Representations of groups and blocks

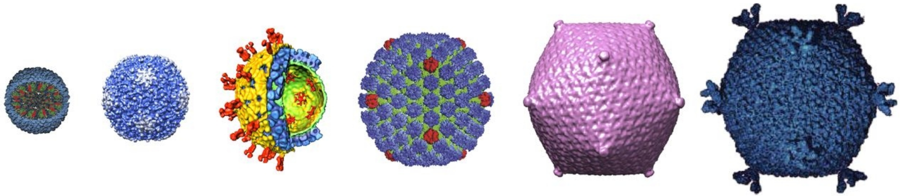
Presentation at the TU Munich

Benjamin Sambale

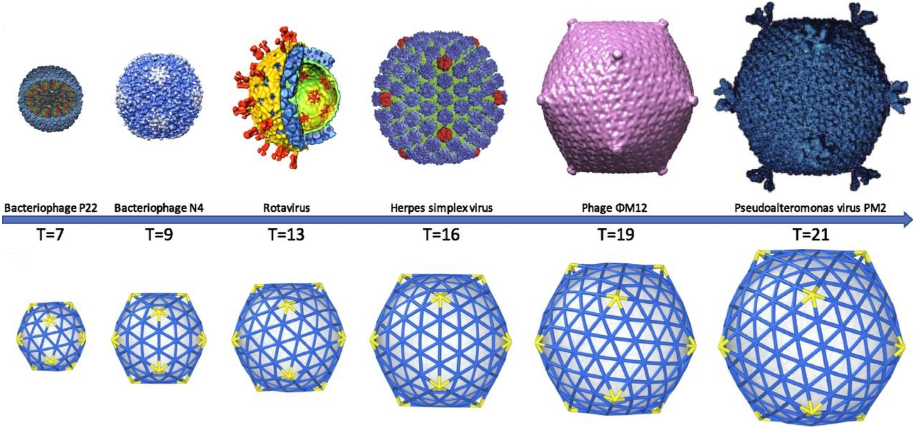
Leibniz Universität Hannover

21. 07. 2022

What is shown?



Viruses!



- Li et al., *Why large icosahedral viruses need scaffolding proteins*, PNAS 115 (2018)
- Peeters, Taormina, *Group theory of icosahedral virus capsid vibrations: A top-down approach*, J. Theoret. Biol. 256 (2009)

Naive symmetry counting

The herpes virus permits the following symmetries:

rotations	60

Naive symmetry counting

The herpes virus permits the following symmetries:

rotations	60
reflections	15
total	75?

Naive symmetry counting

The herpes virus permits the following symmetries:

rotations	60
reflections	15
combinations	45
total	120!

Naive symmetry counting

The herpes virus permits the following symmetries:

rotations	60
reflections	15
combinations	45
total	120!

Group theory simplifies counting!

Introduction

Synopsis

In representation theory, mathematical objects are studied by their actions on sets, vector spaces, graphs, categories etc.

Introduction

Synopsis

In representation theory, mathematical objects are studied by their actions on sets, vector spaces, graphs, categories etc.

Example

- The symmetry group G of the cube permutes the 8 vertices.

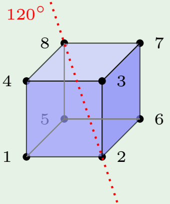
Introduction

Synopsis

In representation theory, mathematical objects are studied by their actions on sets, vector spaces, graphs, categories etc.

Example

- The symmetry group G of the cube permutes the 8 vertices.
- This gives rise to a group homomorphism $\varphi\colon G \to S_8$.

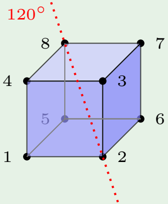


$$\xrightarrow{\varphi} (1,3,6)(4,7,5)$$

Introduction

Example

- There is also a linear action $\psi\colon G \to \mathrm{GL}(3,\mathbb{R})$.

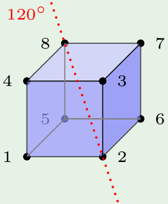


$$\xrightarrow{\psi} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

Introduction

Example

- There is also a linear action $\psi\colon G \to \mathrm{GL}(3,\mathbb{R})$.



$$\xrightarrow{\psi} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

Advantage: Computations are easier inside S_8 or $\mathrm{GL}(3,\mathbb{R})$ than in G.

