## Orthogonality relations for characters and blocks

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# Ordinary orthogonality

- Let G be a finite group.
- Let  $g_1, \ldots, g_k$  be a set of representatives for the conjugacy classes of G.
- Let  $Irr(G) = \{\chi_1, \dots, \chi_k\}$  be the complex irreducible characters of G.
- Then

$$T = (\chi_i(g_j))_{i,j=1}^k$$

is the (ordinary) character table of G.

# Ordinary orthogonality

### Theorem (Orthogonality relations)

We have

$$T^{\mathsf{t}}\overline{T} = \begin{pmatrix} |\mathcal{C}_G(g_1)| & 0\\ & \ddots & \\ 0 & |\mathcal{C}_G(g_k)| \end{pmatrix}$$

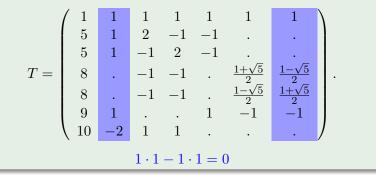
In particular,

$$(\chi_i,\chi_j)_G := \frac{1}{|G|} \sum_{g \in G} \chi_i(g) \overline{\chi_j(g)} = \delta_{ij}.$$

# The character table of $A_6$

### Example

The character table of the alternating group  ${\cal G}={\cal A}_6$  of degree 6 is given by



# Modular representation theory

In the following we fix a prime p.

### Definition

Let  $G_{p'}$  be the set of p'-elements of G. We define a graph  $\mathcal{G}$  with set of vertices  $\mathrm{Irr}(G)$  such that  $[\chi,\psi]$  is an edge iff

$$\sum_{g \in G_{p'}} \chi(g) \overline{\psi(g)} \neq 0.$$

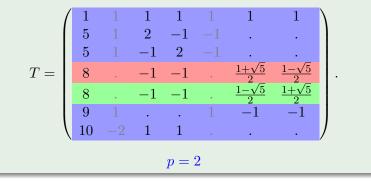
The connected components of  $\mathcal{G}$  are called the (*p*-)blocks of G.

If  $p \nmid |G|$ , then  $G_{p'} = G$  and the orthogonality relations imply that every p-block is a singleton.

## Block distribution for $A_6$

#### Example

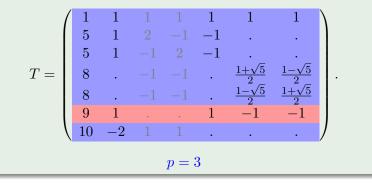
Again let  $G = A_6$ . Then the *p*-blocks are given as follows:



## Block distribution for $A_6$

#### Example

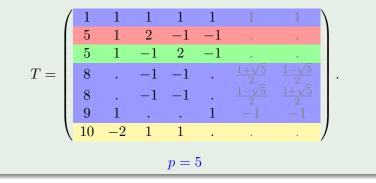
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## Block distribution for $A_6$

#### Example

Again let  $G = A_6$ . Then the *p*-blocks are given as follows:



## p-Factors and p-sections

For  $g \in G$ , the abelian group  $\langle g \rangle$  is a direct product of a *p*-group and a *p'*-group. Therefore, *g* can be written uniquely as  $g = g_p g_{p'}$ where  $g_p \in \langle g \rangle$  is a *p*-element and  $g_{p'} \in G_{p'}$ . Elements  $g, h \in G$  lie in the same (*p*-)section iff  $g_p$  and  $h_p$  are conjugate.

#### Theorem (Brauer)

If  $\chi, \psi \in Irr(G)$  lie in different blocks, then

$$\sum_{g \in S} \chi(g) \overline{\psi(g)} = 0$$

for every section S of G.

# Modular orthogonality

Brauer's Theorem refines the orthogonality of the rows of T. The following dual result deals with the columns of T.

Theorem (Block orthogonality relations)

Let  $g,h\in G$  lie in different sections, then

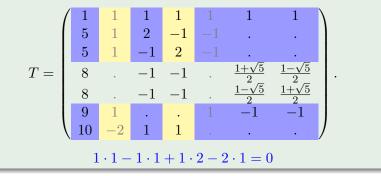
$$\sum_{\chi \in B} \chi(g) \overline{\chi(h)} = 0$$

for every block B of G.

# Block orthogonality for $A_6$

#### Example

Let 
$$G = A_6$$
 and  $p = 2$ .



