Things left to prove

Characters and Blocks of Finite Groups
Gabriel's conference 2025

Benjamin Sambale

31th January 2025

Gabriel likes (to propose) problems

- 1994: *Some Open Problems on Coprime Action and Character Correspondences*, Bull. London Math. Soc. 26, 513–522
- 2004: *Problems on characters and Sylow subgroups*, in: Finite groups, 275–281
- 2004: *The McKay conjecture and Galois automorphisms*, Ann. of Math. 160, 1129–1140 (71 citations)
- 2010: *Problems in character theory*, in: Character theory of finite groups, 97–125
- 2023: *Problems on characters: solvable groups*, Publ. Mat. 67, 173–198
- 2024: "All I would like to prove has been proved!", e-mail correspondence

Automorphisms

Let G be a finite group.

Definition (Automorphism tower)

Let $\mathrm{Aut}^1(G) := \mathrm{Aut}(G)$, $\mathrm{Aut}^2(G) := \mathrm{Aut}(\mathrm{Aut}(G)), \ldots$.

If $\mathrm{Z}(G) = 1$, then $G \cong \mathrm{Inn}(G) \le \mathrm{Aut}^1(G) \le \mathrm{Aut}^2(G) \le \ldots$.

Theorem (Wielandt)

If $\mathrm{Z}(G) = 1$, then $\mathrm{Aut}^n(G) \cong \mathrm{Aut}^{n+1}(G)$ for some n.

Automorphisms

Theorem (Hamkins)

The transfinite automorphism tower of any group is bounded by some cardinal number.

Problem (Kourovka notebook 11.123)

- Is there a constant c such that $|\mathrm{Aut}^n(G)| < c$ for all n?
- Is $\mathrm{Aut}^n(G) \cong \mathrm{Aut}^{n+1}(G)$ for some n?
- Can $\mathrm{Aut}^n(G) \cong \mathrm{Aut}^m(G) \ncong \mathrm{Aut}^{n+1}(G)$ with $n < m$ happen?

Automorphisms

Example

- If G is non-abelian simple, then $\mathrm{Aut}^2(G) \cong \mathrm{Aut}(G)$ (Burnside).
- $\mathrm{Aut}(C_2^n) \cong \mathrm{GL}(n,2)$ is simple (for $n \geq 3$), so $\mathrm{Aut}^3(C_2^n) \cong \mathrm{Aut}^2(C_2^n)$.
- Center can stay non-trivial: $\mathrm{Aut}(D_8) \cong D_8$.
- The sequence can decrease arbitrarily long: $\mathrm{Aut}(C_{2\cdot 3^n}) \cong C_{2\cdot 3^{n-1}}$
- For $G = \mathtt{SmallGroup}(32,13)$ we have

n	1	2	3	4	5
$\lvert\mathrm{Aut}^n(G)\rvert$	2^7	2^{13}	2^{28}	2^{83}	?

Homomorphisms

Theorem (FROBENIUS)

For every n, the number of $x \in G$ such that $x^n = 1$ is divisible by $\gcd(n, |G|)$.

Equivalently,

$$|\mathrm{Hom}(C_n, G)| \equiv 0 \pmod{\gcd(n, |G|)}.$$

Theorem (YOSHIDA)

For every finite abelian group A,

$$|\mathrm{Hom}(A, G)| \equiv 0 \pmod{\gcd(|A|, |G|)}.$$

Homomorphisms

Conjecture (Asai–Yoshida)

For every finite group H,

$$|\mathrm{Hom}(H,G)| \equiv 0 \pmod{\gcd(|H/H'|,|G|)}.$$

Conjugacy classes

- Let $k(G)$ be the number of conjugacy classes of G.
- Landau: $k(G) > f(|G|)$ for some increasing function f.

Theorem (Baumeister–Maróti–Tong-Viet)

For every $\epsilon > 0$ there exists $\delta > 0$ such that

$$k(G) > \frac{\delta \log |G|}{(\log \log |G|)^{3+\epsilon}}.$$

Conjecture (Folklore?)

Is there a constant c such that $k(G) > c \log |G|$ for all G?

Constituents

Let $\mathrm{Irr}(G)$ be the set of irreducible complex characters of G.

Conjecture (Héthelyi–Külshammer)

Let P be a p-group and $\chi \in \mathrm{Irr}(P)$. Then the number of irreducible constituents of $\chi\overline{\chi}$ is 1 mod $p-1$.

Conjecture holds for $|P| \le p^6$.

Problem (Knutson–Murray)

Let P be a p-group and $\chi \in \mathrm{Irr}(P)$. Is there a generalized character ψ such that $\chi\psi$ is the regular character of P?

