Block theory and fusion systems

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Blocks

- Let G be a finite group and p be a prime.
- Let (K, R, F) be a p-modular system, i. e.
  - K is a field of characteristic 0 which contains all $|G|$-th roots of unity.
  - R is a complete discrete valuation ring with quotient field K and maximal ideal (π).
  - $F = R/(\pi)$ is an algebraically closed field of characteristic p.

The block algebra RG splits in a direct sum of minimal twosided ideals

$$RG = B_1 \oplus \ldots \oplus B_n.$$

Definition

The summands B_i are called blocks of RG.

- Every block B of RG is an algebra itself such that the unity element e_B is a primitive idempotent in the center of RG.
- The element e_B is called block idempotent.
- The canonical map $R \to F$ induces a bijection between the blocks of RG and the blocks of FG.

Characters in blocks

- Let $\chi \in \mathrm{Irr}(G)$ be an (ordinary) irreducible character of G over K.
- For a conjugacy class C of G we define the class sum $C^+ := \sum_{x \in C} x \in \mathrm{Z}(FG)$.
- Then for $x \in C$ the map

$$\omega_\chi(C^+) := \frac{|C|}{\chi(1)}\chi(x) + (\pi) \in F$$

defines a homomorphism $\omega_\chi : \mathrm{Z}(FG) \to F$ of algebras.

- There is precisely one block idempotent e_B such that $\omega_\chi(e_B) = 1$. For all other block idempotents $e_{B'}$ we have $\omega_\chi(e_{B'}) = 0$.

- In this case we say that χ belongs to the block B. We write $\chi \in \mathrm{Irr}(B)$.
- If $\chi, \psi \in \mathrm{Irr}(B)$, then $\omega_\chi = \omega_\psi =: \omega_B$ is the central character of B.

Definition

If the trivial character belongs to B, B is called the principal block of RG.

- In a similar way we assign every irreducible Brauer character φ of G to a block B. In this case we write $\varphi \in \mathrm{IBr}(B)$.
- This gives numerical invariants $k(B) := |\mathrm{Irr}(B)|$ and $l(B) := |\mathrm{IBr}(B)|$ for a block B of RG.
- The number $k(B)$ is also the dimension of the center of B and the number $l(B)$ is also the number of simple B-modules.

Defect groups

- Let $C \in \mathrm{Cl}(G)$ be a conjugacy class and $x \in C$. Then a Sylow p-subgroup of $\mathrm{C}_G(x)$ is called defect group of C. We write $\mathrm{Def}(C)$ for the set of defect groups of C.
- For subgroups $S, T \le G$ we write $S \le_G T$ if there exists a $g \in G$ such that $gSg^{-1} \le T$.
- For a p-subgroup $P \le G$ we define

$$I_P(FG) := \mathrm{span}_F\{C^+ : C \in \mathrm{Cl}(G),\ Q \le_G P \text{ for } Q \in \mathrm{Def}(C)\}.$$

- Let B be a block of RG with block idempotent e_B.

- Then there exists a p-subgroup $D \leq G$ such that $e_B \in I_D(FG)$, but $e_B \notin I_Q(FG)$ for all $Q < D$.

Definition

The group D is called defect group of B.

- D is unique up to conjugation and thus up to isomorphism.

Example

The defect groups of the principal block of RG are just the Sylow p-subgroups of G.

- The structure (in particular $k(B)$ and $l(B)$) of B is strongly influenced by D.
- For example B is a simple algebra if and only if D is trivial. In this case we have $k(B) = l(B) = 1$.

The height of a character

- Let D be a defect group of B, and let $\chi \in \mathrm{Irr}(B)$.
- Write $|D| = p^d$ and $|G| = p^a m$ such that $p \nmid m$. Then $p^{a-d} \mid \chi(1)$.

Definition

The largest integer $h(\chi) \in \mathbb{N}_0$ such that $p^{a-d+h(\chi)} \mid \chi(1)$ is called height of χ.

- We set $k_i(B) := |\{\chi \in \mathrm{Irr}(B) : h(\chi) = i\}|$ for $i \in \mathbb{N}_0$.
- It is known that $k_0(B) > 0$ for every block B.

