Rubik's Group

Christmas Lecture on group theory

Benjamin Sambale

Leibniz Universität Hannover

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Disclaimer: I won't spoil how to solve Rubik's cube!

Mechanics

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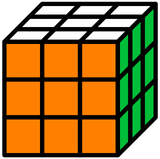

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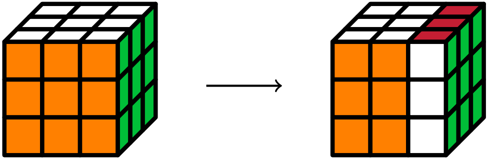

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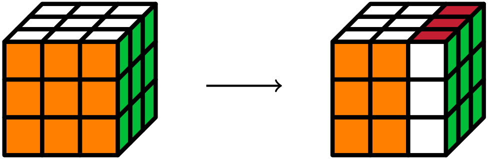

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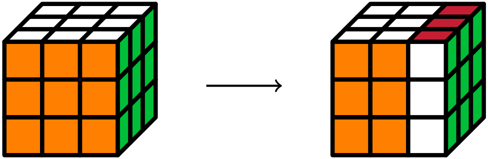

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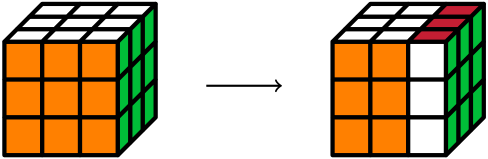

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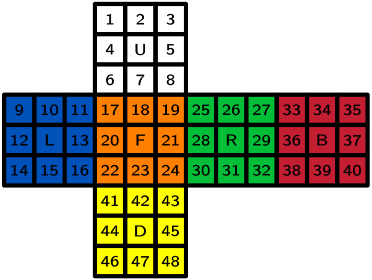
How "big" is the cube?

How many states can we reach by applying an arbitrary number of moves?

Facelets

- Idea: Enumerate the $6 \cdot 8 = 48$ edge and corner facelets:



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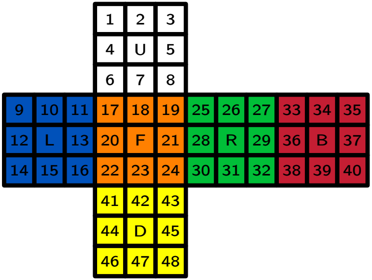

- Every move becomes a permutation in S_{48}, e. g.
  a clockwise $90°$ turn of the front face:

$$f := (6, 25, 43, 16)(7, 28, 42, 13)(8, 30, 41, 11)(17, 19, 24, 22)(18, 21, 23, 30).$$

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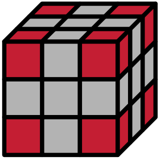
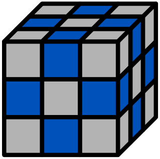
  We can do much better.
- Is G transitive on the 48 facelets?
- No: The $8 \cdot 3 = 24$ corner facelets and the $12 \cdot 2 = 24$ edge facelets form orbits Ω_C and Ω_E.

Ω_C:



Ω_E:



Action on Ω_C

- Hence,

$$G \leq \mathrm{Sym}(\Omega_C) \times \mathrm{Sym}(\Omega_E) \cong S_{24}^2$$

and $|G| \leq (24!)^2 \approx 10^{48}$.

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- No: the three facelets of a corner cubie form a block Δ in Ω_C.

