Fusion systems in representation theory

Three lectures at the University of Valencia

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1 Fusion in groups

Definition 1.1. Let $H \leq G$ be finite groups. Elements $x, y \in H$ (or subsets) are called *fused* in G if they are conjugate in G, but not in H.

Example 1.2.

(i) The permutations $(123), (132) \in A_3$ are fused in S_3.

(ii) Let $X, Y \leq H$ be isomorphic subgroups via an isomorphism $\varphi\colon X \to Y$. We embed H into $G := \mathrm{Sym}(H)$ via the regular representation $\sigma\colon H \to G$, $h \mapsto \sigma_h$ where $\sigma_h(g) = hg$ for $g, h \in H$. Let $\hat{\varphi} \in G$ be any extension of φ. Then for $x \in X$ and $y \in Y$ we compute

$$(\hat{\varphi}\sigma_x\hat{\varphi}^{-1})(y) = \hat{\varphi}(x\varphi^{-1}(y)) = \varphi(x)y = \sigma_{\varphi(x)}(y).$$

Hence, φ is realized by the conjugation with $\hat{\varphi}$ in G.[1]

(iii) A consequence of (ii) is that elements $x, y \in H$ of the same order are conjugate in some finite group $G \geq H$.

Goal: Find "small" subgroups $K \supseteq H$ *controlling fusion* in H, i. e. $x, y \in H$ are fused in G if and only if x, y are fused in K.

Main interest: $H \in \mathrm{Syl}_p(G)$.

In the following let $P \in \mathrm{Syl}_p(G)$. Let $\mathrm{O}_{p'}(G)$ be the largest normal p'-subgroup of G. If no elements of P are fused in G, then G is called *p-nilpotent*.

Theorem 1.3 (Frobenius)**.** *The following assertions are equivalent:*

(1) G *is p-nilpotent.*

(2) $\mathrm{N}_G(Q)/\mathrm{C}_G(Q)$ *is a p-group for all* $Q \leq P$.

(3) $G = \mathrm{O}_{p'}(G)P$.

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[1] This construction fails for infinite groups since for example the isomorphism $\mathbb{Z} \to 2\mathbb{Z}$ does not extend to $\mathbb{Z} \to \mathbb{Z}$. In those situations one can use HNN-*extensions*.

Example 1.4. Every p'-group and every nilpotent group is p-nilpotent.

Theorem 1.5 (Burnside)**.** *$\mathrm{N}_G(P)$ controls fusion in $\mathrm{Z}(P)$.*

Proof. Let $x, y \in \mathrm{Z}(P)$ and $g \in G$ such that ${}^g x := gxg^{-1} = y$. Then $P \le \mathrm{C}_G(y)$ and ${}^g P \le {}^g\mathrm{C}_G(x) = \mathrm{C}_G({}^g x) = \mathrm{C}_G(y)$. By Sylow's theorem, there exists $c \in \mathrm{C}_G(y)$ such that ${}^{cg}P = P$. Now $h := cg \in \mathrm{N}_G(P)$ such that ${}^h x = {}^c({}^g x) = {}^c y = y$. □

Theorem 1.6 (Z^*-theorem[2])**.** *If $z \in \mathrm{Z}(P)$ is not fused to any other element in P, then $G = \mathrm{O}_{p'}(G)\mathrm{C}_G(z)$.*

Proof. Glauberman proved the theorem for $p = 2$ using representation theory, while the only known proof for $p > 2$ is via the classification of finite simple groups (CFSG for short). □

By Burnside's theorem, the Z^*-theorem is equivalent to $G = \mathrm{O}_{p'}(G)\mathrm{C}_G(Z)$ where $Z := \mathrm{Z}(\mathrm{N}_G(P)) \cap P$.

Example 1.7. If P is a (generalized) quaternion 2-group, then $G = \mathrm{O}_{2'}(G)\mathrm{C}_G(\mathrm{Z}(P))$ since $\mathrm{Z}(P)$ is generated by the unique involution in P.[3]

Goldschmidt and Flores–Foote classified more generally groups G with $A \trianglelefteq P$ such that no element of A is fused to an element of $P \setminus A$ (i. e. A is *strongly closed* in P). Let

$$\mathrm{J}(P) := \langle A \le P : A \text{ abelian of maximal order} \rangle$$

be the *Thompson subgroup* of P.[4]

Theorem 1.8 (Thompson)**.** *If $p \ge 5$, then G is p-nilpotent if and only if $\mathrm{N}_G(\mathrm{J}(P))/\mathrm{C}_G(\mathrm{J}(P))$ is a p-group.*

Theorem 1.9 (Glauberman's ZJ-theorem)**.** *Let $p > 2$. Then G is p-nilpotent if and only if $\mathrm{N}_G\big(\mathrm{Z}(\mathrm{J}(P))\big)$ is p-nilpotent. If G has no section isomorphic to $Qd(p) := C_p^2 \rtimes \mathrm{SL}_2(p)$, then $\mathrm{N}_G\big(\mathrm{Z}(\mathrm{J}(P))\big)$ controls fusion in P.*

Example 1.10. For $p \ge 5$, every (p-)solvable group is $Qd(p)$-free.

Theorem 1.11 (Stellmacher)**.** *If $p = 2$ and G has no section isomorphic to $Qd(2) \cong S_4$, then $\mathrm{N}_G(W)$ controls fusion in P for some characteristic subgroup W of P. If $P \ne 1$, then $W \ne 1$.*

Let $G' = [G, G]$ be the commutator subgroup and $\mathrm{O}^p(G) = \langle p'\text{-elements} \rangle$ the p-residue of G.

Theorem 1.12 ((Hyper)focal subgroup theorem)**.**

$$\mathfrak{foc}_G(P) := \langle xy^{-1} : x, y \in P \textit{ are conjugate in } G \rangle = G' \cap P \quad \text{(focal subgroup)},$$
$$\mathfrak{hyp}_G(P) := \langle xy^{-1} : x, y \in P \textit{ are conjugate by a } p'\textit{-element} \rangle = \mathrm{O}^p(G) \cap P \quad \text{(hyperfocal subgroup)}.$$

[2] It is often assumed that x has order p, but this is unnecessary

[3] This special case of the Z^*-theorem was first proved by Brauer–Suzuki.

[4] Several non-equivalent definitions of the Thompson subgroup are used in the literature.

The *transfer map* yields $G/\mathrm{O}^p(G) \cong P/\mathfrak{hyp}_G(P)$.

Theorem 1.13 (GRÜN's theorem).

$$\mathfrak{foc}_G(P) = [\mathrm{N}_G(P), P]\langle P \cap Q' : Q \in \mathrm{Syl}_p(G)\rangle.$$

Let $\Phi(P)$ be the Frattini subgroup of P.

Theorem 1.14. *The following assertions are equivalent:*

(1) G is p-nilpotent.

(2) $\mathfrak{hyp}_G(P) = 1$.

(3) $\mathfrak{hyp}_G(P) \le \Phi(P)$.

