrix C which fulfills all the constraints above.
- This means that the combination of the presented methods should be quite powerful.
- By the way, if you are interested in my book, I have plenty of free copies. Just let me know.