# A converse result

Since  $g\in G_{p'}$  iff  $g_p=1,$  the theorem applies for  $g\in G_{p'}$  and  $h\in G\setminus G_{p'}.$ 

Theorem (Osima)

If  $J \subseteq Irr(G)$  such that

$$\sum_{\chi \in J} \chi(g) \chi(h) = 0 \qquad \forall g \in G_{p'}, \ h \in G \setminus G_{p'},$$

then J is a union of blocks.

It seems that the following stronger result holds.

## A converse result

### Conjecture (Harada, 1981)

If  $J \subseteq Irr(G)$  such that

$$\sum_{\chi \in J} \chi(1)\chi(h) = 0 \qquad \forall h \in G \setminus G_{p'},$$

then J is a union of blocks.

This conjecture holds if G is (p-)solvable or |J| = 1, i. e.  $\{\chi\}$  is a block iff  $\chi(g) = 0$  for all  $g \in G \setminus G_{p'}$ .

 $\chi$ 

# A dual approach

Harada's Conjecture asserts that the block orthogonality relations cannot be refined further. The situation is different for Brauer's Theorem as we will see.

### Definition (S.)

Let  ${\mathcal G}$  be the graph with set of vertices G such that (g,h) is an edge iff there exists a block B with

$$\sum_{\in \operatorname{Irr}(B)} \chi(g)\overline{\chi(h)} \neq 0.$$

The connected components of  $\mathcal{G}$  are called the (*p*-)class blocks of G.

# Examples

- Every class block is a union of conjugacy classes lying in a section of *G*.
- $\bullet~G$  has only one block if and only if the class blocks are the conjugacy classes.
- $\bullet\,$  If G has a normal p-complement, then the class block are the sections.

#### Example

The classes  $(3^2, 1^4)$ ,  $(3^2, 2^2)$ ,  $(6, 2, 1^2)$  and (6, 4) of  $A_{10}$  form a 3-section. The first two and the last two form class blocks.

# A generalization

### The following generalizes Brauer's Theorem.

Proposition (S.)

If  $\chi, \psi \in Irr(G)$  lie in different blocks, then

$$\sum_{g \in C} \chi(g) \overline{\psi(g)} = 0$$

for every class block C of G.

# Dual Osima

There is also a dual version of Osima's Theorem:

Proposition (S.)

Let J be a union of conjugacy classes of G such that

$$\sum_{g \in J} \chi(g) \overline{\psi(g)} = 0 \qquad \forall \chi, \psi \in \operatorname{Irr}(G) \text{ in different blocks.}$$

Then J is a union of class blocks.

The result is false if we fix  $\chi = 1_G$ . Hence, there is no dual version of Harada's Conjecture.

### Brauer characters

- Now let F be an algebraically closed field of characteristic p.
- Then every character χ of G over F determines a Brauer character φ : G<sub>p'</sub> → C by "lifting" χ(g) to C.
- The (finite) set of irreducible Brauer characters of G is denoted by IBr(G).
- The values of these functions can be expressed with the Brauer character table  $T_p = (\varphi_i(g_j))_{i,j}$ . This is again a complex square matrix (the  $g_j$  represent the conjugacy classes inside  $G_{p'}$ ).

• If 
$$p \nmid |G|$$
, then  $Irr(G) = IBr(G)$  and  $T_p = T$ .

# Generalized decomposition numbers

In the following let  $G_p$  be the set of *p*-elements of *G*.

#### Proposition

Let  $u \in G_p$  and let  $\chi \in Irr(G)$ . Then there are uniquely determined algebraic integers  $d^u_{\chi\varphi}$  in the cyclotomic field  $\mathbb{Q}_{|\langle u \rangle|}$  such that

$$\chi(uv) = \sum_{\varphi \in \operatorname{IBr}(\mathcal{C}_G(u))} d^u_{\chi\varphi}\varphi(v) \qquad \forall v \in \mathcal{C}_G(u)_{p'}.$$

The numbers  $d^u_{\gamma\omega}$  are called generalized decomposition numbers.

For u = 1 we obtain a connection between Irr(G) and IBr(G).