Theorem 1.15 (TATE's transfer theorem). *For $P \le H \le G$ we have*

$$\mathfrak{foc}_G(P) = \mathfrak{foc}_H(P) \iff \mathfrak{hyp}_G(P) = \mathfrak{hyp}_H(P) \iff \mathfrak{foc}_G(P)\Phi(P) = \mathfrak{foc}_H(P)\Phi(P).$$

If $\mathfrak{foc}_G(P) = \mathfrak{foc}_H(P)$, we say that H *controls transfer* in P. In this case H determines whether G is p-nilpotent by Theorem 1.14.

Theorem 1.16 (YOSHIDA's transfer theorem). *If P has no quotient isomorphic to $C_p \wr C_p$, then $\mathrm{N}_G(P)$ controls transfer in P.*

Example 1.17.

(i) If $|P| \le p^p$ or $\exp(P) = p$ (exponent) or $c(P) < p$ (nilpotency class), then $\mathrm{N}_G(P)$ controls transfer in P. This follows from the properties of $C_p \wr C_p$.

(ii) Let $p = 2$ and $G = S_4$. Then $\mathrm{N}_G(P) = P \cong D_8 \cong C_2 \wr C_2$ does not control transfer in P since otherwise G would be 2-nilpotent. For $p > 2$ and

$$G = \mathbb{F}_p^p \rtimes \left\langle \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & \ddots & \\ 0 & & & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix} \right\rangle \le \mathrm{ASL}(p, p),$$

again $\mathrm{N}_G(P) = P \cong C_p \wr C_p$ does not control transfer in P.

Theorem 1.18 (GLAUBERMAN). *If $p \ge 5$, then there exists a characteristic subgroup K of P such that $\mathrm{N}_G(K)$ controls transfer in P and $\mathrm{Z}(P) \le K$.*

The simple group $\mathrm{PSL}(2, 17)$ shows that Theorem 1.18 fails for $p = 2$ (here P is a maximal subgroup). It is an open problem whether Theorem 1.18 holds for $p = 3$. For $p \ge 7$ one can take $K = \mathrm{J}(P)$.

2 Fusion systems

For arbitrary groups $S, T \le P$ let $\mathrm{Hom}_P(S,T)$ be the set of homomorphisms $S \to T$ induced by inner automorphisms of P, i. e.

$$\mathrm{Hom}_P(S,T) := \big\{\varphi\colon S \to T : \exists g \in P : \varphi(s) = {}^g s \ \forall s \in S\big\}.$$

Definition 2.1 (Puig[5])**.** A *fusion system* on a finite p-group P is a category $\mathcal{F}$ with objects $\mathrm{Obj}(\mathcal{F}) = \{S : S \le P\}$ and morphisms $\mathrm{Hom}_\mathcal{F}(S,T) \subseteq \{S \to T : \text{injective group homomorphism}\}$ such that

- $\mathrm{Hom}_P(S,T) \subseteq \mathrm{Hom}_\mathcal{F}(S,T)$ for $S, T \le P$,
- $\varphi \in \mathrm{Hom}_\mathcal{F}(S,T) \implies \varphi \in \mathrm{Hom}_\mathcal{F}(S,\varphi(S)),\ \varphi^{-1} \in \mathrm{Hom}_\mathcal{F}(\varphi(S), S)$.

Example 2.2.

(i) Let P be a p-subgroup of a finite group G. Then $\mathrm{Hom}_\mathcal{F}(S,T) := \mathrm{Hom}_G(S,T)$ for $S, T \le P$ defines a fusion system on P, which we denote by $\mathcal{F}_P(G)$. In particular, there is always the *trivial* fusion system $\mathcal{F}_P(P)$, which is a subcategory of every fusion system on P.

(ii) The *universal* fusion system $\mathcal{F} := \mathcal{U}(P)$ on P is defined by

$$\mathrm{Hom}_\mathcal{F}(S,T) := \{S \to T \text{ injective homomorphism}\}.$$

Every fusion system on P is a subcategory of $\mathcal{U}(P)$.

Theorem 2.3 (Park)**.** *For every fusion system $\mathcal{F}$ on P there exists a finite group G containing P such that $\mathcal{F} = \mathcal{F}_P(G)$.*

Theorem 2.3 remains true even for arbitrary finite groups P with appropriate definitions (see Example 1.2(ii) for $\mathcal{F} = \mathcal{U}(P)$).

Definition 2.4. Let $\mathcal{F}$ be a fusion system on P and $S, T \le P$.

- S, T are called *$\mathcal{F}$-conjugate* if there exists an isomorphism $\varphi\colon S \to T$ in $\mathcal{F}$.
- S is called *$\mathcal{F}$-automized* if $\mathrm{Aut}_P(S) \in \mathrm{Syl}_p(\mathrm{Aut}_\mathcal{F}(S))$.
- S is called *$\mathcal{F}$-centralized*[6] if $|\mathrm{C}_P(S)| \ge |\mathrm{C}_P(T)|$ for all $\mathcal{F}$-conjugates T of S.
- S is called *$\mathcal{F}$-normalized* if $|\mathrm{N}_P(S)| \ge |\mathrm{N}_P(T)|$ for all $\mathcal{F}$-conjugates T of S.
- For an isomorphism $\varphi\colon S \to T$ let N_φ be the preimage of $\mathrm{Aut}_P(S) \cap \varphi^{-1}\mathrm{Aut}_P(T)\varphi$ under the conjugation map $\mathrm{N}_P(S) \to \mathrm{Aut}_P(S)$, $x \mapsto c_x$, i. e.

$$N_\varphi := \big\{x \in \mathrm{N}_P(S) : \varphi c_x \varphi^{-1} \in \mathrm{Aut}_P(T)\big\}.$$

- T is called *$\mathcal{F}$-receptive* if every isomorphism $\varphi\colon S \to T$ in $\mathcal{F}$ extends to N_φ (note that $S\mathrm{C}_P(S) \le N_\varphi \le \mathrm{N}_P(S)$).

[5] Puig calls them *Frobenius categories*

[6] often called *fully* $\mathcal{F}$-centralized/normalized

Example 2.5.

(i) If $S, T \le P \le G$ are fused in G, then they are $\mathcal{F}_P(G)$-conjugate.

(ii) If $P \in \mathrm{Syl}_p(G)$, then P is automized in $\mathcal{F}_P(G)$, because $P\mathrm{C}_G(P)/\mathrm{C}_G(P) \in \mathrm{Syl}_p(\mathrm{N}_G(P)/\mathrm{C}_G(P))$.

(iii) Every central subgroup of P is $\mathcal{F}$-centralized and every normal subgroup is $\mathcal{F}$-normalized.