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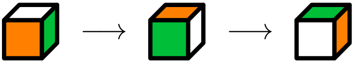
- No: the three facelets of a corner cubie form a block Δ in Ω_C.
- We can permute the three facelets of Δ only cyclically:



Action on Ω_E

From the lecture:

Satz 6.26. *Sei G eine imprimitive Permutationsgruppe auf Ω mit Block Δ. Sei $H := \{g \in G : {}^g\Delta = \Delta\}$ und sei $\varphi : H \to \mathrm{Sym}(\Delta)$ die Operation auf Δ. Sei $\Gamma := \{{}^g\Delta : g \in G\}$ und sei $\psi : G \to \mathrm{Sym}(\Gamma)$ die Operation auf Γ. Dann ist G zu einer Untergruppe von $\varphi(H) \wr \psi(G)$ isomorph.*

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- This gives a homomorphism $G \to C_3 \wr S_8 \leq S_{24}$.
- Similarly, the two facelets of an edge cubie form a block of Ω_E.
- Therefore,

$$G \leq C_3 \wr S_8 \times C_2 \wr S_{12}$$

and $|G| \leq 3^8 8! \cdot 2^{12} 12! \approx 5 \cdot 10^{20}$.

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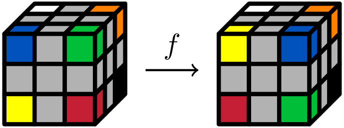

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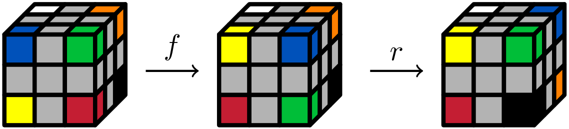

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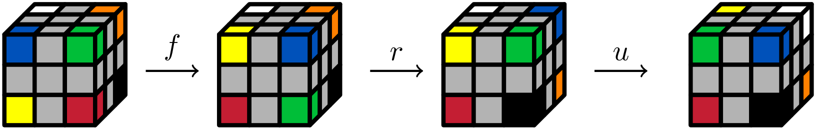

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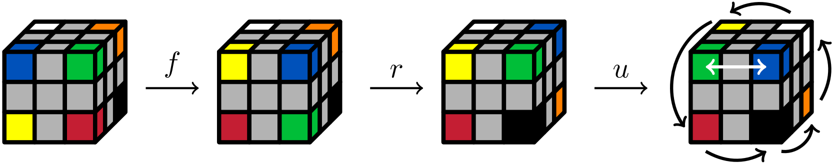

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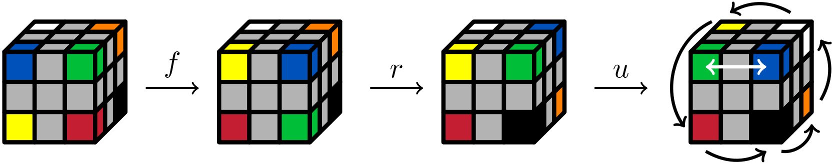

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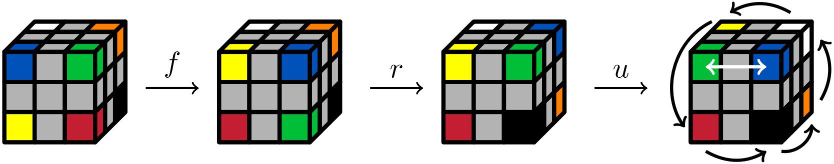

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- By Exercise 31, S_8 is generated by adjacent transpositions. Hence, $\varphi_C(G) = S_8$.

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- By Exercise 31, $A_{12} = \langle (1,2,3), \ldots, (10,11,12) \rangle \subseteq \varphi_E(G_C) \subseteq A_{12}$.

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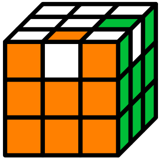
- For this reason it is impossible to flip only one edge and leave everything else fixed.
- Hence, $|N_2| \le 2^{11}$.

Edge flips (realized)

- On the other hand, we can flip just two (adjacent) edges:

$$r^2 f^2 r^{-1} f r f r^2 b^{-1} u^{-1} f^{-1} u f b \quad =$$



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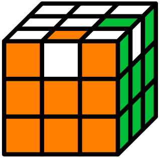

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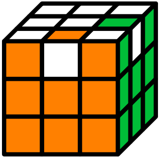

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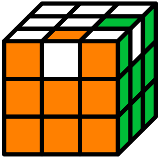

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- The group $N_2 \rtimes S_{12} \leq G$ is the reflection group with Dynkin diagram D_{12}.

