Survey on invariants of blocks of finite groups

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Setting

- Let G be a finite group and p be a prime.
- Let (K, R, F) be a p-modular system, i. e.
  - K is a field of characteristic 0 which contains all $|G|$-th roots of unity.
  - R is a complete discrete valuation ring with quotient field K and maximal ideal (π).
  - $F = R/(\pi)$ is an algebraically closed field of characteristic p.

Blocks

- Let B be a block of the group algebra RG.
- Then one can consider the representation theory of B over K and over F.
- This leads to the number $k(B)$ of ordinary irreducible characters of B, and to the number $l(B)$ of irreducible Brauer characters of B.
- The ordinary characters split into $k_i(B)$ characters of height $i \geq 0$.
- Here the height describes the p-part of the degree of the character.

Defect groups

- These block invariants are usually strongly influenced by the defect group of the block B (Brauer).
- This is a p-subgroup $D \leq G$ which is unique up to conjugation.
- This motivates the following important task in representation theory:

Task

Determine the block invariants $k(B)$, $k_i(B)$ and $l(B)$ with respect to a given defect group.

Fusion systems

- A fixed defect group allows only finitely many block invariants (Brauer-Feit).
- However, in most cases the defect group alone does not determine the block invariants precisely.
- Instead we have to investigate the way how D embeds into the whole group G.
- This information is encoded in the fusion system $\mathcal{F}$ of B (introduced as Frobenius category by Puig).
- In particular one gets the inertial quotient $E(B)$ and its order $e(B) := |E(B)|$.
- For example, B is nilpotent if and only if $\mathcal{F}$ is.

Cohomology

- Sometimes even the fusion system of B is not sufficient to determine the block invariants.
- Then we can study the central linking system associated with $\mathcal{F}$ (Chermak).
- This leads to a certain 2-cocycle on the subcategory of the $\mathcal{F}$-centric subgroups.
- In a similar way one can attach another 2-cocycle on the outer automorphism group of every $\mathcal{F}$-centric subgroup (Külshammer-Puig).
- These cocycles determine the algebra structure of B in the case $D \trianglelefteq G$ completely.

Methods

On the following slides I present a general method to determine the block invariants $k(B)$, $k_i(B)$ and $l(B)$ of a block B with a fixed defect group D.

(1) Determine all (saturated) fusion systems $\mathcal{F}$ on D:

- calculate automorphism groups
- identify candidates for essential subgroups
- apply Alperin's Fusion Theorem
- find concrete examples or prove exoticness of these fusion systems
- if only the nilpotent fusion system exists, we are done (Puig)

(2) Determine the B-subsections:

- find set $\mathcal{R}$ of representatives for the $\mathcal{F}$-conjugacy classes of D
- this gives the B-subsections (u, b_u) for $u \in \mathcal{R}$ up to conjugation
- here b_u is a Brauer correspondent of B in $\mathrm{C}_G(u)$ and one can assume that b_u has defect group $\mathrm{C}_D(u)$
- determine $l(b_u)$ for $u \neq 1$ by considering the dominated block of $\mathrm{C}_G(u)/\langle u \rangle$ with defect group $\mathrm{C}_D(u)/\langle u \rangle$
- compute $k(B) - l(B) = \sum_{1 \neq u \in \mathcal{R}} l(b_u)$ (Brauer)
- if $\mathrm{C}_G(u)$ controls $\mathcal{F}$ for some $u \in \mathcal{R}$, then $l(B) \geq l(b_u)$ and $k(B) \geq k(b_u)$ (Külshammer-Okuyama)
- if D is abelian, we even have $l(B) = l(b_u)$ and $k(B) = k(b_u)$ here (Watanabe)

(3) Determine the decomposition numbers:

- calculate the Cartan matrices C_u of b_u for $u \neq 1$ up to basic sets by induction on $|D|$
- enumerate the possible generalized decomposition matrices D_u corresponding to $u \neq 1$ such that $D_u^\top \overline{D_u} = C_u$ (by computer if necessary).
- here one can use an action of a Galois group on the irreducible characters
- this gives upper bounds for $k(B)$ and $k_0(B)$ which I will present later
- if bounds are sharp, we can stop at this point
- determine the matrix D_1 of ordinary decomposition numbers as the integral orthogonal space of the generalized decomposition matrices

(4) Determine $l(B)$:

- compute the possible Cartan matrices of B as $C_1 = D_1^{\mathrm{T}} D_1$ for all possible decomposition numbers
- determine the elementary divisors of C_1
- find the multiplicities of the nontrivial lower defect groups using results of Brauer, Broué and Olsson
- this gives the multiplicities of the nontrivial elementary divisors of the "right" Cartan matrix
- eliminate the contradictory cases for C_1
- finally $l(B)$ is the dimension of C_1 and $k(B)$ follows as well

(5) Determine $k_i(B)$:

- investigate the contribution matrix $M_u := |D|\overline{D_u}C_u^{-1}D_u^{\mathrm{T}}$ for a major subsection (u, b_u) (i.e. B and b_u have the same defect)
- apply the p-adic valuation on the contributions and use a result of Brauer
- this gives $k_i(B)$ for $i \geq 0$

In many cases the number of possible decomposition matrices is too large to handle. Here one can try the following approach.