The Brauer correspondence

Definition

Let $H \leq G$. Then we define

$$\mathrm{Br}_H^G : \mathrm{Z}(FG) \to \mathrm{Z}(FH),\ \ C^+ \mapsto (C \cap H)^+,$$

where $\varnothing^+ := 0$.

- If H is a p-group, Br_H^G is a homomorphism of algebras, called the Brauer homomorphism.
- Let b be a block of RH. Then $\omega_b \circ \mathrm{Br}_H^G : \mathrm{Z}(FG) \to F$.
- If there exists a block B of RG such that $\omega_b \circ \mathrm{Br}_H^G = \omega_B$, we say that B is a Brauer correspondent of b and conversely. We write $b^G = B$.

Inertial indices

- Let B be a block of RG with defect group D and Brauer correspondent b in $RD\,\mathrm{C}_G(D)$.
- We set $\mathrm{N}_G(D,b) := \{g \in \mathrm{N}_G(D) : gbg^{-1} = b\}$.

Definition

Then $e(B) := |\mathrm{N}_G(D,b) : D\,\mathrm{C}_G(D)|$ is called inertial index of B.

- It is known that $p \nmid e(B) \mid |\mathrm{Aut}(D)|$.

Conjectures

Several open conjectures predict a connection between the block invariants $k(B)$, $k_i(B)$ and $l(B)$ on the one hand and the defect group on the other hand.

Brauer's $k(B)$-Conjecture, 1954

For a block B with defect group D we have $k(B) \leq |D|$.

Olsson's Conjecture, 1975

For a block B with defect group D we have $k_0(B) \leq |D : D'|$.

Brauer's Height Zero Conjecture, 1956

A block B has abelian defect group if and only if $k(B) = k_0(B)$.

Alperin's Weight Conjecture, 1987

For a block B the number $l(B)$ is the number of conjugacy classes of weights for B.

Here a weight for B is a pair of the form (P, β), where $P \leq G$ is a p-subgroup and β is a block of $R[\mathrm{N}_G(P)/P]$ with trivial defect group. Moreover, β is dominated by a Brauer correspondent of B in $R\,\mathrm{N}_G(P)$.

Alperin-McKay Conjecture, 1975

For a block B with defect group D and Brauer correspondent b in $R\,\mathrm{N}_G(D)$ we have $k_0(B) = k_0(b)$.

All these conjectures are known to be true for blocks with cyclic defect groups by the following result of Dade:

Theorem (Dade)

Let B be a block of RG with cyclic defect group D. Then

$$k(B) = k_0(B) = \frac{|D| - 1}{e(B)} + e(B), \qquad l(B) = e(B).$$

Definition of fusion systems

- Let P be a finite p-group, and let $\mathcal{F}$ be a category whose objects are the subgroups of P and whose morphisms are injective group homomorphisms.
- A subgroup $Q \le P$ is called fully $\mathcal{F}$-normalized if $|\mathrm{N}_P(Q)| \ge |\mathrm{N}_P(Q_1)|$ whether Q and Q_1 are $\mathcal{F}$-isomorphic.
- For a morphism $\varphi : S \to P$ in $\mathcal{F}$ we set

$$\mathrm{N}_\varphi := \{y \in \mathrm{N}_P(S) : \exists z \in \mathrm{N}_P(\varphi(S)) : \varphi(yxy^{-1}) = z\varphi(x)z^{-1}\ \forall x \in S\}.$$

Definition

The category $\mathcal{F}$ is called (saturated) fusion system on P if the following properties hold:

(i) For $S \le T \le P$ the inclusion $S \hookrightarrow T$ is a morphism in $\mathcal{F}$.

(ii) For $\varphi \in \mathrm{Hom}_{\mathcal{F}}(S, T)$ we also have $\varphi \in \mathrm{Hom}_{\mathcal{F}}(S, \varphi(S))$ and $\varphi^{-1} \in \mathrm{Hom}_{\mathcal{F}}(\varphi(S), S)$.

(iii) For $S, T \le P$ we have $\mathrm{Hom}_P(S, T) \subseteq \mathrm{Hom}_{\mathcal{F}}(S, T)$.

(iv) $\mathrm{Inn}(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$.

(v) If $\varphi \in \mathrm{Hom}_{\mathcal{F}}(S, T)$ and $\varphi(S)$ is fully $\mathcal{F}$-normalized, then φ extends to a morphism $\mathrm{N}_\varphi \to P$ in $\mathcal{F}$.

