On the Brauer-Feit bound for abelian defect groups

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Introduction

- Let G be a finite group and p be a prime.
- Let B be a p-block of G with defect d.
- We denote the number of irreducible characters of B by $k(B)$, and the number of irreducible Brauer characters by $l(B)$.

Theorem (Brauer-Feit, 1959)

(i) *If* $d \leq 2$, *then* $k(B) \leq p^d$.

(ii) *If* $d > 2$, *then* $k(B) < p^{2d-2}$.

Brauer conjectured that $k(B) \leq p^d$ holds in general (Problem 20).

Abelian defect groups

Theorem (S.)

If B has abelian defect groups of order $p^d > p$, then

$$k(B) < p^{\frac{3}{2}d-\frac{1}{2}}.$$

Robinson already proved $k_0(B) < p^{\frac{3}{2}d-\frac{1}{2}}$ for almost all primes p (depending on d).

Sketch of the proof (I)

Theorem (Halasi-Podoski, 2012)

Let H, K be finite groups such that H acts faithfully on K and $(|H|, |K|) = 1$. Then there exist $x, y \in K$ such that $\mathrm{C}_H(x) \cap \mathrm{C}_H(y) = 1$.

- Let (D, b_D) be a maximal Brauer pair (i. e. D is a defect group of B and b_D is a Brauer correspondent of B in $\mathrm{C}_G(D)$).
- Then the inertial quotient $T(B) := \mathrm{N}_G(D, b_D)/D\,\mathrm{C}_G(D)$ acts faithfully on D and $(|T(B)|, |D|) = 1$.
- It follows that there is a B-subsection (u, b_u) such that $l(b_u) < p^{d-1}$ (i. e. $u \in D$ and b_u is a Brauer correspondent of B in $\mathrm{C}_G(u)$).

Sketch of the proof (II)

- Since D is abelian, we have $k(B) = k_0(B)$ by Kessar-Malle (i. e. all irreducible characters in B have height 0).
- Now apply the following.

Proposition (Brauer, Robinson)

Let (u, b_u) be a B-subsection such that b_u has defect q. Then

$$k_0(B) \le p^q \sqrt{l(b_u)}.$$

Remarks

- The proof relies on the classification of the finite simple groups, since we have used the Kessar-Malle result $k(B) = k_0(B)$.
- In some situations the bound can be slightly improved.
- For example, if the smallest (non-trivial) direct factor of D has order p^n, we obtain

$$k(B) \leq p^{\frac{3}{2}d-\frac{n}{2}}.$$

- Now, let D be an abelian defect group of B of rank r.
- In case $r \leq 2$, Brauer showed $k(B) \leq p^d$.
- For $r = 3$ he proved $k(B) < p^{5d/3}$.
- This can be improved to $k(B) < p^{4d/3}$ using $k(B) = k_0(B)$.

Restrictions on $T(B)$

In the following we restrict $T(B)$ and p in order to obtain stronger results.

Proposition (Robinson)

If D and $T(B)$ are abelian, then $k(B) \leq |D|$.

This can be improved to the following:

Proposition (S.)

If D is abelian and $T(B)$ contains an abelian subgroup of index at most 4, then $k(B) \leq |D|$.

Sketch of the proof (I)

- Let $A \le T(B)$ be abelian such that $|T(B) : A| \le 4$.
- A acts faithfully on the elementary abelian p-group $\Omega(D) := \langle x \in D : x^p = 1\rangle$.
- Moreover, A has a regular orbit on $\Omega(D)$.
- Hence, there exists $x \in D$ such that $\mathrm{C}_A(x) = 1$ and $|\mathrm{C}_{T(B)}(x)| \le 4$.
- Thus, a Brauer correspondent b_x of B in $\mathrm{C}_G(x)$ has inertial index at most 4.

Sketch of the proof (II)

- Results by Usami and Puig imply that b_x is perfectly isometric to a block with normal defect group D.
- In particular, the Cartan matrix (c_{ij}) of b_x can be computed locally.
- Now the claim follows from

$$k(B) \leq \sum_{i=1}^{l(b_x)} c_{ii} - \sum_{i=1}^{l(b_x)-1} c_{i,i+1}.$$

Regular orbits

- One may ask which (non-abelian) groups always provide regular orbits in the situation above.
- Let P be an abelian p-group and let $A \leq \mathrm{Aut}(P)$ be a p'-group.
- Does A have a regular orbit on P?

We deduce a sufficient condition:

- Replace P by $\Omega(P)$.
- By Maschke's Theorem, $P = P_1 \oplus \ldots \oplus P_n$ with irreducible A-invariant subgroups P_i.
- If we have already found $x_i \in P_i$ such that $\mathrm{C}_A(x_i) \subseteq \mathrm{C}_A(P_i)$ for all i, then $\mathrm{C}_A(x_1 \ldots x_n) = 1$ and we are done.