(iv) Every $\mathcal{F}$-receptive subgroup is $\mathcal{F}$-centralized: Let $T \le P$ be receptive and $\varphi\colon S \to T$ an isomorphism in $\mathcal{F}$. Then φ extends to $\hat{\varphi}\colon N_\varphi \to T$. For $s \in S$ and $g \in \mathrm{C}_P(S)$ we have $\hat{\varphi}(g)\varphi(s)\hat{\varphi}(g)^{-1} = \hat{\varphi}(gsg^{-1}) = \varphi(s)$ and $\hat{\varphi}(\mathrm{C}_P(S)) \le \mathrm{C}_P(T)$. Since morphisms are injective, it follows that $|\mathrm{C}_P(S)| \le |\mathrm{C}_P(T)|$.

(v) Every $\mathcal{F}$-centralized, $\mathcal{F}$-automized subgroup $S \le P$ is $\mathcal{F}$-normalized. This follows from $|\mathrm{N}_P(S)| = |\mathrm{Aut}_P(S)||\mathrm{C}_P(S)|$.

(vi) Let $S := \langle(12)(34)\rangle \le P := \langle(1234),(13)\rangle \le G := S_4$ and $\mathcal{F} := \mathcal{F}_P(G)$. Then S is neither $\mathcal{F}$-centralized nor $\mathcal{F}$-normalized since S is $\mathcal{F}$-conjugate to $\mathrm{Z}(P) = \langle(13)(24)\rangle$.

Theorem 2.6. *The following assertions for a fusion system $\mathcal{F}$ on P are equivalent:*

(1) (Roberts–Shpectorov) *Every subgroup of P is $\mathcal{F}$-conjugate to an automized, receptive subgroup.*

(2) P is automized and every subgroup of P is $\mathcal{F}$-conjugate to a normalized, receptive subgroup.

(3) (Stancu) *P is automized and every normalized subgroup of P is receptive.*

(4) (Broto–Levi–Oliver) *Every normalized subgroup of P is centralized and automized and every centralized subgroup is receptive.*

Under these circumstances we call $\mathcal{F}$ saturated.

For a saturated fusion system $\mathcal{F}$ on P and $S \le P$ we have

(i) S is $\mathcal{F}$-centralized if and only if S is $\mathcal{F}$-receptive.

(ii) S is $\mathcal{F}$-normalized if and only if S is $\mathcal{F}$-centralized and $\mathcal{F}$-automized.

Theorem 2.7. *If $P \in \mathrm{Syl}_p(G)$, then $\mathcal{F}_P(G)$ is saturated.*

Proof. We prove Theorem 2.6(1) for $\mathcal{F} := \mathcal{F}_P(G)$. Let $Q \le P$ and $\mathrm{N}_P(Q) \le R \in \mathrm{Syl}_p(\mathrm{N}_G(Q))$. By Sylow's theorem, there exists $g \in G$ such that

$$T := {}^gQ \le {}^gR \le P.$$

Since ${}^gR \in \mathrm{Syl}_p({}^g\mathrm{N}_G(Q)) = \mathrm{Syl}_p(\mathrm{N}_G(T))$, we have ${}^gR = \mathrm{N}_P(T)$ and T is $\mathcal{F}$-automized.

Now let $\varphi\colon S \to T$ be an arbitrary isomorphism in $\mathcal{F}$. Then there exists $a \in G$ with $\varphi(s) = {}^as$ for all $s \in S$. For $x \in N_\varphi$ there exists $y \in \mathrm{N}_P(T)$ such that

$${}^{axa^{-1}}t = (\varphi c_x \varphi^{-1})(t) = {}^yt$$

for all $t \in T$. Hence, $y^{-1}axa^{-1} \in \mathrm{C}_G(T)$ and $axa^{-1} \in \mathrm{N}_P(T)\mathrm{C}_G(T)$. By definition, $N_\varphi \le \mathrm{N}_P(S)$ is a p-group and ${}^aN_\varphi$ is a p-subgroup of $\mathrm{N}_P(T)\mathrm{C}_G(T)$. Since $\mathrm{N}_P(T)$ is a Sylow p-subgroup of $\mathrm{N}_G(T) \ge \mathrm{N}_P(T)\mathrm{C}_G(T)$, there exist $h \in \mathrm{N}_P(T)$ and $z \in \mathrm{C}_G(T)$ with ${}^{hza}N_\varphi \le \mathrm{N}_P(T)$. Then also ${}^{za}N_\varphi \le \mathrm{N}_P(T) \le P$. For $s \in S$ we have ${}^{za}s = {}^z\varphi(s) = \varphi(s)$. Hence, the conjugation with za is an extension of φ to N_φ in $\mathcal{F}$. Consequently, T is $\mathcal{F}$-receptive. □

Example 2.8. Let $|P| > p$. A theorem of Gaschütz' asserts that P has an outer automorphism of p-power order. Hence, P is not automized in $\mathcal{U}(P)$ and $\mathcal{U}(P)$ is not saturated.

Theorem 2.9 (Robinson, Leary–Stancu). *For every saturated fusion system $\mathcal{F}$ on P there exists an* infinite *group G with $P \in \mathrm{Syl}_p(G)$ such that $\mathcal{F} = \mathcal{F}_P(G)$.*

Definition 2.10. A saturated fusion system $\mathcal{F}$ is called *exotic* if there is no *finite* group G with $P \in \mathrm{Syl}_p(G)$ and $\mathcal{F} = \mathcal{F}_P(G)$.

Example 2.11.

(i) For $p = 2$ the only known exotic fusion systems are defined on the Sylow 2-subgroups of $\mathrm{Spin}_7(q) \cong 2.\Omega_7(q)$ where q is an odd prime power. These are called the *Solomon fusion systems*. For $q = 3$ we have $|P| = 2^{10}$.

(ii) For $p > 2$ many families of exotic fusion systems have been discovered recently. For instance, Ruiz–Viruel constructed an exotic fusion system $\mathcal{F}$ on the extraspecial group P of order 7^3 with exponent 7 such that all non-trivial elements of P are $\mathcal{F}$-conjugate.

Most of the fusion and transfer theorems for finite groups stated in Section 1 have been translated to fusion systems. For instance, a saturated fusion system $\mathcal{F}$ is trivial if and only if $\mathrm{Aut}_{\mathcal{F}}(Q)$ is a p-group for every $Q \le P$. This will be generalized in the next section. To state some more theorems, we need the following constructions.

Definition 2.12. Let $\mathcal{F}$ be a saturated fusion system on P and $Q \le P$.

- The fusion system $\mathrm{C}_{\mathcal{F}}(Q)$ on $\mathrm{C}_P(Q)$ consists of the morphisms $\varphi\colon S \to T$ such that there exists a morphism $\psi\colon QS \to QT$ in $\mathcal{F}$ with $\psi_S = \varphi$ and $\psi_Q = \mathrm{id}_Q$.
- The fusion system $\mathrm{N}_{\mathcal{F}}(Q)$ on $\mathrm{N}_P(Q)$ consists of the morphisms $\varphi\colon S \to T$ such that there exists a morphism $\psi\colon QS \to QT$ in $\mathcal{F}$ with $\psi_S = \varphi$ and $\psi(Q) = Q$.
- The fusion system $Q\mathrm{C}_{\mathcal{F}}(Q)$ on $Q\mathrm{C}_P(Q)$ consists of the morphisms $\varphi\colon S \to T$ such that there exists a morphism $\psi\colon QS \to QT$ in $\mathcal{F}$ with $\psi_S = \varphi$ and $\psi_Q \in \mathrm{Inn}(Q)$.