Constituents

Call $\theta \in \mathrm{Irr}(\mathrm{Z}(G))$ fully ramified if θ^G is a multiple of an irreducible character.

Theorem (Howlett–Isaacs)

If $\theta \in \mathrm{Irr}(\mathrm{Z}(G))$ is fully ramified, then G is solvable.

Conjecture (Humphreys, Navarro)

Let $\theta \in \mathrm{Irr}(\mathrm{Z}(G))$ such that all constituents of θ^G have the same degree. Then G is solvable.

Conjecture holds whenever θ^G has at most two constituents (Higgs).

Field of values

For $\chi \in \mathrm{Irr}(G)$ define the abelian number fields

$$\mathbb{Q}(\chi) := \mathbb{Q}\big(\chi(g) : g \in G\big) \subseteq \mathbb{Q}_{|G|} \subseteq \mathbb{C},$$
$$\mathbb{Q}(G) := \mathbb{Q}\big(\chi(g) : \chi \in \mathrm{Irr}(G),\ g \in G\big) \subseteq \mathbb{Q}_{|G|}.$$

By the KRONECKER–WEBER theorem, every abelian number field lies in some $\mathbb{Q}_n$.

Theorem (FEIN–GORDON)

For every abelian number field F there exist a group G and $\chi \in \mathrm{Irr}(G)$ such that $\mathbb{Q}(\chi) = F$.

Field of values

Conjecture

- Not every abelian number field has the form $\mathbb{Q}(G)$.
- For every d, there are only finitely many number fields $\mathbb{Q}(G)$ of degree d.

There are only finitely many fields $\mathbb{Q}(G)$ of degree d where G is solvable or simple (Farias e Soares, Feit–Seitz).

Example (quadratic fields)

Let $d \mid 210$ or $d \in \{-231, -11, 13, 17\}$. Then there exists G with $\mathbb{Q}(G) \cong \mathbb{Q}(\sqrt{d})$.

Field of values

Theorem (Robinson–Thompson)

For $n > 24$ *we have*

$$\mathbb{Q}(A_n) = \mathbb{Q}\Big(\sqrt{(-1)^{\frac{p-1}{2}}p} : p \text{ odd prime } \le n,\ p \neq n-2\Big).$$

In particular, $\mathrm{Gal}(\mathbb{Q}(A_n)|\mathbb{Q})$ *is an elementary abelian* 2*-group.*

Problem

- Is every abelian group the Galois group of some $\mathbb{Q}(G)$?
- Is there a group G such that $|\mathbb{Q}(G) : \mathbb{Q}| = 7$?
- Is there a group G such that $\mathbb{Q}(G) = \mathbb{Q}(\sqrt{11})$?
- Is there a solvable group G such that $\mathbb{Q}(G) = \mathbb{Q}(\sqrt{-5})$?

Field of values

For $\chi \in \mathrm{Irr}(G)$ let $f(\chi)$ be the smallest integer n such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_n$.

Conjecture (Hung–Tiep)

We have $|\mathbb{Q}_{f(\chi)} : \mathbb{Q}(\chi)| \le \chi(1)$ for all $\chi \in \mathrm{Irr}(G)$.

Conjecture holds whenever $\chi(1)$ is a prime (Hung–Tiep–Zalesski).

Problem (Hung–Tiep)

Let $z = \zeta_1 + \ldots + \zeta_n \in \mathbb{C}$ be a sum of roots of unity. Let $c(z)$ be the smallest m such that $z \in \mathbb{Q}_m$. Is $|\mathbb{Q}_{c(z)} : \mathbb{Q}(z)| \le n$?

Algebraic integers

- Let $\mathbb{Z}_G$ be the ring of integers of $\mathbb{Q}(G)$.
- Let $\mathbb{Z}[G] := \mathbb{Z}[\chi(g) : \chi \in \mathrm{Irr}(G),\ g \in G] \subseteq \mathbb{Z}_G$.
- Then $\mathbb{Z}_G/\mathbb{Z}[G]$ is a finite abelian group.

Conjecture (Bächle–S.)

The exponent of $\mathbb{Z}_G/\mathbb{Z}[G]$ divides $|G|$.

Example

Conjecture holds for nilpotent groups. For $G = C_{15} \rtimes D_{16}$,

$$\mathbb{Z}_G/\mathbb{Z}[G] \cong C_{120}^2 \times C_{60}^2 \times C_{12}^4 \times C_4^4 \times C_2^{14}.$$

Weights

- Let π be a set of primes.
- Let $l(G)$ be the number of conjugacy classes of π'-elements of G.
- Let $\chi \in \mathrm{Irr}(G)$ of π-defect 0 if $|G|/\chi(1)$ is a π'-number.
- Let $k^0(G)$ be the number of π-defect 0 characters of G.