## Brauer characters of blocks

#### Definition

Let B be a block of G. We define

$$\operatorname{IBr}(B) := \{ \varphi \in \operatorname{IBr}(G) : d^1_{\chi \varphi} \neq 0 \text{ for some } \chi \in B \}.$$

- This yields a partition of IBr(G), i.e. every irreducible Brauer character belongs to exactly one block.
- In fact, the sets  $\operatorname{IBr}(B)$  are precisely the connected components of the graph  $\mathcal G$  on  $\operatorname{IBr}(G)$  with edges  $[\varphi, \mu]$  where

$$\sum_{g \in G_{p'}} \varphi(g) \overline{\mu(g)} \neq 0.$$

Orthogonality relations Brauer characters and decomposition numbers

### The Brauer character table of $A_6$

#### Example

$$T_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 1 & -2 & -1 & -1 \\ 4 & -2 & 1 & -1 & -1 \\ 8 & -1 & -1 & \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 8 & -1 & -1 & \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} \end{pmatrix}.$$

Orthogonality relations Brauer characters and decomposition numbers

## The Brauer character table of $A_6$

#### Example

$$T_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & -1 & 1 & \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 3 & -1 & 1 & \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} \\ 4 & . & -2 & -1 & -1 \\ 9 & 1 & 1 & -1 & -1 \end{pmatrix}.$$

Orthogonality relations Brauer characters and decomposition numbers

## The Brauer character table of $A_6$

#### Example

$$T_5 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 5 & 1 & 2 & -1 & -1 \\ 5 & 1 & -1 & 2 & -1 \\ 8 & . & -1 & -1 & . \\ 10 & -2 & 1 & 1 & . \end{pmatrix}.$$

## Brauer correspondence

For a character  $\chi$  of G and a subgroup  $H \leq G$ , the restriction  $\chi_H : H \to \mathbb{C}$  is a character of H.

#### Definition

Let  $u \in G_p$ , let b be a block of  $H := C_G(u)$  and let  $\psi \in b$ . Then the Brauer correspondent of b is the unique block  $B := b^G$  of G such that the p-parts of  $\sum_{\chi \in B} \chi(1)(\chi_H, \psi)_H$  and  $|G : H|\psi(1)$  coincide.

#### Theorem (Brauer's Second Main Theorem)

In the situation above we have  $d^u_{\chi\varphi} = 0$  unless  $\varphi \in \operatorname{IBr}(b)$  and  $\chi \in \operatorname{Irr}(b^G)$ .

## Generalized decomposition matrices

### Proposition

Let  $u_1, \ldots, u_r \in G_p$  be representatives for the conjugacy classes in  $G_p$ . We define a matrix

$$Q_p := \left( d_{\chi\varphi}^{u_i} : \chi \in \operatorname{Irr}(G), \ i = 1, \dots, r, \ \varphi \in \operatorname{IBr}(\mathcal{C}_G(u_i)) \right)$$

whose rows are indexed by Irr(G) and the columns are indexed by pairs  $(i, \varphi)$  with  $\varphi \in IBr(C_G(u_i))$ . Then  $Q_p$  is invertible, in particular it has square shape.

## Generalized decomposition matrices of blocks

According to Brauer's second main theorem,  ${\cal Q}_p$  can be arranged in the form

$$Q_p = \begin{pmatrix} W_1 & 0 \\ & \ddots & \\ 0 & & W_n \end{pmatrix}$$

where the  $W_i$  correspond to the blocks  $B_i$  of G. We call  $W_i$  the generalized decomposition matrix of  $B_i$ .

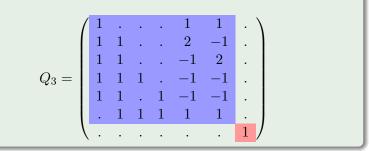
# The generalized decomposition matrix of $A_6$

### Example

$$Q_2 = \begin{pmatrix} 1 & . & . & 1 & 1 & . & . \\ 1 & 1 & . & 1 & -1 & . & . \\ 1 & . & 1 & 1 & -1 & . & . \\ 1 & 1 & 1 & 1 & 1 & . & . \\ 2 & 1 & 1 & -2 & . & . & . \\ . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & 1 \end{pmatrix}$$

## The generalized decomposition matrix of $A_6$

#### Example



# The generalized decomposition matrix of $A_6$

#### Example

$$Q_{5} = \begin{pmatrix} 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} & \cdot & \cdot & \cdot \\ \cdot & 1 & \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & -1 & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 \end{pmatrix}$$

## Another orthogonality

Theorem (Orthogonality of generalized decomposition numbers)

The generalized decomposition matrix  $Q_B$  of a block B can be arranged such that

$$Q_B^t \overline{Q_B} = \begin{pmatrix} C_1 & 0 \\ & \ddots & \\ 0 & & C_l \end{pmatrix}$$

where each  $C_i$  is the Cartan matrix of a block b of  $C_G(u)$  such that  $b^G = B$ . In particular,  $Q_B^t \overline{Q_B}$  is integral and positive definite. Moreover,  $\det(Q_B^t \overline{Q_B})$  is a p-power.