Corner orientations (computed)

- Let $g \in G$ be a generator corresponding to

$$(t, \pi) \in \langle \zeta \rangle^{\mathcal{C}} \rtimes \mathrm{Sym}(\mathcal{C}) \cong C_3 \wr S_8.$$

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$$1 = (t, \pi)^4 = (t \cdot {}^{\pi}t, \pi^2) * (t, \pi) * (t, \pi) = (t \cdot {}^{\pi}t \cdot {}^{\pi^2}t, \pi^3) * (t, \pi)$$
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- In particular,

$$1 = \prod_{c\in\mathcal{C}} (t\,{}^{\pi}t\,{}^{\pi^2}t\,{}^{\pi^3}t)(c) = \prod_{c\in\mathcal{C}} t(c) \prod_{c\in\mathcal{C}} t({}^{\pi^{-1}}c) \prod_{c\in\mathcal{C}} t({}^{\pi^{-2}}c) \prod_{c\in\mathcal{C}} t({}^{\pi^{-3}}c)$$
$$= \Big(\prod_{c\in\mathcal{C}} t(c)\Big)^4 = \prod_{c\in\mathcal{C}} t(c).$$

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- If $(t,\pi),(t',\pi') \in \langle\zeta\rangle^{\mathcal{C}} \rtimes \mathrm{Sym}(\mathcal{C})$ such that $\prod t(c) = \prod t'(c) = 1$, then also
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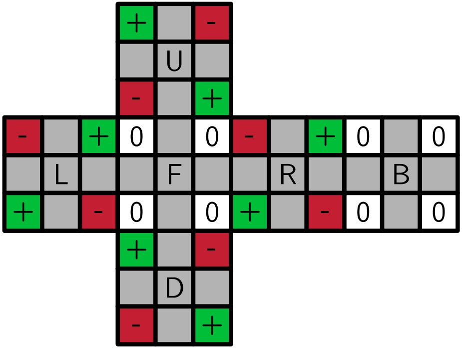
- This can also be visualized as follows.

Corner orientations (visualized)

Fix an orientation of the corner facelets:



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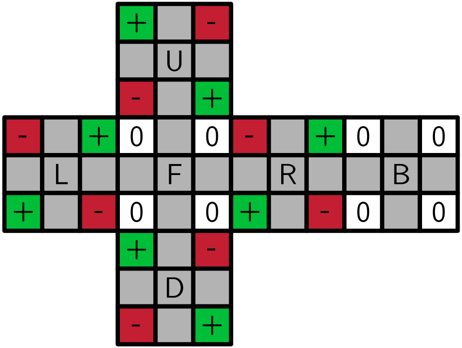

Every move causes one of the following effects:

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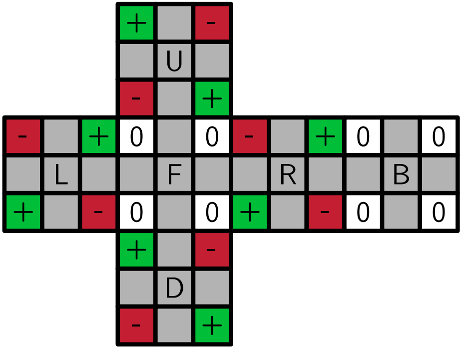

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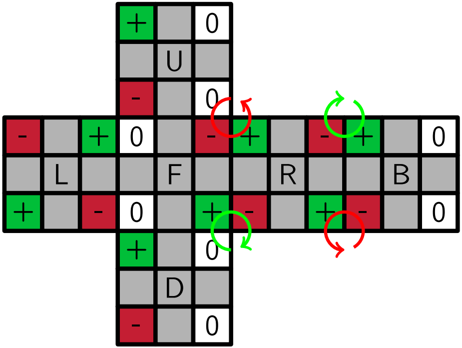