The fusion system of a block

If B is a block of RG with defect group D, one can define a fusion system $\mathcal{F}_D(B)$ on D in the following way:

- If $Q \leq G$ is a p-subgroup and b is a block of $RQ\,\mathrm{C}_G(Q)$ with $b^G = B$, we call the pair (Q, b) a B-subpair.
- For subpairs (S, b_S) and (T, b_T) with $S \trianglelefteq T$ and $b_S^{T\,\mathrm{C}_G(S)} = b_T^{T\,\mathrm{C}_G(S)}$ we write $(S, b_S) \trianglelefteq (T, b_T)$.
- Let $\leq$ be the transitive closure of $\trianglelefteq$ for subpairs.
- Take a Brauer correspondent b_D of B in $RD\,\mathrm{C}_G(D)$.

- Then for each subgroup $Q \le D$ there is a unique block b_Q of $RQ\,\mathrm{C}_G(Q)$ such $(Q, b_Q) \le (D, b_D)$.
- For $S, T \le D$ we define the set of $\mathcal{F}_D(B)$-morphisms as follows

$$\begin{aligned}\mathrm{Hom}_{\mathcal{F}_D(B)}(S,T) := \{\varphi : S \to T : \exists g \in G : {}^g(S, b_S) \le (T, b_T)\\ \wedge\ \varphi(x) = gxg^{-1}\ \forall x \in S\}.\end{aligned}$$

- Here ${}^g(S, b_S) := (gSg^{-1}, gb_Sg^{-1})$ is also a B-subpair.

Examples

Example

If B is the principal block of RG, then $\mathcal{F}_D(B) = \mathcal{F}_D(G)$ is just the fusion system coming from the conjugation action of G (Brauer's third main theorem). In particular every fusion system of a finite group is also a fusion system of a block.

If $\mathcal{F}_D(B) = \mathcal{F}_D(D)$, the block B is nilpotent. Then the structure of B is determined by the following result of Puig:

Theorem (Puig)

If B is a nilpotent block of RG with defect group D, then $B \cong (RD)^{n\times n}$ for some $n \in \mathbb{N}$. In particular

$$k(B) = k(D) := |\mathrm{Irr}(D)|, \qquad k_i(B) = k_i(D), \qquad l(B) = 1.$$

Example

Let B be a block of RG with abelian defect group D. Then B is nilpotent if and only if $e(B) = 1$. In this case we have $k(B) = k_0(B) = |D|$ and $l(B) = 1$.

Alperin's fusion theorem

- Let $\mathcal{F}$ be an arbitrary fusion system on a finite p-group P.
- Then the morphisms of $\mathcal{F}$ are controlled by $\mathcal{F}$-essential subgroups.
- A subgroup $Q \le P$ is called $\mathcal{F}$-essential if the following conditions hold:
  (i) Q is fully $\mathcal{F}$-normalized.
  (ii) Q is $\mathcal{F}$-centric, i. e. $\mathrm{C}_P(Q_1) = \mathrm{Z}(Q_1)$ if Q and Q_1 are $\mathcal{F}$-isomorphic.
  (iii) $\mathrm{Out}_{\mathcal{F}}(Q)$ contains a strongly p-embedded subgroup H, i. e. $p \mid |H|$, $p \nmid |\mathrm{Out}_{\mathcal{F}}(Q) : H| > 1$ and $p \nmid |H \cap xHx^{-1}|$ for all $x \in \mathrm{Out}_{\mathcal{F}}(Q) \setminus H$.

Let $\mathcal{E}$ be a set of representatives for the $\mathrm{Aut}_{\mathcal{F}}(P)$-conjugacy classes of $\mathcal{F}$-essential subgroups.

Theorem (Alperin's Fusion Theorem)

Every isomorphism in $\mathcal{F}$ is a composition of finitely many isomorphisms of the form $\varphi : S \to T$ such that $S, T \le Q \in \mathcal{E} \cup \{P\}$ and there exists $\psi \in \mathrm{Aut}_{\mathcal{F}}(Q)$ with $\psi_{|S} = \varphi$. Moreover, if $Q \neq P$, we may assume that ψ is a p-element.

In many cases we have $\mathcal{E} = \varnothing$. Then $\mathcal{F}$ is controlled by P.