Applications

Representation theory has numerous applications

- within mathematics:
  - group theory (Frobenius kernels, Odd order theorem)
  - combinatorics (Young diagrams, graph automorphisms)
  - number theory (Langlands program, Artin L-series)
  - geometry (Coxeter groups, Lie groups)
  - topology (fundamental groups, classifying spaces)
- outside mathematics:
  - biology (virology, molecular systems)
  - chemistry (crystallography, spectroscopy)
  - physics (particle physics, quantum mechanics)
  - computer science (cryptography, coding theory)

Representations of groups

From now on let G be an abstract finite group.
Let F be a field (e. g. $\mathbb{C}, \mathbb{F}_p, \mathbb{Q}(\zeta), \mathbb{Q}_p, \ldots$).

Representations of groups

From now on let G be an abstract finite group.
Let F be a field (e. g. $\mathbb{C}, \mathbb{F}_p, \mathbb{Q}(\zeta), \mathbb{Q}_p, \ldots$).

Goal

Find a representation $\Delta\colon G \to \mathrm{GL}(d, F)$ such that

- degree d is small (efficient computation).

Representations of groups

From now on let G be an abstract finite group.
Let F be a field (e. g. $\mathbb{C}, \mathbb{F}_p, \mathbb{Q}(\zeta), \mathbb{Q}_p, \ldots$).

Goal

Find a representation $\Delta\colon G \to \mathrm{GL}(d, F)$ such that

- degree d is small (efficient computation).
- kernel $\mathrm{Ker}(\Delta)$ is small (preserving information).

Representations of groups

From now on let G be an abstract finite group.
Let F be a field (e. g. $\mathbb{C}, \mathbb{F}_p, \mathbb{Q}(\zeta), \mathbb{Q}_p, \ldots$).

Goal

Find a representation $\Delta\colon G \to \mathrm{GL}(d, F)$ such that

- degree d is small (efficient computation).
- kernel $\mathrm{Ker}(\Delta)$ is small (preserving information).

Extreme examples

- The trivial representation $\Delta_{\mathrm{tr}}\colon G \to \mathrm{GL}(1, F)$, $g \mapsto 1$ contains no information on G.

Representations of groups

From now on let G be an abstract finite group.
Let F be a field (e. g. $\mathbb{C}, \mathbb{F}_p, \mathbb{Q}(\zeta), \mathbb{Q}_p, \ldots$).

Goal

Find a representation $\Delta\colon G \to \mathrm{GL}(d, F)$ such that

- degree d is small (efficient computation).
- kernel $\mathrm{Ker}(\Delta)$ is small (preserving information).

Extreme examples

- The trivial representation $\Delta_{\mathrm{tr}}\colon G \to \mathrm{GL}(1, F)$, $g \mapsto 1$ contains no information on G.
- The regular representation $\Delta_{\mathrm{reg}}\colon G \to \mathrm{GL}(|G|, F)$, $g \mapsto (\delta_{x,gy})_{x,y \in G}$ is injective, but $d = |G|$ is large.

Irreducible representations

The regular representation decomposes with respect to a suitable basis:

$$\begin{aligned} G &\to \mathrm{GL}(d_1, F) \times \ldots \times \mathrm{GL}(d_k, F), \\ g &\mapsto \begin{pmatrix} A_1 & & * \\ & \ddots & \\ 0 & & A_k \end{pmatrix} \end{aligned}$$

Irreducible representations

The regular representation decomposes with respect to a suitable basis:

$$G \to \mathrm{GL}(d_1, F) \times \ldots \times \mathrm{GL}(d_k, F),$$
$$g \mapsto \begin{pmatrix} A_1 & & * \\ & \ddots & \\ 0 & & A_k \end{pmatrix}$$

Study the irreducible representations $\Delta_i \colon G \to \mathrm{GL}(d_i, F)$, $g \mapsto A_i$.