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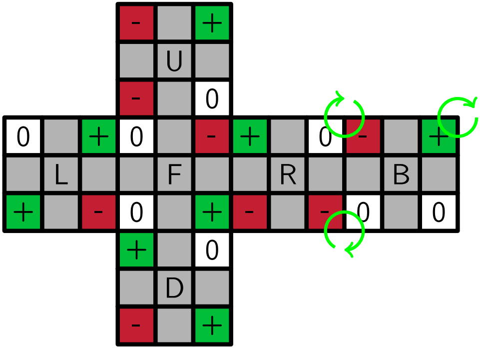

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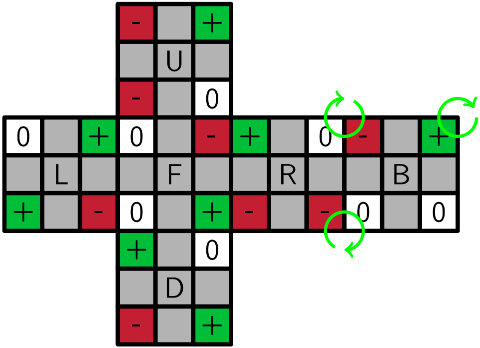

$\Longrightarrow$ The sum of all twists is always 0 modulo 3.

Corner orientations (realized)

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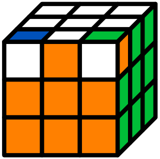
$$u^2bu^2b^{-1}lu^2f^{-1}u^2fl^2b^{-1}lb \quad =$$



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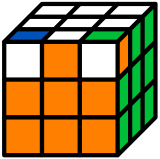

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- This shows $|N_3| = 3^7$.

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An element $(t, \pi, t', \pi') \in (C_3 \wr S_8) \times (C_2 \wr S_{12})$ *belongs to* G *if and only if*

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Hence, the index of G *in* $(C_3 \wr S_8) \times (C_2 \wr S_{12})$ *is* 12 *and*

$$\boxed{|G| = 2^{27} \cdot 3^{14} \cdot 5^3 \cdot 7^2 \cdot 11 = 43.252.003.274.489.856.000.}$$

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Interpretation: After taking apart and reassembling the cubies randomly, the cube is "solvable" in only 1 out of 12 cases.

Consequences

- $G \cong (C_3^7 \times C_2^{11}) \rtimes (A_8 \times A_{12}) \rtimes C_2$. Composition factors: C_2 (12 times), C_3 (7 times), A_8, A_{12}.

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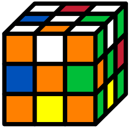
- $\mathrm{Z}(G) = \Phi(G) = \langle s \rangle \cong C_2$ where s is the superflip: 
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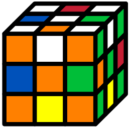

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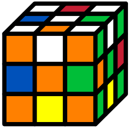

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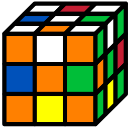

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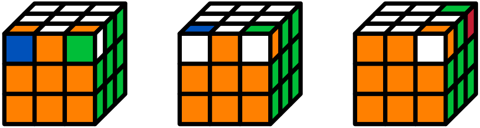
- A chief series: $1 \trianglelefteq \mathrm{Z}(G) \trianglelefteq N_2 \trianglelefteq N \trianglelefteq G_C \trianglelefteq G' \trianglelefteq G$.

Burnside's Lemma

- Some states of the cube are symmetric to each other:



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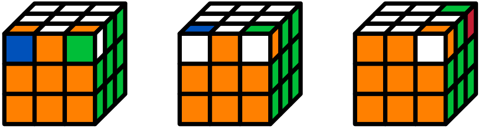

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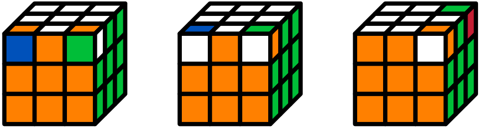

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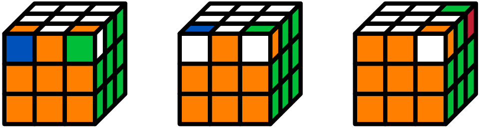

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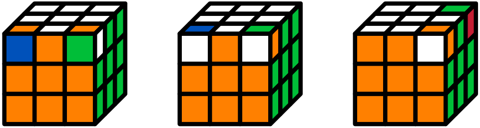

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Using the "symmetry" $g \leftrightarrow g^{-1}$, we get down to $450.541.810.590.509.978$.