Recall that every subgroup $Q \le P$ is $\mathcal{F}$-conjugate to an $\mathcal{F}$-normalized subgroup. In this case, Puig has shown that $\mathrm{C}_{\mathcal{F}}(Q)$, $\mathrm{N}_{\mathcal{F}}(Q)$ and $Q\mathrm{C}_{\mathcal{F}}(Q)$ are saturated.

Example 2.13. If $Q \le P \in \mathrm{Syl}_p(G)$ and $\mathcal{F} = \mathcal{F}_P(G)$. If Q is $\mathcal{F}$-normalized, then $\mathrm{C}_{\mathcal{F}}(Q) = \mathcal{F}_{\mathrm{C}_P(Q)}(\mathrm{C}_G(Q))$, $\mathrm{N}_{\mathcal{F}}(Q) = \mathcal{F}_{\mathrm{N}_P(Q)}(\mathrm{N}_G(Q))$ and $Q\mathrm{C}_{\mathcal{F}}(Q) = \mathcal{F}_{Q\mathrm{C}_P(Q)}(Q\mathrm{C}_G(Q))$.

Theorem 2.14 (Kessar–Linckelmann). *A saturated fusion system $\mathcal{F}$ on P with $p > 2$ is trivial if and only if $\mathrm{N}_{\mathcal{F}}\big(\mathrm{Z}(\mathrm{J}(P))\big)$ is trivial.*

Definition 2.15. For a saturated fusion system $\mathcal{F}$ on P we define

$$\mathrm{Z}(\mathcal{F}) := \big\{x \in P : \varphi(x) = x \ \forall \varphi \in \mathrm{Hom}_{\mathcal{F}}(\langle x \rangle, P)\big\} \qquad (\textit{center}),$$
$$\mathfrak{foc}(\mathcal{F}) := \langle \varphi(x)x^{-1} : x \in P,\ \varphi \in \mathrm{Hom}_{\mathcal{F}}(\langle x \rangle, P)\rangle \qquad (\textit{focal subgroup}),$$
$$\mathfrak{hyp}(\mathcal{F}) := \langle \varphi(x)x^{-1} : x \in Q \le P,\ \varphi \in \mathrm{O}^p(\mathrm{Aut}_{\mathcal{F}}(Q))\rangle \qquad (\textit{hyperfocal subgroup}).$$

Example 2.16.

(i) The center $\mathrm{Z}(\mathcal{F})$ is the largest subgroup $Q \le P$ such that $\mathrm{C}_{\mathcal{F}}(Q) = \mathcal{F}$.

(ii) One can show that $\mathfrak{foc}(\mathcal{F}) = \mathfrak{hyp}(\mathcal{F})P'$ and $\mathfrak{foc}(\mathcal{F}) \cap \mathrm{Z}(\mathcal{F}) = P' \cap \mathrm{Z}(\mathcal{F})$. In particular, the *Fitting decomposition* $P = \mathrm{Z}(\mathcal{F}) \times \mathfrak{foc}(\mathcal{F})$ holds whenever P is abelian.

(iii) If $\mathcal{F} = \mathcal{F}_P(G)$, then $\mathfrak{foc}(\mathcal{F}) = \mathfrak{foc}_P(G)$, $\mathfrak{hyp}(\mathcal{F}) = \mathfrak{hyp}_P(G)$ and $\mathrm{Z}(\mathcal{F}) = \mathrm{Z}(G/\mathrm{O}_{p'}(G))$ by the Z^*-theorem.

Theorem 2.17 (Díaz–Glesser–Park–Stancu). *Let $\mathcal{F}$ be a saturated fusion system on P.*

(i) If $\mathcal{E} \subseteq \mathcal{F}$ is a saturated subsystem (subcategory) on P, then $\mathfrak{foc}(\mathcal{F}) = \mathfrak{foc}(\mathcal{E}) \iff \mathfrak{hyp}(\mathcal{F}) = \mathfrak{hyp}(\mathcal{E})$.

(ii) If P has no quotient isomorphic to $C_p \wr C_p$, then $\mathfrak{foc}(\mathcal{F}) = \mathfrak{foc}(\mathrm{N}_{\mathcal{F}}(P))$. In particular, $\mathcal{F}$ is trivial if and only if $\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Inn}(P)$.

Theorem 2.18 (Díaz–Glesser–Mazza–Park). *Let $\mathcal{F}$ be a saturated fusion system on P with $p \ge 5$. Then $\mathfrak{foc}(\mathcal{F}) = \mathfrak{foc}(\mathrm{N}_{\mathcal{F}}(K))$ where K is the characteristic subgroup from Theorem 1.18.*

Kessar–Linckelmann and Onofrei–Stancu have translated Theorems 1.9 and 1.11 to fusion systems, but this requires the definition of *$Qd(p)$-free* fusion systems.

3 Classification of fusion systems

Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. Let $\mathrm{Out}_{\mathcal{F}}(Q) := \mathrm{Aut}_{\mathcal{F}}(Q)/\mathrm{Inn}(Q)$ for $Q \le P$.

Theorem 3.1 (Glauberman–Thompson). *If $\mathfrak{foc}(\mathcal{F}) = P \ne 1$ and $p \ge 5$, then $\mathrm{Out}_{\mathcal{F}}(P) \ne 1$.*

Definition 3.2. A subgroup $Q \le P$ is called $\mathcal{F}$-*essential* if

- $\mathrm{C}_P(Q) \le Q$,
- Q is $\mathcal{F}$-normalized,
- there exists a *strongly p-embedded* subgroup $H < \mathrm{Out}_{\mathcal{F}}(Q)$, i. e. $p \mid |H|$ and $p \nmid |H \cap H^x|$ for every $x \in \mathrm{Out}_{\mathcal{F}}(Q) \setminus H$ (cf. Frobenius complement).

Example 3.3.

(i) Every $\mathcal{F}$-essential subgroup $Q \le P$ is $\mathcal{F}$-*radical*, i. e. $\mathrm{O}_p(\mathrm{Aut}_{\mathcal{F}}(Q)) = \mathrm{Inn}(Q)$. To prove this, let $H < U := \mathrm{Out}_{\mathcal{F}}(Q)$ be strongly p-embedded. Let $H_p \le U_p$ be Sylow p-subgroups of H and U respectively. For $x \in \mathrm{N}_{U_p}(H_p)$, we have $1 \ne H_p \le H \cap {}^xH$ and therefore $x \in H_p$. Hence, $\mathrm{N}_{U_p}(H_p) = H_p$ and $H_p = U_p$ by standard group theory. It follows that $\mathrm{O}_p(U) \le H \cap {}^uH = 1$ for all $u \in U \setminus H$.