Example

Every fusion system on an abelian p-group P is controlled by P.

Example (Stancu)

Every fusion system on a metacyclic p-group P for an odd prime p is controlled by P.

If $\mathcal{F}$ is controlled by P and $\mathrm{Aut}_{\mathcal{F}}(P)$ is a p-group, then $\mathcal{F} = \mathcal{F}_P(P)$. In particular:

Example

Let B be a block with defect group D such that $\mathcal{F}_D(B)$ is controlled by D (i. e. B is a controlled block) and $\mathrm{Aut}(D)$ is a p-group, then B is nilpotent.

Essential subgroups

We deduce some group theoretical properties of $\mathcal{F}$-essential subgroups.

Proposition

Let $Q \le P$ be $\mathcal{F}$-essential of rank r, i. e. $|Q/\Phi(Q)| = p^r$. Then

$$\mathrm{Out}_{\mathcal{F}}(Q) \le \mathrm{Aut}(Q/\Phi(Q)) \cong \mathrm{GL}(r,p),$$
$$|\mathrm{N}_P(Q)/Q| \le p^{r(r-1)/2},$$
$$[x,Q] \not\subseteq \Phi(Q) \quad \forall x \in \mathrm{N}_P(Q) \setminus Q.$$

Moreover, $\mathrm{N}_P(Q)/Q$ has nilpotency class at most $r-1$ and exponent at most $p^{\lceil \log_p(r) \rceil}$. In particular $|\mathrm{N}_P(Q)/Q| = p$ if $r = 2$.

Proof.

- The kernel of the canonical map $\mathrm{Aut}_{\mathcal{F}}(Q) \to \mathrm{Aut}(Q/\Phi(Q))$ is a p-group containing $\mathrm{Inn}(Q)$.
- On the other hand $\mathrm{O}_p(\mathrm{Aut}_{\mathcal{F}}(Q)) = \mathrm{Inn}(Q)$, since Q is also $\mathcal{F}$-radical.
- This shows $\mathrm{Out}_{\mathcal{F}}(Q) \le \mathrm{Aut}(Q/\Phi(Q)) \cong \mathrm{GL}(r, p)$. In particular $\mathrm{N}_P(Q)/Q \le \mathrm{Out}_{\mathcal{F}}(Q)$ acts faithfully on $Q/\Phi(Q)$.
- Moreover, we can regard $\mathrm{N}_P(Q)/Q$ as a subgroup of the group of upper triangular matrices with ones on the main diagonal.
- The other claims follow from this. □

The case $p = 2$

For $p = 2$ the groups with a strongly p-embedded subgroup are known by the following result of Bender:

Theorem (Bender)

Let H be a finite group with a strongly 2-embedded subgroup. Then one of the following holds:

(i) *The Sylow 2-subgroups of H are cyclic or quaternion. In particular H is not simple.*

(ii) *There exists a normal series $1 \leq M < L \leq H$ such that M and H/L have odd order (and thus are solvable) and L/M is isomorphic to one of the following simple groups:*

$$\mathrm{SL}(2, 2^n), \qquad \mathrm{PSU}(3, 2^n), \qquad \mathrm{Sz}(2^{2n-1}) \qquad (n \geq 2).$$

- The Sylow 2-subgroups of H in Bender's theorem are Suzuki 2-groups, i. e. they admit an automorphism which permutes the involutions transitively.
- Hence, we can apply Higman's results about Suzuki 2-groups.
- Moreover, for an $\mathcal{F}$-essential subgroup $Q \leq P$ we can bound the order of $\mathrm{N}_P(Q)/Q$ by a comparison of the exponent of $\mathrm{SL}(2, 2^n)$, $\mathrm{PSU}(3, 2^n)$, $\mathrm{Sz}(2^{2n-1})$ on the one hand and $\mathrm{GL}(r, 2)$ on the other hand.