Orthogonality relations Brauer characters and decomposition numbers

# Another orthogonality of $A_6$

### Example

Let 
$$G = A_6$$
 and  $p = 2$ .

$$Q_{2} = \begin{pmatrix} 1 & . & . & 1 & 1 & . & . \\ 1 & 1 & . & 1 & -1 & . & . \\ 1 & . & 1 & 1 & -1 & . & . \\ 1 & 1 & 1 & 1 & 1 & . & . \\ 2 & 1 & 1 & -2 & . & . & . \\ . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & 1 \end{pmatrix}$$

Orthogonality relations Brauer characters and decomposition numbers

# Another orthogonality of $A_6$

### Example

Let 
$$G = A_6$$
 and  $p = 2$ .

$$Q_{2} = \begin{pmatrix} 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ 1 & 1 & \cdot & 1 & -1 & \cdot & \cdot \\ 1 & \cdot & 1 & 1 & -1 & \cdot & \cdot \\ 1 & 1 & 1 & 1 & 1 & \cdot & \cdot \\ 2 & 1 & 1 & -2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}$$
$$-1 \cdot 1 + 1 \cdot 1 = 0$$

# Refinements

• By the theorem above, we may restrict ourselves to the matrix

$$Q_B^{(u,b)} := (d^u_{\chi\varphi})_{\chi \in B, \, \varphi \in \mathrm{IBr}(b)}$$

where  $u \in G_p$  and b is a block of  $C_G(u)$  with  $b^G = B$ .

- Let  $p^r$  be the order of u, and let  $\zeta := e^{\frac{2\pi i}{p^r}} \in \mathbb{C}$ .
- Then  $1, \zeta, \ldots, \zeta^s$  where  $s := p^{r-1}(p-1) 1$  is an integral basis for  $\mathbb{Q}_{p^r}$ .
- Hence, for  $\chi \in B$  and  $\varphi \in \text{IBr}(b)$  there exist integers  $a_i(\chi, \varphi)$  such that  $d^u_{\chi\varphi} = \sum_{i=0}^s a_i(\chi, \varphi)\zeta^i$ .

# Discrete Fourier transformation

- Consequently,  $Q_B^{(u,b)}$  can be represented by an integral matrix A of size  $|B| \times s |\text{IBr}(b)|$ . This can be understood as a discrete Fourier transformation.
- By the results above, it is natural to ask if  $A^{\rm t}A$  actually depends on  $Q_B^{(u,b)}.$

### Theorem (S. 2015)

The matrix  $A^{t}A$  only depends on the following local invariants:

• the Cartan matrix  $C_b$  of b,

2 the group 
$$\mathcal{N} := \operatorname{N}_G(\langle u \rangle, b) / \operatorname{C}_G(u)$$
,

**③** the action of  $\mathcal{N}$  on  $\operatorname{IBr}(b)$  by conjugation.

# A special case

Unfortunately,  $A^{t}A$  does not have a "nice" shape in terms of these three ingredients (it is usually not invertible). But in a special case things behave better.

### Proposition (S. 2015)

Suppose that  $\mathcal{N}$  acts trivially on  $\operatorname{IBr}(b)$ . Then  $A^{t}A$  is a Kronecker product of the form  $C_{b} \otimes S$  where S is related to the semidirect product  $\langle u \rangle \rtimes \mathcal{N}$ .

# A trivial case

For u = 1,  $Q_B^{(1,B)}$  is already integral and  $A^{t}A = C_B$ . Here the following is of interest.

#### Basic set conjecture ( $\leq 1991$ )

There exists  $J \subseteq B$  such that  $|J| = |\operatorname{IBr}(B)|$  and the matrix  $(d^1_{\chi\varphi})_{\chi\in J, \varphi\in\operatorname{IBr}(B)}$  has determinant  $\pm 1$ .

This conjecture is satisfied if G is (p-)solvable or |IBr(B)| = 1 (Malle-Navarro-Späth 2015).

## The number of characters in a block

For  $\chi \in Irr(G)$  it is known that  $\chi(1) \in \mathbb{N}$  divides |G|.

#### Definition

Let B be a p-block of G. Then the largest integer d such that  $p^d \mid \frac{|G|}{\chi(1)}$  for some  $\chi \in B$  is called the defect of B. We write d(B) := d.

#### Conjecture (Brauer, 1954)

For every block B we have  $|B| \leq p^{d(B)}$ .