(ii) Part (i) shows that every essential subgroup Q has non-trivial p'-automorphisms and $\mathrm{Out}_{\mathcal{F}}(Q)$ acts faithfully on $Q/\Phi(Q) \cong C_p^r$. Therefore, $\mathrm{Out}_{\mathcal{F}}(Q) \le \mathrm{GL}(r, p)$.

(iii) Since P is $\mathcal{F}$-automized, $\mathrm{Out}_{\mathcal{F}}(P)$ is a p'-group and P is not essential.

(iv) If P is abelian, then there are no essential subgroups, since P is the only self-centralizing subgroup.

(v) Let $G = S_4$, $P \in \mathrm{Syl}_2(G)$ and $\mathcal{F} = \mathcal{F}_P(G)$. Then $V_4 := \langle (12)(34), (13)(24) \rangle \leq P$ is $\mathcal{F}$-essential since $\mathrm{Out}_{\mathcal{F}}(V_4) = G/V_4 \cong S_3$ contains the strongly 2-embedded subgroup $P/V_4 \cong C_2$. On the other hand, $Q := \langle (12), (34) \rangle \cong V_4$ is not $\mathcal{F}$-essential (provided $Q \leq P$).

Theorem 3.4 (ALPERIN–GOLDSCHMIDT's fusion theorem)**.** *Let $\mathcal{E}$ be a set of representatives for the $\mathcal{F}$-conjugacy classes of essential subgroups. Every isomorphism in $\mathcal{F}$ is a composition of isomorphisms of the form $\varphi\colon S \to T$ with the following properties:*

(i) $S, T \leq Q \in \mathcal{E} \cup \{P\}$.

(ii) $\exists \psi \in \mathrm{Aut}_{\mathcal{F}}(Q)$ *such that* $\psi_S = \varphi$,

(iii) If $Q \in \mathcal{E}$, *then* ψ *is a* p*-element.*

The number $|\mathcal{E}|$ in Theorem 3.4 is called the *essential rank* of $\mathcal{F}$.

Theorem 3.5. *A group G contains a strongly p-embedded subgroup if and only if one of the following holds:*

(1) $\mathrm{O}_p(G) = 1$ *and the Sylow p-subgroups of G are cyclic or quaternion groups.*

(2) $\mathrm{O}^{p'}\big(G/\mathrm{O}_{p'}(G)\big)$ *is one of the following:*

- $\mathrm{PSL}(2, p^n)$ *for* $n \geq 2$,
- $\mathrm{PSU}(3, p^n)$ *for* $n \geq 1$,
- $\mathrm{Sz}(2^{2n+1})$ *for* $p = 2$ *and* $n \geq 1$,
- ${}^2G_2(3^{2n-1})$ *for* $p = 3$ *and* $n \geq 1$,
- A_{2p} *for* $p \geq 5$,
- $\mathrm{PSL}_3(4)$, M_{11} *for* $p = 3$,
- $\mathrm{Aut}(\mathrm{Sz}(32))$, ${}^2F_4(2)'$, McL, Fi_{22} *for* $p = 5$,
- J_4 *for* $p = 11$.

Proof. The proof of $p = 2$ is due to Bender, while the case $p > 2$ was established during the CFSG. □

Example 3.6.

(i) In the situation of Theorem 3.5(1), every $P \in \mathrm{Syl}_p(G)$ has a unique subgroup $\Omega(P)$ of order p. It is easy to see that $\mathrm{N}_G(\Omega(P))$ is strongly p-embedded in G.

(ii) The groups in Theorem 3.5(2) apart from A_{2p}, ${}^2G_2(3) \cong \mathrm{PSL}(2, 8).3$ and $\mathrm{Aut}(\mathrm{Sz}(32)) \cong \mathrm{Sz}(32).5$ are precisely the simple groups G with a non-cyclic *trivial intersection* (TI) Sylow p-subgroup P, i. e. $P \cap {}^gP = 1$ for all $g \in G \setminus \mathrm{N}_G(P)$. Thus, $\mathrm{N}_G(P)$ is strongly p-embedded in this case.

(iii) Let $p \geq 5$ and $G = A_{2p}$. Then $H := G \cap (S_p \wr C_2)$ is strongly p-embedded in G.

Corollary 3.7. *Let $Q \leq P$ be $\mathcal{F}$-essential with $p \geq 5$. Then one of the following holds for $N := \mathrm{N}_P(Q)/Q$:*

(1) N is cyclic or elementary abelian.

(2) $\exp(N) = p$ *and* $\mathrm{Z}(N) = N' = \Phi(N) \cong C_p^n$ *where* $|N| = p^{3n}$ *(i. e. N is* special).

Alperin–Goldschmidt's fusion theorem and Theorem 3.5 make it feasible to determine all saturated fusion systems on a given p-group. Parker–Semeraro have developed a MAGMA algorithm for this purpose and discovered fusion systems overlooked in previous work.[7] Since "most" p-groups do not have non-trivial p'-automorphisms, there are very few essential subgroups and "most" fusion systems are trivial.

Definition 3.8.

- We call $\mathcal{F}$ *controlled* if there are no essential subgroups.
- We call P *resistant*[8] if every fusion system on P is controlled.
- We call P *fusion-trivial* if every fusion system on P is trivial.

Example 3.9.

(i) Let $P \in \mathrm{Syl}_p(G)$. Then $\mathcal{F}_P(G)$ is controlled if and only if $\mathrm{N}_G(P)$ controls fusion in P.

(ii) By the Schur–Zassenhaus theorem, $\mathrm{Inn}(P)$ has a complement A in $\mathrm{Aut}_{\mathcal{F}}(P)$ since P is automized. If $\mathcal{F}$ is controlled, then $\mathcal{F} = \mathcal{F}_P(P \rtimes A)$. In particular, $\mathcal{F}$ is not exotic.

(iii) Every abelian p-group is resistant by Example 3.3.

(iv) Stancu proved that every metacyclic p-group for $p > 2$ is resistant. I proved that metacyclic 2-groups apart from D_{2^n}, Q_{2^n}, SD_{2^n} and $C_{2^n}^2$ are fusion-trivial.

(v) Every 2-group of the form $C_{2^{a_1}} \times \ldots \times C_{2^{a_n}}$ with $a_1 < \ldots < a_n$ is fusion-trivial. The smallest non-trivial fusion-trivial p-group of odd order is $\texttt{SmallGroup}(3^6, 46)$.

(vi) Let $\mathcal{F}$ be a saturated fusion system on $P = \langle x, y : x^4 = y^2 = 1,\ {}^y x = x^{-1} \rangle \cong D_8$. There are three cases:

   (a) $\mathcal{F}$ is controlled and therefore trivial since $\mathrm{Aut}(P) \cong D_8$ is a 2-group.

   (b) There is exactly one essential subgroup, say $\langle x^2, y \rangle$. Then $\mathcal{F} = \mathcal{F}_P(S_4)$.