Irreducible representations

The regular representation decomposes with respect to a suitable basis:

$$G \to \mathrm{GL}(d_1, F) \times \ldots \times \mathrm{GL}(d_k, F),$$
$$g \mapsto \begin{pmatrix} A_1 & & * \\ & \ddots & \\ 0 & & A_k \end{pmatrix}$$

Study the irreducible representations $\Delta_i \colon G \to \mathrm{GL}(d_i, F)$, $g \mapsto A_i$.
Extend linearly to a representation of algebras:

$$\widehat{\Delta}_i \colon FG \to F^{d_i \times d_i}$$

where $FG = \sum_{g \in G} Fg$ is the group algebra of G.

Ordinary representation theory

- Suppose that $|G| \neq 0$ in F (i. e. $\mathrm{char}(F) \nmid |G|$).

Ordinary representation theory

- Suppose that $|G| \neq 0$ in F (i. e. $\mathrm{char}(F) \nmid |G|$).
- Then FG is semisimple by Maschke's Theorem, i. e.

$$\mathrm{Ker}(\widehat{\Delta}_1) \cap \ldots \cap \mathrm{Ker}(\widehat{\Delta}_k) = 0.$$

Ordinary representation theory

- Suppose that $|G| \neq 0$ in F (i. e. $\mathrm{char}(F) \nmid |G|$).
- Then FG is semisimple by Maschke's Theorem, i. e.

$$\mathrm{Ker}(\widehat{\Delta}_1) \cap \ldots \cap \mathrm{Ker}(\widehat{\Delta}_k) = 0.$$

- If additionally F is algebraically closed (e. g. $F = \mathbb{C}$), then $\widehat{\Delta}_i$ is surjective and we obtain the Artin–Wedderburn isomorphism

$$\boxed{FG \cong F^{d_1 \times d_1} \times \ldots \times F^{d_l \times d_l}}$$

(not all $\widehat{\Delta}_i$ are needed).

Ordinary representation theory

- Suppose that $|G| \neq 0$ in F (i. e. $\mathrm{char}(F) \nmid |G|$).
- Then FG is semisimple by Maschke's Theorem, i. e.

$$\mathrm{Ker}(\widehat{\Delta}_1) \cap \ldots \cap \mathrm{Ker}(\widehat{\Delta}_k) = 0.$$

- If additionally F is algebraically closed (e. g. $F = \mathbb{C}$), then $\widehat{\Delta}_i$ is surjective and we obtain the Artin–Wedderburn isomorphism

$$\boxed{FG \cong F^{d_1 \times d_1} \times \ldots \times F^{d_l \times d_l}}$$

(not all $\widehat{\Delta}_i$ are needed).

- This situation is well-understood.

Modular representation theory

- From now on assume that $p := \mathrm{char}(F)$ is a prime dividing $|G|$ and F is algebraically closed.

Modular representation theory

- From now on assume that $p := \mathrm{char}(F)$ is a prime dividing $|G|$ and F is algebraically closed.
- Decompose FG into indecomposable algebras

$$\boxed{FG = B_1 \times \ldots \times B_n.}$$

Modular representation theory

- From now on assume that $p := \mathrm{char}(F)$ is a prime dividing $|G|$ and F is algebraically closed.
- Decompose FG into indecomposable algebras

$$\boxed{FG = B_1 \times \ldots \times B_n.}$$

- Call $B_1, \ldots, B_n$ the (p-)blocks of FG.

Modular representation theory

- From now on assume that $p := \mathrm{char}(F)$ is a prime dividing $|G|$ and F is algebraically closed.
- Decompose FG into indecomposable algebras

$$\boxed{FG = B_1 \times \ldots \times B_n.}$$

- Call $B_1, \ldots, B_n$ the (p-)blocks of FG.
- Each irreducible representation belongs to exactly one block.

Modular representation theory

- From now on assume that $p := \mathrm{char}(F)$ is a prime dividing $|G|$ and F is algebraically closed.
- Decompose FG into indecomposable algebras

$$\boxed{FG = B_1 \times \ldots \times B_n.}$$

- Call $B_1, \ldots, B_n$ the (p-)blocks of FG.
- Each irreducible representation belongs to exactly one block.
- The block containing Δ_{tr} is called the principal block.