Regular orbits

- Hence, replace P by P_i and A by $A/\operatorname{C}_A(P)$.
- If A has no regular orbit, then

$$P = \bigcup_{x \in A \setminus \{1\}} \operatorname{C}_P(x)$$

  and $p < |A|$, since P cannot be the union of p proper subgroups.
- Thus, for fixed A there are only finitely many possibilities for P which can be handled by a computer.
- It turns out that 84% of the groups A of order less than 128 give regular orbits.

Small inertial indices

This implies the following result on the inertial index $|T(B)|$.

Proposition (S.)

Let B be a block with abelian defect group D and $|T(B)| \leq 255$. Then $k(B) \leq |D|$.

For $|T(B)| = 256$ and $|D| = 81$ the method does not work anymore.

2-blocks

Let C_n be a cyclic group of order n and let $C_n^m := C_n \times \ldots \times C_n$ (m copies).

Theorem (S.)

Let B be a 2-block with defect group $D \cong \prod_{i=1}^n C_{2^i}^{m_i}$. Assume that one of the following holds:

(i) *For some $i \in \{1, \ldots, n\}$ we have $m_i \le 4$ and $m_j \le 2$ for all $j \ne i$.*

(ii) *D has rank 5.*

Then $k(B) \le |D|$.

Sketch of the proof (I)

Lemma

Let A be a p'-automorphism group of an abelian p-group $P \cong \prod_{i=1}^{n} C_{p^i}^{m_i}$. Then A is isomorphic to a subgroup of $\prod_{i=1}^{n} \mathrm{GL}(m_i, p)$ where $\mathrm{GL}(0, p) := 1$.

- Hence, in case (i) we have $T(B) \leq \mathrm{GL}(4,2) \times \mathrm{GL}(2,2) \times \ldots \times \mathrm{GL}(2,2)$.
- Since $|T(B)|$ is odd and $\mathrm{GL}(2,2) \cong S_3$, the projection of $T(B)$ onto $\mathrm{GL}(2,2)$ is abelian.

Sketch of the proof (II)

- In order to show that $T(B)$ has an abelian subgroup of index at most 4, we may assume that $T(B) \leq \mathrm{GL}(4,2) \cong A_8$.
- Then $|T(B)| \in \{1,3,5,7,9,15,21\}$ and the claim follows.
- In part (ii) we have $T(B) \leq \mathrm{GL}(5,2)$ and $|T(B)| \leq 255$.

2-blocks of defect 6

- Let us consider abelian defect groups D of order 64.
- In order to show that $k(B) \leq 64$, we may assume that D is elementary abelian (otherwise D has rank at most 5).
- We may also assume that $T(B) \leq \mathrm{GL}(6,2)$ has order at least 256.
- As an odd order group, $T(B)$ is solvable.
- It turns out that $T(B) \cong (C_7 \rtimes C_3)^2$.

2-blocks of defect 6

This implies:

Proposition

Let B be a 2-block with abelian defect groups of order 64. Then $k(B) \leq 3 \cdot 64$.

... which is better than the (improved) Brauer-Feit bound.

More on 2-blocks

For $p = 2$, Robinson's inequality can be slightly improved:

Proposition (S.)

Let (u, b_u) be a subsection of a 2-block B such that b_u has defect q. Set

$$\alpha := \begin{cases} \left\lfloor \sqrt{l(b_u)} \right\rfloor & \text{if } \left\lfloor \sqrt{l(b_u)} \right\rfloor \text{ is odd,} \\ \frac{l(b_u)}{\left\lfloor \sqrt{l(b_u)} \right\rfloor + 1} & \text{otherwise.} \end{cases}$$

Then $k_0(B) \leq 2^q \alpha$.

More on 2-blocks

This gives another improvement of the Brauer-Feit bound:

Proposition

Let B be a 2-block with abelian defect groups and odd defect $d > 1$. Then

$$k(B) \leq 2^d(2^{\frac{d-1}{2}} - 1).$$

More on 2-blocks

The following result holds for arbitrary defect groups.

Proposition (Robinson, S.)

Let B be a 2-block with defect d. Then

$$k(B) \leq \begin{cases} 2^d & \text{if } d \leq 5, \\ 3 \cdot 2^{2d-4} - 8 & \text{if } d > 5. \end{cases}$$

3-blocks and 5-blocks

Theorem (S.)

Let B be a 3-block with defect group $D \cong \prod_{i=1}^{n} C_{3^i}^{m_i}$ such that for two $i, j \in \{1, \ldots, n\}$ we have $m_i, m_j \leq 3$, and $m_k \leq 1$ for all $i \neq k \neq j$. Then $k(B) \leq |D|$.

Theorem (S.)

Let B be a 5-block with abelian defect group D of rank 3. Then $k(B) \leq |D|$.

For the defect group C_7^3 the method does not work anymore.

Theorem

Happy Birthday, Geoff!