   (c) There are two essential subgroups $\langle x^2, y \rangle$ and $\langle x^2, xy \rangle$. Then $\mathcal{F} = \mathcal{F}_P(\mathrm{GL}(3, 2))$. In contrast to S_4, all involutions in $\mathrm{GL}(3, 2)$ are conjugate, namely to the rational canonical form

$$\begin{pmatrix} 1 & . & . \\ . & . & 1 \\ . & 1 & . \end{pmatrix}.$$

Definition 3.10. We call $Q \trianglelefteq P$ *normal* in $\mathcal{F}$ (and write $Q \trianglelefteq \mathcal{F}$) if $\mathrm{N}_{\mathcal{F}}(Q) = \mathcal{F}$.

Let $Q, R \trianglelefteq \mathcal{F}$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(S, T)$. Then there exist $\psi \in \mathrm{Hom}_{\mathcal{F}}(RS, RT)$ and $\tau \in \mathrm{Hom}_{\mathcal{F}}(QRS, QRT)$ such that $\psi(R) = R$, $\psi_S = \varphi$, $\tau(Q) = Q$ and $\tau_{RS} = \psi$. Hence, $\tau(QR) = \tau(Q)\psi(R) = QR$ and $\tau_S = \psi_S = \varphi$. This shows that $\varphi \in \mathrm{N}_{\mathcal{F}}(QR)$ and $QR \trianglelefteq \mathcal{F}$. The following definition is therefore justified.

Definition 3.11.

- The (unique) largest normal subgroup of $\mathcal{F}$ is denoted by $\mathrm{O}_p(\mathcal{F})$.

[7] `https://github.com/chris1961parker/fusion-systems`

[8] sometimes called *Swan group*

- We call $\mathcal{F}$ *constrained* if $\mathrm{C}_P(\mathrm{O}_p(\mathcal{F})) \le \mathrm{O}_p(\mathcal{F})$.

Example 3.12.

(i) If $Q \le P \in \mathrm{Syl}_p(G)$ and $Q \trianglelefteq G$, then $Q \trianglelefteq \mathcal{F}_P(G)$. On the other hand, if P is abelian, then $P \trianglelefteq \mathcal{F}_P(G)$, but not necessarily $P \trianglelefteq G$.

(ii) Every essential subgroup contains $\mathrm{O}_p(\mathcal{F})$ and $\mathrm{Z}(\mathcal{F}) \le \mathrm{O}_p(\mathcal{F})$.

(iii) Every controlled fusion system $\mathcal{F}$ on P is constrained with $\mathrm{O}_p(\mathcal{F}) = P$. On the other hand, $\mathcal{F} := \mathcal{F}_{D_8}(S_4)$ is constrained with $\mathrm{O}_2(\mathcal{F}) = V_4$, but not controlled.

(iv) Let $G = \mathrm{GL}(3, 2)$ and $P \in \mathrm{Syl}_2(G)$. Then $\mathcal{F}_P(G)$ is not constrained since the two essential subgroups intersect in $\mathrm{Z}(P)$ (cf. Example 3.9(vi)). Moreover, $\mathfrak{foc}(\mathcal{F}) = P \ntrianglelefteq \mathcal{F}$.

(v) A group G is called *p-constrained* if $\mathrm{C}_{\overline{G}}(\mathrm{O}_p(\overline{G})) \le \mathrm{O}_p(\overline{G})$ where $\overline{G} := G/\mathrm{O}_{p'}(G)$. In this case $\mathcal{F} := \mathcal{F}_P(G)$ is constrained with $\overline{\mathrm{O}_p(\mathcal{F})} = \overline{\mathrm{O}_p(G)}$. By Theorem 3.13 below every constrained fusion system arises in this way. The Hall–Higman lemma asserts that every (p-)solvable group is p-constrained.

Theorem 3.13 (Model theorem)**.** *For every constrained fusion system $\mathcal{F}$ on P there exists a unique finite group G (called* model*) such that*

(i) $P \in \mathrm{Syl}_p(G)$ *and* $\mathcal{F} = \mathcal{F}_P(G)$.

(ii) $\mathrm{O}_{p'}(G) = 1$ *and* $\mathrm{C}_G(\mathrm{O}_p(G)) \le \mathrm{O}_p(G)$.

In particular, $\mathcal{F}$ is not exotic.

Let G be a model for the constrained fusion system $\mathcal{F}$ on P with $|P| = p^n$. A theorem of Hall shows that

$$|G| \le |G/\mathrm{O}_p(G)||P| \le |\mathrm{Aut}(\mathrm{O}_p(G))|p^n \le |\mathrm{GL}(n,p)|p^n = (p^n - 1)\dots(p^n - p^{n-1})p^n.$$

In particular, there are only finitely many choices when P is given.

Example 3.14. If $\mathcal{F}$ is controlled, then $P \rtimes A$ is the model for $\mathcal{F}$ where $A \cong \mathrm{Out}_{\mathcal{F}}(P)$ as in Example 3.9.

Theorem 3.15 (Glesser)**.** *Let $p > 2$ and $\mathcal{F}$ a non-trivial fusion system on P. Then $\mathcal{F}$ contains (as a subcategory) a non-trivial constrained fusion system on P.*

One can use Theorem 3.15 and the model theorem to decide whether a given group P is fusion-trivial. The fusion system $\mathcal{F}_{D_{16}}(\mathrm{PGL}(2, 7))$ (found by Craven) shows that Glesser's theorem fails for $p = 2$. In order to classify non-constrained fusion systems (especially exotic fusion systems), Oliver has introduced *reduced* and *tame* fusion systems. In an ongoing effort to simplify the CFSG, Aschbacher has investigated *simple* fusion systems. Unfortunately, fusion systems of simple groups are not always simple, but well-studied nevertheless.

4 Representation theory

Let F be an algebraically closed field of characteristic $p > 0$. Let B be a (p-)*block* of FG, i. e. an indecomposable direct summand. We fix a *defect group* $D \le G$ of B.

Definition 4.1 (Alperin–Broué, Olsson).

- We call (Q, b_Q) a *B-subpair* if $Q \le D$ and b_Q is a Brauer correspondent of B in $Q\mathrm{C}_G(Q)$, i. e. $b_Q^G = B$. For subpairs we write $(S, b_S) \trianglelefteq (T, b_T)$ if $S \trianglelefteq T$ and $b_S^{T\mathrm{C}_G(S)} = b_T^{T\mathrm{C}_G(S)}$.[9] Let $\le$ be the transitive closure of $\trianglelefteq$, i. e.

$$(S, b_T) \le (T, b_T) \iff (S, b_T) = (T_1, b_1) \trianglelefteq \ldots \trianglelefteq (T_n, b_n) = (T, b_T).$$

- We fix a B-subpair (D, b_D) (by Brauer's extended first main theorem, (D, b_D) is unique up to conjugation). It can be shown that for every $Q \le D$ there exists a unique subpair of the form $(Q, b_Q) \le (D, b_D)$. We fix those in the following. The fusion system $\mathcal{F} = \mathcal{F}_D(B)$ on D is defined by
$$\mathrm{Hom}_{\mathcal{F}}(S, T) := \bigl\{\varphi\colon S \to T : \exists g \in G : {}^g(S, b_S) \le (T, b_T) \wedge \varphi(s) = {}^g s\, \forall s \in S\bigr\}.$$

Theorem 4.2 (Puig). *The fusion system $\mathcal{F}_D(B)$ is saturated.*

We call B *nilpotent* (*controlled*, *constrained*) if $\mathcal{F}_D(B)$ is trivial (controlled, constrained). The irreducible ordinary and modular characters of G can be distributed into blocks. We set $k(B) := |\mathrm{Irr}(B)|$ and $l(B) := |\mathrm{IBr}(B)|$. Moreover, let $\mathfrak{foc}(B) := \mathfrak{foc}(\mathcal{F}_D(B))$.