A comparison

Example

- For the symmetry group of the cube $G \cong S_4 \times C_2$ we have

$$\mathbb{C}G \cong \mathbb{C}^4 \times \left(\mathbb{C}^{2\times 2}\right)^2 \times \left(\mathbb{C}^{3\times 3}\right)^4.$$

A comparison

Example

- For the symmetry group of the cube $G \cong S_4 \times C_2$ we have

$$\mathbb{C}G \cong \mathbb{C}^4 \times \left(\mathbb{C}^{2\times 2}\right)^2 \times \left(\mathbb{C}^{3\times 3}\right)^4.$$

- On the other hand, $\overline{\mathbb{F}_2}G$ is just the principal block.

A comparison

Example

- For the symmetry group of the cube $G \cong S_4 \times C_2$ we have

$$\mathbb{C}G \cong \mathbb{C}^4 \times \left(\mathbb{C}^{2\times 2}\right)^2 \times \left(\mathbb{C}^{3\times 3}\right)^4.$$

- On the other hand, $\overline{\mathbb{F}_2}G$ is just the principal block.
- For $G = S_{20}$ and $F = \overline{\mathbb{F}_2}$ not even the degrees $d_1, \ldots, d_k$ are known!

Defect groups

The algebra structure of a block B is measured by its defect group D (a p-subgroup of G).

Defect groups

The algebra structure of a block B is measured by its defect group D (a p-subgroup of G).

Theorem (Brauer)

B is a simple algebra iff $D = 1$. In this case, $B \cong F^{d\times d}$ for some $d \geq 1$.

Defect groups

The algebra structure of a block B is measured by its defect group D (a p-subgroup of G).

Theorem (Brauer)

B is a simple algebra iff $D = 1$. In this case, $B \cong F^{d\times d}$ for some $d \geq 1$.

- The defect group of the principal block is a Sylow p-subgroup of G. In particular, not all blocks are simple.

Defect groups

The algebra structure of a block B is measured by its defect group D (a p-subgroup of G).

Theorem (Brauer)

B is a simple algebra iff $D = 1$. In this case, $B \cong F^{d\times d}$ for some $d \geq 1$.

- The defect group of the principal block is a Sylow p-subgroup of G. In particular, not all blocks are simple.
- In general the isomorphism type of B (even its dimension) cannot be described by D alone.

Defect groups

The algebra structure of a block B is measured by its defect group D (a p-subgroup of G).

Theorem (Brauer)

B is a simple algebra iff $D = 1$. In this case, $B \cong F^{d \times d}$ for some $d \geq 1$.

- The defect group of the principal block is a Sylow p-subgroup of G. In particular, not all blocks are simple.
- In general the isomorphism type of B (even its dimension) cannot be described by D alone.
- Instead, classify blocks up to Morita equivalence, i. e. determine the module category B-mod.

Finiteness conjectures

Motivation:

Finiteness conjectures

Motivation:

Conjecture (Donovan)

For every p-group D there exist only finitely many Morita equivalence classes of blocks with defect group D.

Finiteness conjectures

Motivation:

Conjecture (Donovan)

For every p-group D there exist only finitely many Morita equivalence classes of blocks with defect group D.

Conversely, many features of D can be read off from B-mod.

Finiteness conjectures

Motivation:

Conjecture (Donovan)

For every p-group D there exist only finitely many Morita equivalence classes of blocks with defect group D.

Conversely, many features of D can be read off from B-mod. However:

Theorem (García–Margolis–Del Río, 2021)

There exist p-groups $P \ncong Q$ such that $FP \cong FQ$.

Representation type

Theorem (Hamernik, Dade, Janusz, Kupisch)

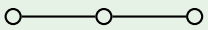
B has finite representation type iff D is cyclic. In this case, B-mod is determined by the Brauer tree of B.

Representation type

Theorem (Hamernik, Dade, Janusz, Kupisch)

B has finite representation type iff D is cyclic. In this case, B-mod is determined by the Brauer tree of B.

Example

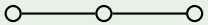
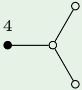
- The principal 3-block of $G = S_4$ has Brauer tree 

Representation type

Theorem (Hamernik, Dade, Janusz, Kupisch)

B has finite representation type iff D is cyclic. In this case, B-mod is determined by the Brauer tree of B.