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- Obviously, $s_0 = 1$ and $s_1 = 3 \cdot 6 = 18$.
- On the second move, it makes no sense to turn the same face again. This leaves 15 moves.

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Proof (continued).

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- Solving the recurrence yields $\sum_{n=0}^{17} s_n < |G|$. □

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Theorem (Rokicki–Kociemba–Davidson–Dethridge, 2010)

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Sponsored by Google. More info at `https://www.cube20.org`. □

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- Most states require 18 moves and the average is slightly below 18.
- If only quarter turn moves are allowed, God's number increases to 26.

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- There is a zero-knowledge AI algorithm in the spirit of AlphaZero which finds solution with 30 quarter turn moves on average.

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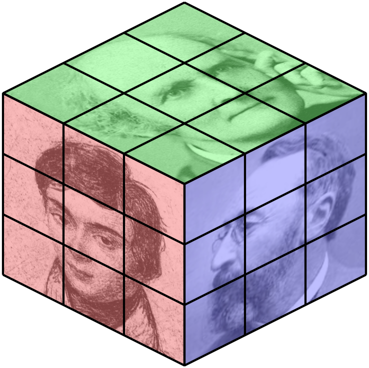
Visit: `https://www.worldcubeassociation.org`, `www.speedsolving.com`

Variations

Is the following cube any different?



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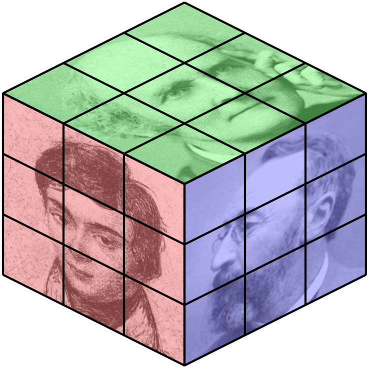


Yes, but not so much harder to solve ($\rightarrow$ Christmas exercise!).

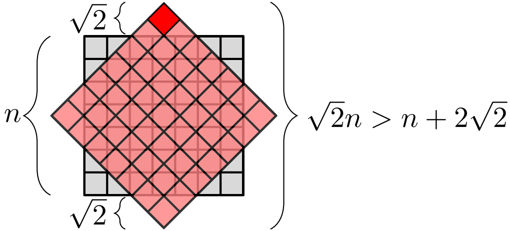
Variations

The invention of $n \times n \times n$-cubes:

n	Inventor	Product name	Year
2	Larry D. Nichols	Pocket Cube	1970
3	Ernő Rubik	Rubik's Cube	1974
4	Péter Sebestény	Rubik's Revence	1981
5	Udo Krell	Professor's Cube	1981
6	Panagiotis Verdes	V-Cube 6	2004

Variations

- For $n \geq 7$ there is a fundamental design problem:



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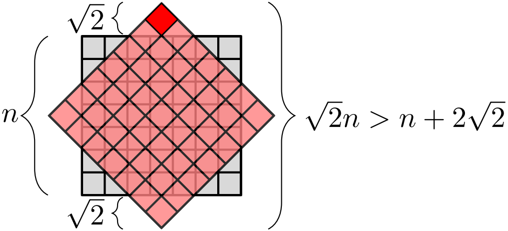

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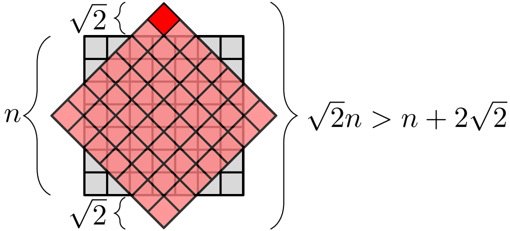