Example 4.3.

(i) The *principal* block $B = B_0(G)$ contains the trivial character of G. In this case $D \in \mathrm{Syl}_p(G)$ and $\mathcal{F}_D(B) = \mathcal{F}_D(G)$. In particular, G is p-nilpotent if and only if B is nilpotent. In this case, all blocks of G are nilpotent.

(ii) If $\mathrm{C}_G(\mathrm{O}_p(G)) \le \mathrm{O}_p(G)$, then $B_0(G)$ is the only block of G.

(iii) In the context Definition 4.1, $\mathrm{Out}_{\mathcal{F}}(D) = \mathrm{N}_G(D, b_D)/D\mathrm{C}_G(D)$ is called the *inertial quotient* of B and its order is the *inertial index*, which is coprime to p by Theorem 4.2.

(iv) The dihedral group $G = D_{24}$ has a nilpotent 3-block with defect group $D \cong C_3$, while the principal 3-block is not nilpotent. This shows that D alone does not determine the fusion system of a block.

Conjecture 4.4. *For every block B of G with defect group D there exists a finite group H such that $D \in \mathrm{Syl}_p(H)$ and $\mathcal{F}_D(B) = \mathcal{F}_D(H)$.*

Theorem 4.5.

(i) Let B be a block of S_n with defect group D. Then there exists an integer $w \ge 0$ (called the weight *of B) such that $D \in \mathrm{Syl}_p(S_{pw})$ and $\mathcal{F}_D(B) = \mathcal{F}_D(S_{pw})$.*

(ii) Let B be a block of A_n with defect group D. Then $\mathcal{F}_D(B) \in \{\mathcal{F}_D(S_{pw}), \mathcal{F}_D(A_{pw})\}$ for some $w \ge 0$.

[9] Alperin–Broué require additionally that b_S is T-invariant, but Olsson showed that this is unnecessary.

Theorem 4.6 (Humphreys, An–Dietrich). *Let B be a block of a group G of Lie type in characteristic p with defect group D. Then $D = 1$ or $D \in \mathrm{Syl}_p(G)$ and $\mathcal{F}_D(B) = \mathcal{F}_D(G)$.*

It has been shown that there is no block with the exotic fusion systems mentioned in Example 2.11.

Theorem 4.7 (Puig). *Let B be nilpotent. Then $B \cong (FD)^{n\times n}$ for some $n \ge 1$. In particular, B and FD are* Morita equivalent*, i. e. they have equivalent module categories. Moreover, $k(B) = k(D)$ and $l(B) = 1$.*

Theorem 4.8 (Fong–Reynolds). *Let b be a block of $N \trianglelefteq G$ with inertial group G_b. Then the Brauer correspondence $C \mapsto C^G$ gives a bijection between the blocks of G_b covering b and the blocks of G covering b. Moreover, C and C^G are Morita equivalent and have the same fusion system.*

Theorem 4.9 (Second Fong Reduction). *Let B be a block of G covering a G-invariant block of $N \trianglelefteq G$ with defect 0. Then B is Morita equivalent to a block of a finite group H with the same fusion system. Moreover, there exists a cyclic p'-subgroup $Z \le \mathrm{Z}(H)$ such that $H/Z \cong G/N$.*

The block of H in the situation of Theorem 4.9 is Morita equivalent to a twisted group algebra $F_\alpha[G/N]$ where $\alpha \in \mathrm{H}^2(G/N, F^\times)$. Conversely, every such twisted group algebra is Morita equivalent to a block of a suitable central extension. If B is the principal block or if G/N has trivial Schur multiplier, then $\alpha = 1$ and B is Morita equivalent to $F[G/N]$. This applies also to the following two theorems.

Theorem 4.10 (Külshammer). *If $D \trianglelefteq G$, then B is controlled and Morita equivalent to a twisted group algebra $F_\alpha[D \rtimes \mathrm{Out}_{\mathcal{F}}(D)]$ where $\alpha \in \mathrm{H}^2(\mathrm{Out}_{\mathcal{F}}(D), F^\times)$.*

Theorem 4.11 (Külshammer). *If G is p-solvable, then B is constrained and Morita equivalent to $F_\alpha H$ where H is the model for $\mathcal{F}_D(B)$ from Theorem 3.13 and $\alpha \in \mathrm{H}^2(H, F^\times)$.*

Theorem 4.12 (Eaton–Kessar–Külshammer–Sambale). *Every 2-block B with a metacyclic defect group D belongs to one of the following cases:*

(1) B is nilpotent.

(2) D is dihedral, semidihedral or quaternion and B has tame *representation type (Morita equivalence classes classified up to scalars).*

(3) $D \cong C_{2^n}^2$ and B is Morita equivalent to $F[D \rtimes C_3]$.

(4) $D \cong C_2^2$ and B is Morita equivalent to $B_0(A_5)$.

Conjecture 4.13 (Blockwise Z*-conjecture). *Let B be a block with fusion system $\mathcal{F}$ and $Z := \mathrm{Z}(\mathcal{F})$. Then B is Morita equivalent to its Brauer correspondent b_Z in $\mathrm{C}_G(Z)$.*

Since $\mathrm{N}_G(D, b_D) \le \mathrm{C}_G(Z)$, b_Z is indeed the unique Brauer correspondent of B by the Brauer's first main theorem. Conjecture 4.13 holds for principal blocks by Example 2.16.

Theorem 4.14 (Külshammer–Okuyama, Watanabe). *In the situation of Conjecture 4.13 we have $k(B) \ge k(b_Z)$ and $l(B) \ge l(b_Z)$ with equality in both cases if D is abelian.*

Conjecture 4.15 (Rouquier). *If $Q := \mathfrak{hyp}(\mathcal{F}_D(B))$ is abelian, then B is derived equivalent to its Brauer correspondent B_Q in $\mathrm{N}_G(Q)$.*

Example 4.16. Suppose that B has abelian defect group D. *Broué's conjecture* predicts that B and B_Q are derived equivalent to their common Brauer correspondent in $\mathrm{N}_G(D)$. This implies Rouquier's conjecture for B. Conversely, if Rouquier's conjecture and the blockwise Z*-conjecture hold for B, then B is derived equivalent to its Brauer correspondent in $\mathrm{N}_G(Q, b_Q) \cap \mathrm{C}_G(Z) = \mathrm{N}_G(D, b_D)$ since $D = Q \times Z$ by the Fitting decomposition (Example 2.16). Thus, Broué's conjecture holds for B.