Example

- The principal 3-block of $G = S_4$ has Brauer tree 
- No block with Brauer tree 

is known!

Tame blocks

Theorem (Bondarenko–Drozd)

B has tame representation type iff $p = 2$ and D is a dihedral, semidihedral or quaternion group.

Tame blocks

Theorem (Bondarenko–Drozd)

B has tame representation type iff $p = 2$ and D is a dihedral, semidihedral or quaternion group.

Erdmann described tame blocks as path algebras. For dihedral D, mod-B was determined by Macgregor (2021).

Tame blocks

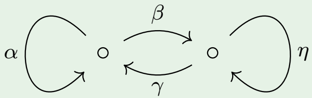
Theorem (Bondarenko–Drozd)

B has tame representation type iff $p = 2$ and D is a dihedral, semidihedral or quaternion group.

Erdmann described tame blocks as path algebras. For dihedral D, mod-B was determined by Macgregor (2021).

Example

The principal 2-block of $G = S_4$ has defect group $D \cong D_8$ and quiver/relations



$$\beta\eta = \eta\gamma = \gamma\beta = \alpha^2 = 0,$$
$$\alpha\beta\gamma = \beta\gamma\alpha, \qquad \eta^2 = \gamma\alpha\beta.$$

Some wild blocks

Very little is known for blocks of wild representation type.

Some wild blocks

Very little is known for blocks of wild representation type.
A cyclic extension of a cyclic group is called metacyclic.

Some wild blocks

Very little is known for blocks of wild representation type.
A cyclic extension of a cyclic group is called metacyclic.

Theorem (Eaton–Kessar–Külshammer–S.)

If D is a metacyclic 2-group, then one of the following holds:

1. *B has tame representation type.*
2. *B is nilpotent. Then $B \cong (FD)^{d\times d}$ for some $d \geq 1$.*
3. *$D \cong C_{2^d} \times C_{2^d}$ with $d \geq 2$ and B is Morita equivalent to $F[D \rtimes C_3]$.*

Methods

- Characters are the “shadows” of representations:
  $\chi\colon G \to \mathbb{C},\ g \mapsto \mathrm{tr}(\Delta(g))$.

Methods

- Characters are the "shadows" of representations:
  $\chi\colon G \to \mathbb{C}$, $g \mapsto \mathrm{tr}(\Delta(g))$.
- Fusion systems are categories which describe the embedding $D \hookrightarrow G$.

Methods

- Characters are the “shadows” of representations:
  $\chi\colon G \to \mathbb{C}$, $g \mapsto \mathrm{tr}(\Delta(g))$.
- Fusion systems are categories which describe the embedding $D \hookrightarrow G$.
- Cartan matrices encode the decomposition of the regular representation into indecomposable summands.

Methods

- Characters are the "shadows" of representations:
  $\chi\colon G \to \mathbb{C},\ g \mapsto \mathrm{tr}(\Delta(g))$.
- Fusion systems are categories which describe the embedding $D \hookrightarrow G$.
- Cartan matrices encode the decomposition of the regular representation into indecomposable summands.
- They give rise to positive definite quadratic forms and can be simplified by Minkowski reduction or the LLL algorithm.

Methods

- Characters are the "shadows" of representations:
  $\chi \colon G \to \mathbb{C},\ g \mapsto \mathrm{tr}(\Delta(g))$.
- Fusion systems are categories which describe the embedding $D \hookrightarrow G$.
- Cartan matrices encode the decomposition of the regular representation into indecomposable summands.
- They give rise to positive definite quadratic forms and can be simplified by Minkowski reduction or the LLL algorithm.
- Clifford theory reduces problems to (quasi)simple groups. They can be settled using the classification of finite simple groups.

Methods

- Characters are the "shadows" of representations: $\chi\colon G \to \mathbb{C},\ g \mapsto \mathrm{tr}(\Delta(g))$.
- Fusion systems are categories which describe the embedding $D \hookrightarrow G$.
- Cartan matrices encode the decomposition of the regular representation into indecomposable summands.
- They give rise to positive definite quadratic forms and can be simplified by Minkowski reduction or the LLL algorithm.
- Clifford theory reduces problems to (quasi)simple groups. They can be settled using the classification of finite simple groups.
- Computer algebra systems like GAP, Magma, Maple, Oscar, Chevie help to generate data and to formulate conjectures.