(6) Reduce to quasisimple groups:

- apply Fong Reduction and the Külshammer-Puig Theorem until we can assume

$$Z(G) = O_{p'}(G) = F(G)$$

- consider a block of the layer $E(G)$ which is covered by B
- deduce that G has only one component, so that $E(G)$ is quasisimple
- apply the classification of the finite simple groups
- use methods of Deligne, Lusztig and many others

For abelian defect groups one can use a method of Puig and Usami.

(7) Construct perfect isometries:

- construct a twisted group algebra L on $D \rtimes E(B)$ using the cocycle mentioned earlier
- show that a given isometry on a certain space of generalized characters which vanish on the p-regular elements can be extended to all generalized characters
- this gives a so-called local system in the sense of Broué
- the existence of a perfect isometry between the generalized characters of B and the generalized characters of L follows at once
- deduce the block invariants from L

More notation

- Let C_n be a cyclic group of order $n \geq 1$.
- We set $C_n^m := \underbrace{C_n \times \ldots \times C_n}_{m \text{ copies}}$.
- Let D_8 (resp. Q_8) be the dihedral (resp. quaternion) group of order 8. Let S_3 be the symmetric group of degree 3.
- We denote a central product of groups by $G_1 * G_2$.
- By p_-^{1+2} we describe the extraspecial group of order p^3 and exponent p^2 where p is odd.
- The following table lists many cases where the block invariants are known.

Results

p	D	$E(B)$	classification used?
arbitrary	cyclic	arbitrary	no
arbitrary	abelian	$e(B) \leq 4$	no
arbitrary	abelian	S_3	no
≥ 7	abelian	$C_4 \times C_2$	no
$\notin \{2, 7\}$	abelian	C_3^2	no
2	metacyclic	arbitrary	no
2	maximal class $*$ cyclic, incl. $* = \times$	arbitrary	only for $D \cong C_2^3$
2	minimal nonabelian	arbitrary	only for one family where $\lvert D\rvert = 2^{2r+1}$

Results

p	D	$E(B)$	classification used?
2	minimal nonmetacyclic	arbitrary	only for $D \cong C_2^3$
2	$\lvert D\rvert \le 16$	$\ncong C_{15}$	yes
2	$C_4 \wr C_2$	arbitrary	no
2	$D_8 * Q_8$	C_5	yes
2	$C_{2^n} \times C_2^3$, $n \ge 2$	arbitrary	yes
3	C_3^2	$\notin \{C_8, Q_8\}$	no
3	3_-^{1+2}	arbitrary	no
5	5_-^{1+2}	C_2	no

Remarks

- All these results were obtained without any restrictions on G.
- If one considers only p-solvable groups or blocks with maximal defect (for example), then much more can be proven.
- The table also shows that the method described above works better if the p-rank of D is small.

Conjectures

Many open conjectures in representation theory concern the relation between a block and its defect group. We list some of them:

- Alperin's Weight Conjecture predicts $l(B)$ as the number of B-weights.
- Brauer's $k(B)$-Conjecture asserts $k(B) \leq |D|$.
- Brauer's Height Zero Conjecture states that D is abelian if and only if $k(B) = k_0(B)$.
- Olsson's Conjecture predicts that $k_0(B) \leq |D : D'|$ where D' is the commutator subgroup of D.
- The Alperin-McKay-Conjecture asserts $k_0(B) = k_0(b)$ where b is the Brauer correspondent of B in $\mathrm{N}_G(D)$.

Remarks

- One implication of the Height Zero Conjecture was recently proven by Kessar and Malle using the classification.
- It is often possible to verify some of these conjectures without the precise knowledge of the block invariants.
- In particular for Brauer's and Olsson's Conjecture only bounds on the invariants are necessary.
- In this sense the following result is an important tool.

Theorem (S.)

Let (u, b_u) be a B-subsection such that b_u has Cartan matrix $C_u = (c_{ij})$ up to basic sets. Then for every positive definite, integral quadratic form $q(x_1, \ldots, x_{l(b_u)}) = \sum_{1 \le i \le j \le l(b_u)} q_{ij} x_i x_j$ we have

$$k_0(B) \le \sum_{1 \le i \le j \le l(b_u)} q_{ij} c_{ij}.$$

In particular

$$k_0(B) \le \sum_{i=1}^{l(b_u)} c_{ii} - \sum_{i=1}^{l(b_u)-1} c_{i,i+1}.$$

If (u, b_u) is major, we can replace $k_0(B)$ by $k(B)$ in these formulas.