Theorem 4.17 (Watanabe). *If Q is cyclic in the situation of Rouquier's conjecture, then $\mathcal{F}$ is controlled with $\mathrm{Out}_{\mathcal{F}}(D) \le C_{p-1}$ and*

$$k(B) = k(B_Q) = k(D \rtimes \mathrm{Out}_{\mathcal{F}}(D)),$$
$$l(B) = l(B_Q) = |\mathrm{Out}_F(D)|.$$

If $p > 2$ and D is non-abelian metacyclic, then Theorem 4.17 applies.

Definition 4.18. Let $\mathcal{F}$ be a saturated fusion system on P and $Q \trianglelefteq \mathcal{F}$. Then the (saturated) fusion system $\mathcal{F}/Q$ on P/Q consists of the morphism $\varphi\colon S/Q \to T/Q$ such that there exists a morphism $\psi\colon S \to T$ in $\mathcal{F}$ with $\varphi(xQ) = \psi(x)Q$ for all $x \in S$.

Theorem 4.19. *Let B be a block of G with defect group D and $\mathcal{F} = \mathcal{F}_D(B)$. Let (Q, b_Q) be a B-subpair such that Q is $\mathcal{F}$-normalized. Then*

(i) b_Q has defect group $Q\mathrm{C}_D(Q)$ and fusion system $Q\mathrm{C}_{\mathcal{F}}(Q)$.

(ii) $b_Q^{\mathrm{N}_G(Q)}$ has defect group $\mathrm{N}_D(Q)$ and fusion system $\mathrm{N}_{\mathcal{F}}(Q)$.

(iii) b_Q dominates a unique block $\overline{b_Q}$ of $\mathrm{C}_G(Q)Q/Q$ with defect group $\mathrm{C}_P(Q)Q/Q$ and fusion system $Q\mathrm{C}_{\mathcal{F}}(Q)/Q$. Moreover, $l(b_Q) = l(\overline{b_Q})$.

In the situation of Theorem 4.19 the map $S \to S/Q$ is a bijection between the set of $\mathrm{C}_{\mathcal{F}}(Q)Q$-essential subgroups and the set of $\mathrm{C}_{\mathcal{F}}(Q)Q/Q$-essential subgroups. This allows inductive arguments.

Theorem 4.20 (Brauer). *Let B be a block of G with defect group D and $\mathcal{F} = \mathcal{F}_D(B)$. Let $\mathcal{X} \subseteq D$ be a set of representatives for the $\mathcal{F}$-conjugacy classes of D such that $\langle x \rangle$ is $\mathcal{F}$-normalized for $x \in \mathcal{X}$. Then*

$$k(B) = \sum_{x \in \mathcal{X}} l(b_x) = \sum_{x \in \mathcal{X}} l(\overline{b_x}),$$

where $b_x := b_{\langle x \rangle}$. In particular, $k(B) - l(B)$ is locally determined.

The fusion system of a block does not determine $k(B)$ or $l(B)$. For example, the group

$$G = \texttt{SmallGroup}(72, 23) \cong C_3^2 \rtimes D_8$$

with $|\mathrm{Z}(G)| = 2$ from the *small groups library* has two 3-blocks B_0, B_1 with defect group $D = C_3^2$ and fusion system $\mathcal{F}_D(S_3^2)$, but $l(B_0) = 4$ and $l(B_1) = 1$. We need an additional ingredient: For an F-algebra A let $z(A)$ be the number of simple projective A-modules up to isomorphism.

Conjecture 4.21 (Alperin's weight conjecture). *Let B be a block G with defect group D and $\mathcal{F} = \mathcal{F}_D(B)$. Let $\mathcal{R}$ be a set of representatives for the $\mathcal{F}$-conjugacy classes of self-centralizing, $\mathcal{F}$-centralized subgroups of D. Then*

$$l(B) = \sum_{Q \in \mathcal{R}} z\big(F_{\gamma_Q} \mathrm{Out}_{\mathcal{F}}(Q)\big)$$

where $\gamma_Q \in \mathrm{H}^2(\mathrm{Out}_{\mathcal{F}}(Q), F^\times)$ is the so-called Külshammer–Puig class.

Example 4.22.

(i) Suppose that B is controlled in the situation of Conjecture 4.21. Then $z(F_{\gamma_Q}\mathrm{Out}_{\mathcal{F}}(Q)) = 0$ for $Q < D$, since $\mathrm{N}_D(Q)/Q$ is a non-trivial normal p-subgroup of $\mathrm{Out}_{\mathcal{F}}(Q)$. Hence, Alperin's conjecture becomes $l(B) = z(F_{\gamma_D}\mathrm{Out}_{\mathcal{F}}(D))$. If in addition B is the principal block (or $\mathrm{Out}_{\mathcal{F}}(D)$ has trivial Schur multiplier), then $l(B) = z(F\mathrm{Out}_{\mathcal{F}}(D)) = k(\mathrm{Out}_{\mathcal{F}}(D))$.

(ii) Let B be the principal 2-block of S_4 with $D = \langle x, y\rangle$ as in Example 3.9. The self-centralizing, $\mathcal{F}$-centralized subgroups are $Q_1 = \langle x^2, y\rangle$, $Q_2 = \langle x^2, xy\rangle$, $Q_3 = \langle x\rangle$ and $Q_4 = D$. Alperin's conjecture becomes

$$l(B) = \sum_{i=1}^{4} z\big(F_{\gamma_{Q_i}}\mathrm{Out}_{\mathcal{F}}(Q_i)\big) = z(FS_3) + 2z(FC_2) + z(F) = 1 + 0 + 1 = 2.$$

Definition 4.23. The *height* $h \geq 0$ of $\chi \in \mathrm{Irr}(B)$ is defined by $\chi(1)_p = p^h|G : D|_p$. Let $k_h(B)$ be the number of $\chi \in \mathrm{Irr}(B)$ with height h.

Theorem 4.24 (Broué–Puig, Robinson). *Let B be a block with defect group D. Then*

(i) $|D/\mathfrak{foc}(B)|$ divides $k_0(B)$ with equality if and only if B is nilpotent.

(ii) $|\mathrm{Z}(D)\mathfrak{foc}(B)/\mathfrak{foc}(B)|$ divides $k_h(B)$ for all $h \geq 0$.

If D is abelian, then $|\mathrm{Z}(\mathcal{F}_D(B))|$ divides $k(B)$, because $D = \mathfrak{foc}(B) \times \mathrm{Z}(\mathcal{F}_D(B))$.

Dade's conjecture, expressing $k_h(B)$ in terms of alternating sums, has been reformulated in terms of fusion systems by Robinson (*ordinary weight conjecture*). Kessar–Linckelmann–Lynd–Semeraro have generalized this and other conjectures in block theory to statements on abstract fusion systems.

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