The character table of S_4

S_4	1	(12)	$(12)(34)$	(123)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	2	-1	0
χ_4	3	1	-1	0	-1
χ_5	3	-1	-1	0	1

The character table of S_4

S_4	1	(12)	$(12)(34)$	(123)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	2	-1	0
χ_4	3	1	-1	0	-1
χ_5	3	-1	-1	0	1

Theorem (S.)

The character table of a group determines the representation type of a given block.

The Cartan matrix of the principal 2-block of $\mathrm{SL}(3,4)$

$$\begin{pmatrix} 34 & 23 & 23 & 16 & 16 \\ 23 & 17 & 16 & 12 & 12 \\ 23 & 16 & 17 & 12 & 12 \\ 16 & 12 & 12 & 10 & 9 \\ 16 & 12 & 12 & 9 & 10 \end{pmatrix} \xrightarrow{\text{LLL}} \begin{pmatrix} 2 & 1 & 1 & . & 1 \\ 1 & 2 & 1 & . & . \\ 1 & 1 & 3 & 1 & 1 \\ . & . & 1 & 2 & 1 \\ 1 & . & 1 & 1 & 7 \end{pmatrix} =: \tilde{C}$$

The Cartan matrix of the principal 2-block of $\mathrm{SL}(3,4)$

$$\begin{pmatrix} 34 & 23 & 23 & 16 & 16 \\ 23 & 17 & 16 & 12 & 12 \\ 23 & 16 & 17 & 12 & 12 \\ 16 & 12 & 12 & 10 & 9 \\ 16 & 12 & 12 & 9 & 10 \end{pmatrix} \xrightarrow{\text{LLL}} \begin{pmatrix} 2 & 1 & 1 & . & 1 \\ 1 & 2 & 1 & . & . \\ 1 & 1 & 3 & 1 & 1 \\ . & . & 1 & 2 & 1 \\ 1 & . & 1 & 1 & 7 \end{pmatrix} =: \tilde{C}$$

Theorem (S.)

If $\tilde{C}$ is equivalent (as quadratic form) to the Cartan matrix of a block B, then $\dim_F \mathrm{Z}(B) \leq \mathrm{tr}(\tilde{C})$.

The Cartan matrix of the principal 2-block of $\mathrm{SL}(3,4)$

$$\begin{pmatrix}34&23&23&16&16\\23&17&16&12&12\\23&16&17&12&12\\16&12&12&10&9\\16&12&12&9&10\end{pmatrix}\xrightarrow{\text{LLL}}\begin{pmatrix}2&1&1&.&1\\1&2&1&.&.\\1&1&3&1&1\\.&.&1&2&1\\1&.&1&1&7\end{pmatrix}=:\tilde{C}$$

Theorem (S.)

If $\tilde{C}$ is equivalent (as quadratic form) to the Cartan matrix of a block B, then $\dim_F \mathrm{Z}(B) \le \mathrm{tr}(\tilde{C})$.

In the example above we obtain $\dim \mathrm{Z}(B) \le 16 \le 64 = |D|$.

The Cartan matrix of the principal 2-block of $\mathrm{SL}(3,4)$

$$\begin{pmatrix} 34 & 23 & 23 & 16 & 16 \\ 23 & 17 & 16 & 12 & 12 \\ 23 & 16 & 17 & 12 & 12 \\ 16 & 12 & 12 & 10 & 9 \\ 16 & 12 & 12 & 9 & 10 \end{pmatrix} \xrightarrow{\text{LLL}} \begin{pmatrix} 2 & 1 & 1 & . & 1 \\ 1 & 2 & 1 & . & . \\ 1 & 1 & 3 & 1 & 1 \\ . & . & 1 & 2 & 1 \\ 1 & . & 1 & 1 & 7 \end{pmatrix} =: \tilde{C}$$

Theorem (S.)

If $\tilde{C}$ is equivalent (as quadratic form) to the Cartan matrix of a block B, then $\dim_F \mathrm{Z}(B) \le \mathrm{tr}(\tilde{C})$.