Theorem

If $p = 2$ and $Q \le P$ is $\mathcal{F}$-essential of rank r, then one of the following holds for $N := \mathrm{N}_P(Q)/Q$:

(i) *N is cyclic of order at most $2^{\lceil \log_2(r) \rceil}$.*
(ii) *N is quaternion of order at most $2^{\lceil \log_2(r) \rceil + 1}$.*
(iii) *N is elementary abelian of order at most $2^{\lfloor r/2 \rfloor}$.*
(iv) $\Omega(N) = \mathrm{Z}(N) = \Phi(N) = N'$ *and* $|N| = |\Omega(N)|^2 \le 2^{\lfloor r/2 \rfloor}$.
(v) $\Omega(N) = \mathrm{Z}(N) = \Phi(N) = N'$ *and* $|N| = |\Omega(N)|^3 \le 2^{\lfloor r/2 \rfloor}$.

In particular N has nilpotency class $1, 2$ or maximal class. Moreover, N has exponent 2, 4, $|N|/2$ or $|N|$.

Proposition

If $p = 2$ and $Q \leq P$ is $\mathcal{F}$-essential of rank at most 3, then $|\mathrm{N}_P(Q)/Q| = 2$ and $\mathrm{Out}_{\mathcal{F}}(Q) \cong S_3$.

Proposition

If $p = 2$ and $Q \leq P$ is $\mathcal{F}$-essential of rank 4, then $|\mathrm{N}_P(Q)/Q| \leq 4$ and $|\mathrm{Out}_{\mathcal{F}}(Q)| \in \{6, 10, 18, 20, 30, 36, 60, 180\}$.

Proposition

Let $\mathcal{F}$ be a fusion system on a finite 2-group P with nilpotency class 2. Then every $\mathcal{F}$-essential subgroup $Q \le P$ is normal and P/Q is cyclic or elementary abelian.

Proof.

- Since $P' \subseteq \mathrm{Z}(P) \subseteq \mathrm{C}_P(Q) \subseteq Q$, we have $Q \trianglelefteq P$ and P/Q is abelian.
- By the previous theorem P/Q is cyclic or elementary abelian.

□

Proposition

If $Q \in \{C_2 \times C_2, D_8, Q_8\}$ is a self-centralizing subgroup of P, then P has maximal class, i. e. P is a dihedral, semidihedral or quaternion group. This holds in particular if Q is $\mathcal{F}$-essential.

Metacyclic defect groups

Theorem

Let B be a 2-block of RG with metacyclic defect group D. Then one of the following holds:

(1) *B is nilpotent.*

(2) *D is a dihedral group of order $2^n \geq 8$. Then $k(B) = 2^{n-2} + 3$, $k_0(B) = 4$ and $k_1(B) = 2^{n-2} - 1$. According to two different fusion systems, $l(B)$ is 2 or 3.*

(3) *D is a quaternion group of order 8. Then $k(B) = 7$, $k_0(B) = 4$ and $k_1(B) = l(B) = 3$.*

Theorem (continuation)

(4) *D is a quaternion group of order $2^n \geq 16$. Then $k_0(B) = 4$ and $k_1(B) = 2^{n-2} - 1$. According to two different fusion systems, one of the following holds*

(a) $k(B) = 2^{n-2} + 4$, $k_{n-2}(B) = 1$ *and* $l(B) = 2$.

(b) $k(B) = 2^{n-2} + 5$, $k_{n-2}(B) = 2$ *and* $l(B) = 3$.

(5) *D is a semidihedral group of order $2^n \geq 16$. Then $k_0(B) = 4$ and $k_1(B) = 2^{n-2} - 1$. According to three different fusion systems, one of the following holds*

(a) $k(B) = 2^{n-2} + 3$ *and* $l(B) = 2$.

(b) $k(B) = 2^{n-2} + 4$, $k_{n-2}(B) = 1$ *and* $l(B) = 2$.

(c) $k(B) = 2^{n-2} + 4$, $k_{n-2}(B) = 1$ *and* $l(B) = 3$.

(6) *D is a direct product of two isomorphic cyclic groups. Then* $k(B) = k_0(B) = \frac{|D|+8}{3}$ *and* $l(B) = 3$.

Sketch of the proof

Lemma

If P is a metacyclic 2-group such that $\operatorname{Aut}(P)$ is not a 2-group, then $P \cong Q_8$ or $P \cong C_{2^m} \times C_{2^m}$ for some $m \in \mathbb{N}$.

- Let $\mathcal{F} := \mathcal{F}_D(B) \neq \mathcal{F}_D(D)$.
- If D is abelian, the Lemma implies $D \cong C_{2^m} \times C_{2^m}$. Then by the work of Usami and Puig there exists a perfect isometry between B and its Brauer correspondent. The claim follows in this case.
- Hence, assume that D is nonabelian.