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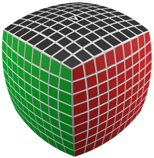
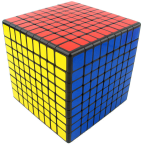
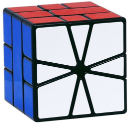
- Remedy:

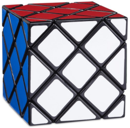
V-Cube 9: 

ShengShou 9: 

Endless other variations

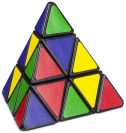


Square-1

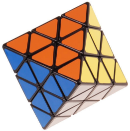


Skewb Master

Ghost cube



Pyraminx



Skewb Diamond

Skewb Ultimate

Gigaminx

Hypercube

Visit: `www.thecubicle.com`, `ruwix.com`, `www.cubikon.de`

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```

With the computer. . .

Lets use the open-source computer algebra system GAP. Rubik's group can be copied from
http://www.gap-system.org/Doc/Examples/rubik.html

GAP-Code

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f:=(6,25,43,16)...;
G:=Group(f,b,l,r,u,d);
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G=Group(u*l,f*r*b); #returns true
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Order(G);
G=Group(u*l,f*r*b); #returns true
```

Interpretation: Every state can be solved using only the two sequences ul and frb (never turning the down face)!

With the computer...

GAP-Code

```
orb:=Orbits(G);
```

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GAP-Code

```
orb:=Orbits(G);
corners:=Blocks(G,orb[1]);
```

With the computer. . .

GAP-Code

```
orb:=Orbits(G);
corners:=Blocks(G,orb[1]);
edges:=Blocks(G,orb[2]);
```

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orb:=Orbits(G);
corners:=Blocks(G,orb[1]);
edges:=Blocks(G,orb[2]);
phiC:=ActionHomomorphism(G,corners,OnSets);
```

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orb:=Orbits(G);
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StructureDescription(Image(phiC)); #returns "S8"
```

With the computer. . .

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orb:=Orbits(G);
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phiE:=ActionHomomorphism(G,edges,OnSets);
StructureDescription(Image(phiC)); #returns "S8"
StructureDescription(Image(phiE,Kernel(phiC))); #"A12"
```

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ZG:=Center(G);
```

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phiE:=ActionHomomorphism(G,edges,OnSets);
StructureDescription(Image(phiC)); #returns "S8"
StructureDescription(Image(phiE,Kernel(phiC))); #"A12"
ZG:=Center(G);
s:=ZG.1; #first generator = superflip
```

With the computer. . .

GAP-Code

```
FG:=FreeGroup("f","b","l","r","u","d");
```

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GAP-Code

```
FG:=FreeGroup("f","b","l","r","u","d");
hom:=GroupHomomorphismByImages(FG,G,GeneratorsOfGroup(FG),
 GeneratorsOfGroup(G)); #Satz 8.7
```

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PreImagesRepresentative(hom,s); #solution of the superflip
```

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```

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PreImagesRepresentative(hom,s); #solution of the superflip
Length(last); #number of quarter turn moves
PreImagesRepresentative(hom,Random(G));
```

With the computer. . .

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PreImagesRepresentative(hom,s); #solution of the superflip
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PreImagesRepresentative(hom,Random(G));
StringTime(time); #how long did it take?
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 GeneratorsOfGroup(G)); #Satz 8.7
PreImagesRepresentative(hom,s); #solution of the superflip
Length(last); #number of quarter turn moves
PreImagesRepresentative(hom,Random(G));
StringTime(time); #how long did it take?
BrowseRubikCube(); #interactive mode
```

My personal advise

- The $2 \times 2 \times 2$-cube or the Pyraminx are fair Christmas presents (can even be solved by luck).

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Merry Christmas!