Further evidence for conjectures in block theory

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Introduction

- Let G be a finite group and p be a prime.
- Let B be a p-block of G with defect d.
- We denote the number of irreducible characters of B by $k(B)$, and the number of irreducible Brauer characters by $l(B)$.

Theorem (Olsson, 1981)

(i) *If* $l(B) \leq 2$, *then* $k(B) \leq p^d$.

(ii) *If* $p = 2$ *and* $l(B) \leq 3$, *then* $k(B) \leq 2^d$.

In particular Brauer's $k(B)$*-Conjecture holds in these cases.*

Remarks

- Usually the knowledge of $l(B)$ implies the exact value of $k(B)$.
- Hence, Olsson's result is more of theoretical nature.
- In order to improve Olsson's theorem, the idea is to replace $l(B)$ by something "local".
- Let D be a defect group of B, and let $u \in \mathrm{Z}(D)$.
- Then there is a Brauer correspondent b_u of B in $\mathrm{C}_G(u)$.
- The pair (u, b_u) is called major subsection (for B).

A generalization

Theorem (S., 2012)

Let $p = 2$, and let (u, b_u) a major B-subsection such that $l(b_u) \leq 3$. Then

$$k(B) \leq k_0(B) + \frac{2}{3}\sum_{i=1}^{\infty} 2^i k_i(B) \leq 2^d.$$

In particular Brauer's $k(B)$-Conjecture holds for B.

Here $k_i(B)$ denotes the number of irreducible characters of height $i \geq 0$ of B.

Remarks

- The proof relies on calculations with so-called "contributions" which were introduced by Brauer.
- In contrast to Olsson's proof, the contributions are not always integers in this general setting.
- Applying Galois theory fixes this issue.
- However, for odd primes p things are more difficult, since the cyclotomic fields behave differently.

A similar inequality

- Olsson also proved the $k(B)$-Conjecture under the hypothesis $k(B) - l(B) \leq 3$ for $p = 2$.
- This can also be carried over to $k(B) - l(b_u) \leq 3$.
- However, in many cases we have $k(B) - l(B) \leq k(B) - l(b_u)$; so the general result adds little.
- Let us go over to arbitrary subsections, i. e. $u \in D$ does not necessarily belong to the center of D.

Characters of height 0

Theorem (Robinson, 1992)

Let (u, b_u) be a B-subsection such that b_u has defect q. Then

$$k_0(B) \leq p^q \sqrt{l(b_u)}.$$

- This is useful for proving Olsson's Conjecture $k_0(B) \leq |D : D'|$.
- For $p = 2$ this can be slightly improved to:

Characters of height 0

Theorem (S., 2012)

Let (u, b_u) be a subsection of a 2-block B such that b_u has defect q. Set

$$\alpha := \begin{cases} \lfloor \sqrt{l(b_u)} \rfloor & \text{if } \lfloor \sqrt{l(b_u)} \rfloor \text{ is odd,} \\ \frac{l(b_u)}{\lfloor \sqrt{l(b_u)} \rfloor + 1} & \text{otherwise.} \end{cases}$$

Then $k_0(B) \le 2^q \alpha$. In particular $k_0(B) \le 2^q$ if $l(b_u) \le 3$.

The Cartan method

Last year I developed a similar method for bounding $k(B)$ and $k_0(B)$ using the Cartan matrix:

Theorem (S., 2011)

Let (u, b_u) be a B-subsection such that b_u has Cartan matrix $C_u = (c_{ij})$ up to basic sets. Then

$$k_0(B) \leq \sum_{i=1}^{l(b_u)} c_{ii} - \sum_{i=1}^{l(b_u)-1} c_{i,i+1}.$$

If (u, b_u) is major, we can replace $k_0(B)$ by $k(B)$ in this formula.

Examples

- So in case $p = 2$ and $l(b_u) \leq 3$ we do not need the Cartan matrix anymore.
- This implies Brauer's $k(B)$-Conjecture in many more cases which will be shown later.

Theorem (S., 2011)

Let B be a 2-block with defect group $M \times C$ or $M * C$ where C is cyclic and M is a nonabelian group of maximal class. Then $l(B) \leq 3$.

Noncommutative versions

Recently, I extended this result to the following similar defect groups:

$$\begin{aligned}
&\langle v, x, a \mid v^{2^n} = x^2 = a^{2^m} = 1,\ {}^x v = {}^a v = v^{-1},\ {}^a x = vx \rangle \\
&\qquad \cong D_{2^{n+1}} \rtimes C_{2^m} \quad (n, m \ge 2), \\
&\langle v, x, a \mid v^{2^n} = 1,\ a^{2^m} = x^2 = v^{2^{n-1}},\ {}^x v = {}^a v = v^{-1},\ {}^a x = vx \rangle \\
&\qquad \cong D_{2^{n+1}}.C_{2^m} \cong Q_{2^{n+1}}.C_{2^m} \quad (n, m \ge 2 \text{ and } m \ne n), \\
&\langle v, x, a \mid v^{2^n} = a^{2^m} = 1,\ x^2 = v^{2^{n-1}},\ {}^x v = {}^a v = v^{-1},\ {}^a x = vx \rangle \\
&\qquad \cong Q_{2^{n+1}} \rtimes C_{2^m} \quad (n, m \ge 2).
\end{aligned}$$

In fact all block invariants ($k(B)$, $k_i(B)$ and $l(B)$) could be determined precisely. This gives new evidence for Alperin's Weight Conjecture.