Theorem (Navarro–S.)

If G is π-separable with solvable Hall π-subgroups, then

$$l(G) = \sum_P k^0(\mathrm{N}_G(P)/P)$$

where P runs through the nilpotent π-subgroups of G up to conjugation.

For $\pi = \{p\}$, this is Alperin's weight conjecture for p-solvable groups.

Weights

Definition

Define the weight of a group P by

$$\mu(P) = 1 - \sum_{\substack{Q<P \\ \mathrm{N}_P(Q)=Q}} \mu(Q) \qquad (Q \text{ up to conjugation}).$$

Conjecture (Navarro–S.)

For every π-separable group G,

$$l(G) = \sum_P \mu(P) k^0(\mathrm{N}_G(P)/P)$$

where P runs through the π-subgroups of G up to conjugation.

Weights

Example

- If P is nilpotent, then $\mu(P) = 1$.
- If P is solvable, but not nilpotent, then $\mu(P) = 0$ (Carter subgroups).
- On the other hand,

P	A_5	A_6	A_7	A_8	$\mathrm{PSL}(3,4)$	A_5^2	$S_n (3 \leq n \leq 11)$
$\mu(P)$	1	-2	1	-1	6	-1	0

Characterization of blocks

Let G^0 be the set of p-regular elements of G.

Theorem (Osima)

A subset $J \subseteq \operatorname{Irr}(G)$ is a union of p-blocks if and only if

$$\sum_{\chi \in J} \chi(g)\chi(h) = 0 \qquad (\forall g \in G^0,\ h \in G \setminus G^0).$$

Conjecture (Harada)

It is enough to fix $g = 1$ in Osima's theorem.

Basic sets

- Let B be a p-block of G.
- For $\chi \in \mathrm{Irr}(G)$ let χ^0 be the restriction to G^0.
- The Brauer characters in p-solvable groups have the form χ^0 (Fong–Swan).

Conjecture (ordinary basic set, Geck)

There exist $\chi_1, \ldots, \chi_l \in \mathrm{Irr}(B)$ such that $\chi_1^0, \ldots, \chi_l^0$ is a $\mathbb{Z}$-basis for the ring of generalized Brauer characters of B.

This holds in a strong sense for "most" groups of Lie type (Brunat–Dudas–Taylor).

Number of characters

- For $\chi, \psi \in \mathrm{Irr}(G)$ let

$$[\chi, \psi]^0 = \frac{1}{|G|} \sum_{g \in G^0} \chi(g)\overline{\psi(g)}.$$

- Let d be the defect of B. Then $p^d[\chi, \chi]^0 \in \mathbb{N}$ for all $\chi \in \mathrm{Irr}(B)$.

Conjecture (Navarro-S.)

We have $p^d[\chi, \chi]^0 \geq l(B)$ for all $\chi \in \mathrm{Irr}(B)$.

- This implies Brauer's conjecture $k(B) \leq p^d$ with equality only if B has abelian defect groups.
- The case $\chi = 1_G$ was conjectured by Murai and implies Frobenius conjecture: G is p-nilpotent iff $|G^0| = |G|_{p'}$ (known via CFSG).

Fusion numbers

- Let
$$X_B := \big(\chi(g_i) : \chi \in \mathrm{Irr}(B),\ i = 1, \ldots, k(G)\big)$$
be a submatrix of the character table.
- The non-zero elementary divisors of $X_B^{\mathrm{t}}\overline{X_B}$ over a complete discrete valuation ring are p-powers $e_1, \ldots, e_t$.

Conjecture (S.)

We have $\gamma(B) := \frac{1}{e_1} + \ldots + \frac{1}{e_t} \geq 1$ with equality if and only if B is nilpotent.

Example

Conjecture holds for symmetric groups and "ATLAS groups". For the principal 2-block B of the Monster, $\gamma(B) \approx 39.5$.

Some memories

- *I read your question just before going to bed and you have ruined my night. Good question. I dreamt that I had an example.*
- *I am two days, 18 hours each, trying to clean up the inductive McKay, no waving hands.* [finishing his second book]
- *If these two guys are not Morita equivalent then I won't say the word Morita again!* [they weren't]
- *Silly? I am in the middle of a restaurant.*
- *Seriously, I have thought on this for 3 minutes.*
- *Zero.* [answer to: How many examples did you check?]
- *I am also amazed by my intuition, sorry to say!*
- *25th anniversary of my wedding. I have promised my wife not to think on mathematics for a couple of days. Now you are making my promise easy to break...*