- The case $D \cong Q_8$ was done by Olsson. Thus, we may assume that $\mathrm{Aut}(D)$ is a 2-group and the inertial index $e(B)$ equals 1.
- Then there exists an $\mathcal{F}$-essential subgroup $Q \leq D$.
- Q is also metacyclic and $\mathrm{Out}_{\mathcal{F}}(Q)$ (and so $\mathrm{Aut}(Q)$) is not a 2-group.
- Moreover, $\mathrm{C}_D(Q) = \mathrm{Z}(Q)$.
- In the case $Q \cong Q_8$ it is easy to see that D must be a quaternion or semidihedral group.
- This case was also done by Olsson.

- Thus, assume $Q \cong C_{2^m} \times C_{2^m}$.
- If $m \geq 2$, one can show that $\mathrm{N}_D(Q)/Q$ does not act faithfully on $Q/\Phi(Q)$. This contradicts $\mathrm{Out}_{\mathcal{F}}(Q) \leq \mathrm{Aut}(Q/\Phi(Q))$.
- Hence, we have $Q \cong C_2 \times C_2$.
- Then $D \cong D_{2^n}$ or $D \cong SD_{2^n}$ for some $n \in \mathbb{N}$ by one of the previous propositions.
- In the case $D \cong D_{2^n}$ the result follows from a work by Brauer. $\square$
- All major conjectures are satisfied for 2-blocks with metacyclic defect groups.

Defect group $D_{2^n} \times C_{2^m}$

Theorem

Let B be a 2-block of RG with defect group $D_{2^n} \times C_{2^m}$ for $n \geq 3$ and $m \geq 0$. Then

$$k(B) = 2^m(2^{n-2} + 3), \qquad k_0(B) = 2^{m+2},$$
$$k_1(B) = 2^m(2^{n-2} - 1), \qquad l(B) \in \{1, 2, 3\}.$$

Alperin's weight conjecture and Robinson's ordinary weight conjecture are satisfied for B. Moreover, the gluing problem for B has a unique solution.

Sketch of the proof

- Let

$$D := \langle x, y \mid x^{2^{n-1}} = y^2 = 1,\ yxy^{-1} = x^{-1} \rangle \times \langle z \mid z^{2^m} = 1 \rangle$$

and $\mathcal{F} := \mathcal{F}_D(B)$.

- There are two candidates for $\mathcal{F}$-essential subgroups up to conjugation:

$$Q_1 := \langle x^{2^{n-2}}, y, z \rangle \cong C_2 \times C_2 \times C_{2^m},$$
$$Q_2 := \langle x^{2^{n-2}}, xy, z \rangle \cong C_2 \times C_2 \times C_{2^m}.$$

- This gives four cases:
  (aa) Q_1 and Q_2 are both $\mathcal{F}$-essential.
  (ab) Q_1 is $\mathcal{F}$-essential and Q_2 is not.
  (ba) Q_1 is not $\mathcal{F}$-essential, but Q_2 is.
  (bb) There are no $\mathcal{F}$-essential subgroups.
- Case (ab) is symmetric to case (ba) (replace y by xy).
- In case (bb) the block B is nilpotent, since $\mathrm{Aut}(D)$ is a 2-group.
- In the next step we determine a set of representatives $\mathcal{R}$ for the conjugacy classes of B-subsections, i. e. pairs (α, b_α) such that $(\langle\alpha\rangle, b_\alpha)$ is a B-subpair.

- A result by Brauer shows that

$$k(B) = \sum_{(\alpha, b_\alpha) \in \mathcal{R}} l(b_\alpha).$$

- For $\alpha \neq 1$ we have $l(b_\alpha) = l(\overline{b_\alpha})$, where $\overline{b_\alpha}$ is a block of $R[\mathrm{C}_G(\alpha)/\langle\alpha\rangle]$.
- Using induction we can determine $l(b_\alpha)$ for $\alpha \neq 1$ and thus also $k(B) - l(B)$.
- The final conclusion follows from considerations of generalized decomposition numbers and lower defect groups.
- We have $l(B) = 1$, 2 or 3 according to the cases (bb), (ab) or (aa) respectively. □