In the example above we obtain $\dim \mathrm{Z}(B) \le 16 \le 64 = |D|$.
This confirms Brauer's $k(B)$-Conjecture for B.

The classification of finite simple groups

Theorem (CFSG)

Every finite simple group belongs to one of the following families:

- *cyclic groups of prime order,*
- *alternating groups of degree* ≥ 5*,*
- *matrix groups of Lie type,*
- *26 sporadic groups.*

The classification of finite simple groups

Theorem (CFSG)

Every finite simple group belongs to one of the following families:

- *cyclic groups of prime order,* (*trivial*)
- *alternating groups of degree* ≥ 5*,*
- *matrix groups of Lie type,*
- *26 sporadic groups.*

The classification of finite simple groups

Theorem (CFSG)

Every finite simple group belongs to one of the following families:

- *cyclic groups of prime order,* (*trivial*)
- *alternating groups of degree* ≥ 5*,* (*combinatorics*)
- *matrix groups of Lie type,*
- *26 sporadic groups.*

The classification of finite simple groups

Theorem (CFSG)

Every finite simple group belongs to one of the following families:

- *cyclic groups of prime order,* (*trivial*)
- *alternating groups of degree* ≥ 5*,* (*combinatorics*)
- *matrix groups of Lie type,* (*algebraic geometry*)
- *26 sporadic groups.*

The classification of finite simple groups

Theorem (CFSG)

Every finite simple group belongs to one of the following families:

- *cyclic groups of prime order,* (*trivial*)
- *alternating groups of degree* ≥ 5*,* (*combinatorics*)
- *matrix groups of Lie type,* (*algebraic geometry*)
- *26 sporadic groups.* (*computer algebra*)

A GAP code to compute Cartan matrices

```
Cartanmatrix:=function(ct,p,b)
local chars,classes,orders,i,A,Q,C;
  chars:=Positions(PrimeBlocks(ct,p).block,b);
  orders:=OrdersClassRepresentatives(ct);
  classes:=PositionsProperty(orders,i->i mod p=0);
  A:=Irr(ct){chars}{classes}; #partial character table
  Q:=NullspaceIntMat(IntegralizedMat(A).mat);
  C:=Q*TransposedMat(Q); #Cartan matrix up equivalence
  return LLLReducedGramMat(C).remainder; #LLL reduction
end;
```

Definition of defect groups

- Let
$$1_B = \sum_{g \in G} \lambda_g g \in FG$$
be the identity element of a p-block B of FG.

Definition of defect groups

- Let
$$1_B = \sum_{g \in G} \lambda_g g \in FG$$
be the identity element of a p-block B of FG.
- A defect group of B is p-subgroup $D \leq G$ maximal such that there exists $g \in \mathrm{C}_G(D)$ with $\lambda_g \neq 0$.

Definition of defect groups

- Let
$$1_B = \sum_{g \in G} \lambda_g g \in FG$$
be the identity element of a p-block B of FG.
- A defect group of B is p-subgroup $D \leq G$ maximal such that there exists $g \in \mathrm{C}_G(D)$ with $\lambda_g \neq 0$.
- One can show that D is unique up to conjugation in G.

Definition of defect groups

- Let
$$1_B = \sum_{g \in G} \lambda_g g \in FG$$
be the identity element of a p-block B of FG.
- A defect group of B is p-subgroup $D \leq G$ maximal such that there exists $g \in \mathrm{C}_G(D)$ with $\lambda_g \neq 0$.
- One can show that D is unique up to conjugation in G.
- In particular, the isomorphism type of D is uniquely determined by B.

References

1. (with C. W. Eaton, R. Kessar and B. Külshammer) *2-blocks with abelian defect groups*, Adv. Math. **254** (2014), 706–735
2. *On the Brauer-Feit bound for abelian defect groups*, Math. Z. **276** (2014), 785–797
3. *Solution of Brauer's $k(B)$-Conjecture for π-blocks of π-separable groups*, Forum Math. **30** (2018), 1061–1064
4. *Bounding the number of characters in a block of a finite group*, Adv. Math. **358** (2019), 106861
5. (with G. Navarro), *Weights and nilpotent subgroups*, Int. Math. Res. Not. **2021** (2021), 2526–2538