#### Theorem (Brauer-Feit)

For every block B we have  $|B| \leq p^{2d(B)}$ .

# Non-zero decomposition numbers

#### Proposition

Let b be a block of  $C_G(u)$  such that b and  $B := b^G$  have the same defect. Then for every  $\chi \in B$  there exists a  $\varphi \in IBr(b)$  such that  $d^u_{\chi\varphi} \neq 0$ .

• It follows that

$$|B| \le \sum_{\chi \in B} \sum_{\varphi \in \mathrm{IBr}(b)} |d^u_{\chi \varphi}|^2 = \mathrm{tr}(C_b).$$

- The proposition applies with u = 1 and B = b. In this case we also have  $|B| \leq \det(C_B)$  (S. 2015).
- To improve these bounds we apply the discrete Fourier transformation introduced earlier.

# A global-local bound

#### Theorem (S. 2015)

Let  $A^{t}A$  be the integral matrix coming from the discrete Fourier transformation of  $Q_{B}^{(u,b)}$ . Let  $m \in \mathbb{N}$  be maximal with the property that there exists an integral matrix M with m non-zero rows such that  $M^{t}M = A^{t}A$ . Then  $|B| \leq m$ .

- The importance of the theorem is that  $A^{t}A$  is locally determined and thus easier to compute than A itself.
- There is an algorithm by Plesken which finds all matrices M such that  $M^{t}M = A^{t}A$ . This can be used to compute m in the theorem above.

# A bigger example

### Example

• Let  $G = {}^2F_4(2)'$ . This is a simple group of Lie type of order

$$|G| = 17,971,200 = 2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$$

- Let B be the principal 3-block  $(1_G \in B)$ .
- Then d(B) = 3 and Brauer's conjecture asserts that  $|B| \le 27$ .
- Let  $u \in G$  be of order 3 and let b be a block of  $C_G(u)$  such that  $b^G = B$ .

# A bigger example

### Example (continued)

• Then d(b) = 3 and

$$C_b = 3 \begin{pmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{pmatrix}$$

- In fact,  $C_b$  is determined by  $C_G(u)/\langle u \rangle \cong Z_3^2 \rtimes Z_4$ , a group with only 36 elements.
- This implies  $|B| \leq 18$  (use something more clever than  $tr(C_b)$ ).

# A bigger example

### Example (continued)

The matrix  $A^{t}A$  is given by

$$A^{\mathsf{t}}A = \begin{pmatrix} 8 & 1 & 7 & -1 & 6 & . & 6 & . \\ 1 & 2 & -1 & -2 & . & . & . \\ 7 & -1 & 8 & 1 & 6 & . & 6 & . \\ -1 & -2 & 1 & 2 & . & . & . & . \\ 6 & . & 6 & . & 9 & . & 6 & . \\ . & . & . & . & . & . & . & . \\ 6 & . & 6 & . & 6 & . & 9 & . \\ . & . & . & . & . & . & . & . \end{pmatrix}$$

Plesken's algorithm gives  $|B| \le 15$ . In fact, it is known that |B| = 13.

An application of the global-local bound together with other ideas leads to the following.

### Theorem (S. 2015)

Let B be a p-block with  $d(B) \leq 3$  (or  $d(B) \leq 5$  if p = 2). Then  $|B| \leq p^{d(B)}$ , i. e. Brauer's conjecture holds for B.

# Defect groups

### Definition

Let B be a block of G. A defect group of B is a maximal p-subgroup  $D \leq G$  such that B has a Brauer correspondent in  $N_G(D)$ .

- One can show that D is unique up to conjugation and  $|D|=p^{d(B)}.$
- When we study the matrix  $Q_B^{(u,b)}$ , we may always assume that u lies in a defect group of B.

# Abelian defect groups

#### Proposition

Let B be a block with defect group D, let  $u \in Z(D)$  and let b be a block of  $C_G(u)$  such that  $b^G = B$ . Then d(b) = d(B) and the global-local bound established above applies.

If D is abelian, then  $\mathbf{Z}(D)=D$  and the methods are particularly strong.

Theorem (S. 2014)

Let B be a block with abelian defect group. Then  $|B| \leq p^{3d(B)/2}$ .

Brauer's conjecture A global-local bound Defect groups

## Abelian defect groups

### Theorem (S. 2015)

Let B be a block with abelian defect group of rank  $\leq 3$  (or  $\leq 7$  if p = 2). Then  $|B| \leq p^{d(B)}$ .

Both results rely implicitly on the classification of the finite simple groups.