The proof of this theorem relies on the following proposition.

Proposition (Broué)

If $\chi \in \mathrm{Irr}(B)$ has height 0, then the contribution of χ does not vanish for all (u, b_u).

If the Cartan matrix C_u is not known, one can use the following weaker bound.

Theorem (Robinson)

Let (u, b_u) be a B-subsection such that b_u has defect d. Then $k_0(B) \leq p^d\sqrt{l(b_u)}$. If (u, b_u) is major and $l(b_u) = 1$, we have

$$\sum_{i=0}^{\infty} p^{2i} k_i(B) \leq |D|.$$

We present some corollaries.

Corollary

Let (u, b_u) be a B-subsection such that b_u has defect group Q. Then the following hold:

(i) *If $Q/\langle u\rangle$ is cyclic, we have*

$$k_0(B) \le \left(\frac{|Q/\langle u\rangle| - 1}{l(b_u)} + l(b_u)\right)|\langle u\rangle| \le |Q|.$$

(ii) *If $|Q/\langle u\rangle| \le 9$, we have $k_0(B) \le |Q|$.*

(iii) *Suppose $p = 2$. If $Q/\langle u\rangle$ is metacyclic or minimal nonabelian, we have $k_0(B) \le |Q|$.*

If (u, b_u) is major, we can replace $k_0(B)$ by $k(B)$ in all these formulas.

Brauer's $k(B)$-Conjecture

If the defect group is "small", the $k(B)$-Conjectures follow at once.

Corollary

Brauer's $k(B)$-Conjecture holds for all 2-blocks of defect at most 5 except possibly the extraspecial defect group $D_8 * D_8$.

Corollary

Brauer's $k(B)$-Conjecture holds for all 3-blocks of defect at most 3.

Theorem (Gao for $p > 2$)

Brauer's $k(B)$-Conjecture holds for all blocks with metacyclic defect groups.

Brauer's $k(B)$-Conjecture

The following older results were obtained similarly.

Theorem (Brauer)

If $|D| \leq p^2$, then $k(B) = k_0(B) \leq |D|$.

Theorem (Olsson)

If $l(B) \leq 2$ or $k(B) - l(B) \leq 2$, then

$$k(B) \leq \sum_{i=0}^{\infty} p^i k_i(B) \leq |D|.$$

Olsson's Conjecture

As another application we have verified Olsson's Conjecture under certain circumstances.

Theorem (Héthelyi-Külshammer-S.)

Let $p > 3$. Then Olsson's Conjecture holds for all p-blocks with defect groups of p-rank 2 and for all p-blocks with minimal nonabelian defect groups.

Using the action of a Galois group as mentioned earlier we obtained even stronger bounds.

More inequalities

Theorem (S.)

Let $p = 2$, and let (u, b_u) be a B-subsection such that $\langle u \rangle$ is fully $\mathcal{F}$-normalized and u is conjugate to u^{-5^n} for some $n \in \mathbb{Z}$ in D. If $l(b_u) \leq 2$, then

$$k_0(B) \leq 2|\mathrm{N}_D(\langle u \rangle)/\langle u \rangle|.$$

Here fully $\mathcal{F}$-normalized means that $|\mathrm{N}_D(\langle u \rangle)|$ is as large as possible among all $\mathcal{F}$-conjugates of u.

The case $p > 2$

The analogous result for odd primes gives a weaker bound.

Theorem (S.)

Let $p > 2$, and let (u, b_u) be a B-subsection such that $l(b_u) = 1$ and b_u has defect d. Moreover, let $|\mathrm{Aut}_{\mathcal{F}}(\langle u \rangle)| = p^s r$ where $p \nmid r$ and $s \geq 0$. Then we have

$$k_0(B) \leq \frac{|\langle u \rangle| + p^s(r^2 - 1)}{|\langle u \rangle| r} p^d \leq p^d.$$

If (in addition) (u, b_u) is major, we can replace $k_0(B)$ by $\sum_{i=0}^{\infty} p^{2i} k_i(B)$.

Height Zero Conjecture

An application of the formula gives the following recent theorem.

Theorem (S.)

Brauer's Height Zero Conjecture holds for all blocks with defect group p_-^{1+2}.

A related result

The next related result was obtained using the theory of integral quadratic forms.

Theorem (S.)

Let C be the Cartan matrix of B. If $l(B) \leq 4$ and $\det C = |D|$, then

$$k(B) \leq \frac{|D|-1}{l(B)} + l(B).$$

Moreover, this bound is sharp.

Remarks

- It should be pointed out that the knowledge of $l(B)$ usually implies the knowledge of $k(B)$ anyway. So this theorem has more theoretical value.
- The next theorem gives a useful sufficient condition for $\det C = |D|$.

Theorem (Fujii)

Let C be the Cartan matrix of B. If $l(b_u) = 1$ for all nontrivial B-subsections (u, b_u), then $\det C = |D|$.