Bicyclic 2-groups

- These groups are examples of bicyclic 2-groups, i. e. they can be written in the form $D = \langle x\rangle\langle y\rangle$ for some $x, y \in D$.
- I also classified all saturated fusion systems on bicyclic 2-groups using results of Janko.
- It turns out that they are not exotic, i. e. they occur in finite groups.
- Instead of the whole classification, I state some corollaries.

Theorem (S., 2012)

Let P be a bicyclic, nonmetacyclic 2-group. Then P admits a non-trivial fusion system if and only if P' is cyclic. The number of these groups (and fusion systems) grows with $\log^2|P|$.

Corollaries

Corollary

Let G be a finite group with bicyclic Sylow 2-subgroup P. If P' is noncyclic, then P has a normal complement in G.

Theorem (Yang for odd primes, 2011)

Olsson's Conjecture holds for all blocks with bicyclic defect groups.

Here it is important to observe that bicyclic p-groups for odd primes p are always metacyclic (Huppert).

Theorem (S., 2012)

Let D be a cyclic central extension of one of the following groups

1. *a metacyclic group,*
2. *a minimal nonabelian group,*
3. *a group of order at most 16,*
4. $M \overset{\times}{*} C$ *where M has maximal class and C is cyclic,*
5. $D_{2^n} \rtimes C_{2^m}$, $Q_{2^n} \rtimes C_{2^m}$ *and* $D_{2^n}.C_{2^m}$ *as above,*
6. $\prod_{i=1}^n C_{2^{m_i}}$ *where* $|\{m_i : i = 1, \ldots, n\}| \ge n - 1$,
7. $\mathtt{SmallGroup}(32, i)$ *for* $i \in \{11, 22, 28, 29, 33, 34\}$,
8. *a group which admits only the trivial fusion system.*

Then Brauer's $k(B)$-Conjecture holds for every 2-block with defect group D.

Sketch of the proof (1)

- Let B be a 2-block of a finite group G with defect group D as above.
- Let $u \in \mathrm{Z}(D)$ such that $D/\langle u \rangle$ has one of the stated isomorphism types.
- Let (u, b_u) be the corresponding (major) subsection.
- Then b_u dominates a block $\overline{b_u}$ of $\mathrm{C}_G(u)/\langle u \rangle$ with defect group $D/\langle u \rangle$.
- Moreover, the Cartan matrices of b_u and $\overline{b_u}$ differ only by the factor $|\langle u \rangle|$. In particular $l(b_u) = l(\overline{b_u})$.

Sketch of the proof (2)

- In most cases Brauer's $k(B)$-Conjecture follows from $l(b_u) = l(\overline{b_u}) \leq 3$.
- In case $D/\langle u\rangle \cong C_2^4$ we can apply the "inverse Cartan method" introduced by Brauer.
- In the remaining cases we can compute (by computer) a list of possible Cartan matrices for $\overline{b_u}$ and the Cartan method implies the result. □
- For defect groups of order at most 32 the $k(B)$-Conjecture follows at once.
- Using the result above, I verified the $k(B)$-Conjecture for 244 of the 267 defect groups of order 64.

Applications

Corollary

Let B be a 2-block with defect group D of order at most 64. If D is generated by two elements, then Brauer's $k(B)$-Conjecture holds for B.

Corollary

Let D be a 2-group containing a cyclic subgroup of index at most 4. Then Brauer's $k(B)$-Conjecture holds for every block with defect group D.

For every $n \geq 6$ there are exactly 33 groups of order 2^n satisfying the hypothesis of the last corollary (Ninomiya).

Olsson's Conjecture

A similar theorem for arbitrary subsections yields the following result.

Theorem (S., 2012)

Olsson's Conjecture holds for all 2-blocks of defect at most 5.

Metacyclic defect groups

- The invariants of 2-blocks with metacyclic defect groups are known by work of several authors.
- In particular most of the conjectures are fulfilled.
- For odd primes the situation is more complicated.
- We already saw that Olsson's Conjecture holds for metacyclic defect groups (even bicyclic).

Theorem (Gao for odd primes, 2011)

Brauer's $k(B)$-Conjecture is satisfied for all blocks with metacyclic defect groups.

Metacyclic defect groups

Brauer's Height Zero Conjecture asserts that $k(B) = k_0(B)$ if and only if B has abelian defect groups.

Theorem (S., 2012)

Brauer's Height Zero Conjecture is satisfied for all blocks with metacyclic defect groups.

Sketch of the proof (1)

- Let B be a p-block with metacyclic defect group D.
- We may assume that p is odd and D is nonabelian (Kessar-Malle).
- Then the fusion of subsections is controlled by the inertial group of B (Stancu).
- The theory of lower defect groups implies $l(B) \geq e(B) \mid p-1$ where $e(B)$ is the inertial index of B.
- If $\mathcal{R}$ is a set of representatives of the G-conjugacy classes of subsections, then we have

$$k(B) = \sum_{(u,b_u)\in\mathcal{R}} l(b_u).$$

Sketch of the proof (2)

- This gives a lower bound for $k(B)$.
- We can always find a subsection (u, b_u) such that $|\mathrm{C}_D(u)| = |D : D'|$ and $\mathrm{C}_D(u)/\langle u\rangle$ is cyclic.
- Since $\overline{b_u}$ has defect group $\mathrm{C}_D(u)/\langle u\rangle$, the Cartan method implies an upper bound for $k_0(B)$.
- Now $k_0(B) < k(B)$ follows. $\square$